SECTION – A

1 UNITS AND DIMENSIONS

LEARNING OBJECTIVES

- What is a unit?
- Characteristics of units selected for measurement.
- What is the need of a system of units?
- What are fundamental and derived physical quantities?
- What are fundamental and derived units?
- What is International System of units?
- Fundamental units of SI system.
- Derived units in International System.
- Some Important non-SI units.
- Dimensions of physical quantities.
- Dimensional formula.
- Dimensional equation.
- Dimensional formulae of some physical quantities.
- Principle of homogeneity of dimensions.
- Checking the dimensional correctness of physical equation.
- Derivation of simple physical relations.
- Limitations of dimensional analysis.
- What are significant figures?
- Rules or significant figure.
- Arithmetic operations with significant figures.
- What is error?
- Categories of errors.
- How to express an error?
- Propagation of error.

(A) UNITS

1.1 CHEMISTRY AND ITS IMPORTANCE

In the measurement of any *physical quantity, we require some 'reference standard'. This reference e standard of measurement is called a unit. **This unit of a physical quantity is defined as the reference standard used to measure it.**

1.2 CHARACTERISTICS OF UNITS SELECTED FOR MEASUREMENT

(i) The unit should be well-defined. (ii) The unit should be neither too small nor too large in comparison with the physical quantity to be measured. In simple words, the unit should be of some suitable size. (iii) The unit should be imperishable. (iv) The unit should neither change with time nor with time nor with physical condition like pressure, temperature etc. (v) The unit should be easily reproducible.

1.3 WHAT IS THE NEED OF A SYSTEM OF UNITS?

There are as many as there are independent quantities. Let us consider three physical quantities mass, length and time. These quantities are independent of each other. So, three separate units are requited for the measurement of these quantities. Thus, it become important to establish a system of units. Historically, the choice of every unit has been changing. As an example, the metre was originally defined in terms of the distance from the North Pole to the equator. This distance is very nearly equal to 10^7m. Until recently, the standard metre of the world was the distance between two scratches on a platinum-iridium alloy bar that has been kept at the International Bureau of Weights and Measures in France. Presently, the standard metre in France has been specified in term of the number of wavelengths of light of a specified spectral line of the isotope Kr – 86.

1.4 WHAT ARE FUNDAMENTAL AND DERIVED PHYSICAL QUATITIES?

Fundamental physical quantities are those which cannot be defined in terms of other quantities. Derived physical quantities are those which can be defined in terms of fundamental physical quantities.

1.5 WHAT ARE FUNDAMENTAL AND DERIVED UNITS?

Units are classified into two categories – basic or fundamental units and derived units.

The basic or fundamental units are the units of fundamental quantities. These units are so named because they can neither be derived from one another not can be further resolved into other more simpler units.

In Physics, we deal with a very large number of physical quantities. But the minimum convenient number of units is only seven. In other words, the number of basic units or base units is only seven. These are the units of length, mass, time, electric current, temperature, intensity of light and amount of substance.

Derived units are those units which are derived from basic units. As an example, the unit of speed is cm s^{-1} and the unit of acceleration of cm s^{-2}. There are derived units.

1.6 WHAT IS INTERNATIONAL SYSTEM OF UNITS?

The French name for this system is *"System International d' Unites"*. It is abbreviated as S.I. This system is in-fact the improved and extended version of M.K.S. system of units.

In the year 1960, the Eleventh General Conference of Weights and Measures made an attempt to improve the two metric systems – C.G.S. system and M.K.S. system. This attempt was made to meet the varied needs of scientists, technologists and engineers. This conference introduced the International System of Units. The recommendations of this conference was endorsed by both the International Standards Organization (I.S.O.) and International Electrochemical Commission (I.E.C.) in 1962.

1.7 FUNDAMENTAL UNITS IN SI SYSTEM

Following are the *seven fundamental units* of 'International System of Units':

(i) metre (m). *It is defined as the distance occupied by 1,650,763.73 wavelengths in vacuum of the radiation* emitted by the krypton-86 atom in its transition between the states $2p_{10}$ and $5d_5$.*

(ii) kilogram (kg). *It is defined as the mass of a platinum-iridium cylinder of diameter equal to its height which is preserved in a vault at International Bureau of Weights and Measures at Sevres near Paris.*

➢ **Q.** Why the cylinder used in defining kilogram made of platinum-iridium alloy ?

Ans. This is because platinum-iridium alloy is least affected by environment and time.

(iii) second (s). *It is the duration of 9,192,631.770.0 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of cesium-133 atom.*

It may also be defined as the time taken by cesium-133 atom to make 9,192,631,770 vibrations.

(iv) ampere (A). *It is that constant current which, when flowing in two straight parallel conductors of infinite length and of negligible area of cross-section and placed one metre apart in vacuum, would produce between the conductors a force equal to* 2×10^{-7} *newton per metre of length.*

(v) kelvin (K). *It is equal to* $\frac{1}{273.16}$ *of the thermodynamic temperature of triple point of water.*

> ➢ **Q.** What is triple point of water?
>
> **Ans.** It is the temperature at which ice, water and water vapours co-exist.

(vi) candela (od). *It is the luminous intensity, in a direction at right angles to a surface of* $\frac{1}{600{,}000}$ *square metre area of a black body, at a temperature of freezing platinum under a pressure of 101,325 newton per square metre.*

(vii) mole (mol). *It is the amount of substance which contains as many elementary units as there are carbon atoms in exactly 0.012 kg of* C^{12}. *In addition to seven fundamental units, there are following* ***two supplementary units:***

(a) radian (rad). *It is defined as the plane angle between the two radii of a circle which cut off on the circumference an arc of length equal to the radius of that circle.*

The angle subtended at the centre of n circle of radius r by an are of length l is given by

$$\theta = \frac{l}{r} \text{ radian}$$

(b) steradian (sr). *It is defined as the solid angle subtended at the centre of a sphere by an area of its surface equal to the square of the radius of the sphere.*

The solid angle subtended at the centre of a sphere of radius r by an area ΔS on the surface of the sphere is given by

$$\Delta\Omega = \frac{\Delta S}{r^2} \text{ steradian}$$

1.8 DERIVED UNITS IN INTERNATIONAL SYSTEM

S. No.	Name of Physical Quantity	Relationship with other physical quantities	derived unit in terms of SI fundamental units	Symbolic representation	Equivalent
1.	area	length × breath	square metre	m^2	
2.	Volume	length × breath × height	cubic metre	m^3	
3.	Density (Mass density)	$\frac{\text{mass}}{\text{volume}}$	kilogram per cubic metre	$kg\ m^{-3}$	
4.	Specific Gravity	$\frac{\text{Density of body}}{\text{Density of water at } 4^\circ \text{ C}}$	No units		
5.	Speed	$\frac{\text{Distance covered}}{\text{Time taken}}$	metre per second	$m\ s^{-1}$	
6.	Velocity	$\frac{\text{Change in displacement}}{\text{Time taken}}$	metre per second	$m\ s^{-1}$	
7.	Velocity gradient	$\frac{\text{Change in velocity}}{\text{Distance}}$	per second	s^{-1}	
8.	Momentum	Mass × Velocity	kilogram metre per second	$kg\ m^{-1}$	Ns
9.	Acceleration	$\frac{\text{Change in velocity}}{\text{Time}}$	metre per square second	$m\ s^{-2}$	
10.	Acceleration due to gravity		metre per square second	$m\ s^{-2}$	

1.9 SOME IMPORTANCE NON-SI UNITS

S.No.	Name of unit	value in SI units
1.	Light year	9.5×10^{15} m
2.	Astronomical unit	1.5×10^{11} m
3.	Parsec	3.08×10^{16} m
4.	Micron	10^{-6} m
5.	Nanometre	10^{-9} m
6.	Angstrom	10^{-10} m
7.	Fermi or femtometre	10^{-15} m
8.	Atomic mass unit	1.66×10^{-27} kg

(B) DIMENSIONS

1.10 DIMENSIONS OF PHYSICAL QUANTITIES

The dimensions of a physical quantity are the powers to which the fundamental quantities are to be raised to represent that physical quantity.

We know that speed. $\frac{\text{distance}}{\text{time}} = \frac{[L]}{[T]} = [M^0LT^{-1}]$

So speed is said to posses zero dimension in mass, one dimension in length and -1 dimension in time.

1.11 DIMENSIONAL FORMULA

Dimensional formula is an expression which shows how and which of the fundamental units are required to represent the unit of a physical quantity.

Example. $[M^0LT^{-1}]$ is the dimensional formula of speed.

1.12 DIMENSIONAL EQUATION

Dimensional equation is the equation obtained by equating the physical quantity with its dimensional formula.

The dimensional equation of speed is given as under:

Speed $[M^0LT^{-1}]$

1.13 DIMENSIONAL FORMULAE OF SOME PHYSICAL QUANTITIES

S.No.	Physical Quantity	Dimensional Formula
1.	Area	$[M^0L^2T^0]$
2.	Volume	$[M^0L^3T^0]$
3.	Density (Mass density)	$[ML^{-3}T^0]$
4.	Specific Gravity	$[M^0L^0T^0]$
5.	Speed	$[M^0LT^{-1}]$
6.	Velocity	$[M^0LT^{-1}]$
7.	Velocity gradient	$\frac{[M^0LT^{-1}]}{[L]}$ $[M^0L^0T^{-1}]$
8.	Momentum	$[MLT^{-1}]$
9.	Acceleration	$\frac{[M^0LT^{-1}]}{[L]}$ $[M^0LT^{-2}]$
10.	Acceleration due to gravity	$[M^0LT^{-2}]$
11.	Force	$[MLT^{-2}]$

12.	Weight	$[MLT^{-2}]$
13.	Tension	$[MLT^{-2}]$
14.	Normal reaction	$[MLT^{-2}]$
15.	Coefficient of friction	$[M^0L^0T^0]$
16.	Force constant	$\frac{[MLT^{-2}]}{[L]}$ $[ML^0T^{-2}]$
17.	Work	$[MLT^{-2}]$ $[L]$ $[ML^2T^{-2}]$

Sample Problem 1.1. Name the quantities represented by the dimensional formulae given below:

(i) $[M^1L^2T^{-1}]$ (ii) $[ML^2T^{-2}]$ (iii) $[ML^{-3}T^2]$.

Solution. (i) Planck's constant or Angular momentum (ii) Energy or Work (iii) Density.

1.14 PRINCIPLE OF HOMOGENEITY OF DIMENSIONS

A given physical relation in dimensionally correct if the dimension of the various terms on either side of the relation are the same.

Thin principle is based on the fact that two quantities of the same nature only can be added up. The resulting quantity is also of the same nature.

1.15 CHECKING THE DIMENSIONAL CORRECTNESS OF PHYSICAL EQUATION

[An application of dimensional analysis]

This can be done by applying the principle of homogeneity of dimensions.

Illustration. Let us check the dimensional correctness of the relation $v = u +$ at.

Here 'u' represents the initial velocity, 'v' represents the final velocity, 'a' the uniform acceleration and 't' the time.

Dimensional formula of 'u' is $[M^0LT^{-1}]$.

Dimensional formula of 'v' in $[M^0LT^{-1}]$.

Dimensional formula of 'at' is $((M^0LT^{-2}]$ $[T].=$ $(M^0LT^{-1}]$.

The dimensions of every term in the given physical relation are the same. So, according to the principle of homogeneity of dimensions, the given physical relation in dimensionally correct.

Sample Problem 1.2. Consider an equation.

$$\frac{1}{2} \mathbf{mv}^{2} = \mathbf{mgh}$$

where m is the mass of the body, v its velocity, g is the acceleration due to gravity and he is the height. Check whether this equation is dimensionally correct.

Solution. The dimensions of L.H.S. are

$$[M][LT^{-1}]^2 = [M]\ [L^2T^{-2}] = [ML^2T^{-2}]$$

The dimension of R.H.S. are

$[M][LT^{-2}] = [M]\ [L^2T^{-2}] = [ML^2T^{-2}]$

The dimensions of L.H.S. And R.H.S. are the same and hence the equation is dimensionally correct.

EXERCISES

1. Check the correctness of the following formulae by dimensional analysis.

   (i) $F = \frac{mv^2}{r}$ (ii) $T = 2\pi \sqrt{\frac{l}{g}}$

   where all the notations have their usual meaning.

   [**Ans.** (i) correct (1) correct]

2. Check the accuracy of the following relations.

   (i) $E = mgh + \frac{1}{2} mv^{2}$ (ii) $v^2 - u^2 = 2aS^2$

   [**Ans.** (i) correct (ii) wrong]

1.16 DERIVATION OF SIMPLE PHYSICAL RELATIONS

[An application of dimensional analysis]

This is possible by making use of the principle of homogeneity of dimensions.

Illustration. Let us find us expression for the time period *t of a simple pendulum. The time period t may possibly depend upon (i) mass m of the bob of the pendulum, (ii) length l of the pendulum, (iii) acceleration due to gravity g at the place where the pendulum is suspended, (iv) angel of swing θ.

Let (i) $t \propto m^a$ (ii) $t \propto l^b$ (iii) $t \propto g^c$ (iv) $t \propto \theta^d$

Combining all the four factors, we get

$$t \propto m^a\, l^b\, g^c\, \theta^d \qquad \text{or} \qquad t = \mathrm{K} m^a\, l^b\, g^c\, \theta^d \qquad \ldots\text{(i)}$$

where K is a **dimensionless** constant of proportionality.

Writing down the dimensions on either side of equation (i), we get

$$[\mathrm{T}] = [\mathrm{M}^a][\mathrm{L}^b][\mathrm{LT}^{-2}]^c = [\mathrm{M}^a\mathrm{L}^{b+c}\mathrm{T}^{-2c}]$$

Comparing dimensions, $a = 0,\ b + c = 0,\ -2c = 1$

$$\therefore \qquad a = 0,\ c = -\frac{1}{2},\ b = \frac{1}{2}$$

From equation (i), $t = \mathrm{K} m^0\, l^{1/2}\, g^{-1/2}$ or $t = \mathrm{K}\left(\frac{l}{g}\right)^{1/2} = \mathrm{K}\sqrt{\frac{l}{g}}$

The value of K, as found by experiment or mathematical investigation, comes out to be 2π.

$$\therefore \qquad \boxed{t = 2\pi\sqrt{\frac{l}{g}}}$$

Sample Problem 1.3. The wavelength λ associated with a moving electron depends upon its mass m, its velocity v and Plank's constant h. Prove dimensionally that $\lambda = \frac{h}{mv}$.

Solution. Let (i) $\lambda \propto m^a$ (ii) $\lambda \propto v^b$ (iii) $\lambda \propto h^c$

Combining the three factors, $\lambda \propto m^a\, v^b\, h^c$

or $$\lambda \propto k m^a\, v^b\, h^c$$

where k is a dimensionless constant of proportionally.

Writing down the dimensions on either side, we get

$$[\mathrm{M}^0\mathrm{LT}^0] = [\mathrm{M}]^0[\mathrm{LT}^{-1}]^b\, [\mathrm{ML}^2\mathrm{T}^{-1}]^c$$

or $$[\mathrm{M}^0\mathrm{LT}^0] = [\mathrm{M}^{a+b}\mathrm{L}^{b+2c}\mathrm{T}^{-b-c}]$$

Equating dimensions, we get $a + c = 0$,

$$b + 2c = 1,\ -b - c = 0$$

On simplification, $a = -1,\ c = 1,\ b = -1$.

From equation (1), $\lambda = k m^{-1}\, v^{-1}\, h$ or $\lambda = k\,\frac{h}{mv}$

or $$\lambda = \frac{h}{mv}$$

EXERCISES

1. A body of mass m is moving in a circle of radius r with angular velocity ω. Find expression for centripetal force F acting on it using method of dimensional analysis.

   [Ans. $F = mr\omega^2$]

2. Consider a simple pendulum, having a bob attached to a string that oscillates under the action of the force of gravity. Suppose that the period of oscillation of the simple pendulum depends on its length (l), mass of the bob (m) and acceleration due to gravity (g). Derive the expression for its time period using methods of dimensions.

   $\left[\text{Ans. } T = 2\pi\sqrt{\frac{l}{g}}\right]$

1.17 LIMITATIONS OF DIMENSIONAL ANALYSIS

(i) It supplies no information about dimensionless constants. They have to be determined either by experiment or by mathematical investigation.

(ii) This method is applicable only in the case of power functions. It fails in the case of exponential and trigonometric relations.

(iii) This method fails to drive directly a relation which contains two or more than two quantities of like nature.

(iv) It fails to derive the exact from of a physical relation, if a physical quantity depends upon more than three other physical quantities. This is because by equating powers of M, L and T, we can obtain only three equations. Three equations cannot determine more than three 'unknowns'.

(v) If we cannot identify all the factors on which a physical quantity depends, then the method of dimensional analysis cannot be used to derive expression for a physical quantity.

(vi) It can only check whether a physical relation is dimensionally correct or not. It cannot tell whether the relation in absolutely correct or not. Let us consider the equation: $S = ut + \frac{1}{4}at^2$

Through the equation is dimensionally correct but it is actually wrong equation because the correct equation is $S = ut + \frac{1}{4}at^2$

(C) SIGNIFICANT FIGURES

1.18 WHAT ARE SIGNIFICANT FIGURES?

The digits which tell us the number of units we are reasonably sure of having counted in making a measurement are called significant figures.

Significant figures are those digits in a measurement that are known reliably plus the first digit that is uncertain.

When we say that the length of a table is 1.2 m, it has two significant figure. The 1 is reliable while the second figure 2 is uncertain. However, it may be clearly noted here that the decimal does not separate the reliable and unreliable digits.

The greater the number of significant figures obtained when making a measurement the more accurate is the measurement. Conversely, a measurement made only to a few significant figures is not a very accurate one even if all care is taken to make the measurement.

1.19 RULES FOR SIGNIFICANT FIGURES

Following are the **rules** *for determining the number of significant figures.*

1. All non-zero digits are significant.
As an example, 132.73 contains *five* significant figures.

2. All zeros between two non-zero digits are significant.
As an example, 207.009 contains *six* significant figures.

3. *(a) If there is no decimal point, all zeros to the right of the right-most non-zero digit are not significant.*

As an example, 307000 contains only *three* significant figures.

(b) If there is no decimal point, all zeros to the right of the right-most non-zero digit are significant if they come from a measurement.

As an example, consider a distance of 400 m. This distance is measured to the nearest meter. Both the zeros in this value are significant.

Rules 3(a) and 3(b) can be summarized as follow:

If there is no decimal point, zeros to the right of the right-most non-zero digit are significant only if they come from a measurement.

4. All Zeros to the left of an expressed decimal point and to the right of a non-zero digit are significant.

As an example, 307000. Contains *six* significant figures.

5. All zeros to the left of the left-most non-zero digit are not significant.

As an example, 0.000345 m contains only *three* significant figures.

It may be noted here that th*e single zero conventionally placed to the left of the decimal point is never significant.*

6. All zeros to the right of a decimal point and to the right of a non-zero digit are significant.

As an example, 0.03040 and 40.00 m each contain four significant figures.

7. The number of significant figures does not vary with the choice of different units.

Consider a value 200 m. It is a distance which has been measured to the nearest metre. Since the given value has come from a measurement therefore both the zeros are significant. So, the given value has three significant figures. If the given distance is expressed as 20,000 cm, then the last two zeros cannot be counted as significant because these have come as a result of multiplication by a factor of 100 and not from measurement. But a reader cannot know which zeros have come from a measurement or otherwise. So, a convention is adopted. Only such zeros as are the result of a measurement are put to the right of the rightmost non-zero digit. Other zeros (which have come as a result of choice of different units) are included in powers of 10. *The powers of 10 do not influence the accuracy of the measurement.* So, 200 m, 2.00×10^2 m, 2.00×10^4 cm, $2{,}00 \times 10^5$ mm are all equivalent and each of these has 3 significant figure accuracy.

Sample Problem 1.4. State the number of significant figures in the following:

(i) 600900 (ii) 95300 (iii) 648700 (iv) 400 m

Solution

(i) 4 [Rules 1, 2, 3(*a*)] (ii) 3 [Rules 1, 3(*a*)]

(iii) 6 [Ruler 1, 4] (iv) 3 [Rules 1, 3(b)]

1.20 ARITHMETIC OPERATIONS WITH SIGNIFICANT FIGURES

1. Addition and Subtraction

In addition or subtraction, the number of decimal places which are significant in the result is the same as the smallest number of significant decimal places in any number used in the sum or difference.

Let us find the sum of the following measurements of length,

3.7 m, 13.07 m, 0.311 m.

In any number obtained by measurement, all the digits beyond the last significant figure are unknown. As an example, in the first measurement above, we do not know the second and third decimal places. In the second measurement, we do not know the third decimal place. Let us represent the unknown digits by question marks and try to find the sum of the given three measurements. It is to be kept in mind that by adding a unknown quantity to a known quantity, we get an unknown quantity.

	3.	7	?	?
1	3.	0	7	?
	0.	3	1	1
1	7.	0	?	?

It is clear from here that the accuracy of a sum is limited to the accuracy of the least accurate term. In the given illustration, the least accurate term is the first measurement which is known to tenth of a metre only. So, we should *either round off our result to the nearest tenth or we may round off the individual values to tenth before adding.*

3.7 m			3.7 m	
13.7 m	or		13.1 m	
0.377 m			0.3 m	
17.081 m	or	17.1 m	17.1 m	

A similar procedure is followed in the case of subtraction. First of all, we round off to the number of decimal places in the measurement of least accuracy and then we perform the operation of subtraction. A special case is that of subtraction of quantities of nearly equal magnitude.

As an example, 4.48 cm - 4.41 cm = 0.07 cm.

The result has only one significant figure and not three as in original measurements. This lends us to very important conclusion that the subtraction of nearly equal quantities destroys accuracy.

2. Multiplication and Division

In multiplication or division, the number of significant figures in the product or the quotient is the same as the smallest number of significant figures in any of the factors.

Illustration. Let us multiply 11.21 and 4.31. While multiplying, the following rules are to be used:

(i) The last significant figure is uncertain. [*The uncertain digits have been 1.22 encircled in this illustration.*]

(ii) Addition of uncertain digit to certain/uncertain digit is uncertain.

(iii) Product of uncertain digit and certain/uncertain digit is uncertain.

	1	1		2	1
		4		3	1
		1	1	2	1
	3	3	6	3	
4	4	8	4		
4	8	3	1	5	1

Since only one uncertain number is to be retained in the final result therefore the final result is **48.3.**

We follow a similar procedure for division.

Sample Problem 1.5. Add 17.35 g, 25.6 g and 8.498 g.

Solution. Out of the three given measurements, 25.6 g is the least accurately known. So, the other two measurements have to be rounded. 17.35 g becomes 17.4 g and 8.498 g becomes 8.5 g.

Now, 17.4 g + 25.6 g + 8.5 g = **51.5 g**

EXERCISES

Solve the following with due regard to significant figures.

1. 5.8 + 0.125 [**Ans.** 6.9]
2. 9.15 + 3.8 [**Ans.** 13.0]
3. 3.5 - 2.51 [**Ans.** 1.0]
4. $\sqrt{3.5 - 3.31}$ [**Ans.** 0.4]
5. 3.9×10^5 - $2,5 \times 10^4$ [**Ans.** 3.6×10^5]

(D) ERROR ANALYSIS

1.21 WHAT IS ERROR?

*The uncertainty in a measurement is called '**error**'. It is the difference between the measured and the true values of a physical quantity.*

The concept of error is different from that of discrepancy. Discrepancy merely the difference between the two measured values of a physical quantity.

1.22 CATEGORIES OF ERRORS

1. Constant Errors. *If the same error in repeated every time in a series of observations, the error is said to be constant error.*

Constant error is due to faulty calibration of the scale of a measuring instrument.

In order to minimise constant error, measurements are made with all possible different methods. The mean value to obtained is regarded as the true value.

II. Systematic Errors. Systematic errors are those errors which occur according to a certain pattern or system.

These errors are due to known reasons.

These errors can be minimised by locating the source of error.

Systematic errors can be classified into following four main categories.

(1) Instrumental errors. All measurements are carried out with the help of instruments. If an instrument is faulty or inaccurate, then even repeated careful measurements will fail to reveal the presence of these errors. Either the interchange of two similar instruments or the use of different methods to measure the same quantity can be of some help in minimizing instrumental errors.

(2) Personal errors. Sometimes, errors in the measurement are due to the individual qualities of the experimenter himself. These errors may arise due to the lack of attentiveness, bad sight, habits and peculiarities of the observer. As an example, the observer may be in the habit of holding his head always a bit too far to the left while reading a needle on the scale. In order to eliminate personal errors, the measurements are repeated by different observers.

(3) Errors due to external sources. These errors are caused due to change in external conditions like pressure, temperature, wind, etc.

These errors can be minimined by keeping control over external conditions in which experiment is being performed.

(4) Errors due to internal sources or errors due to imperfection. These errors are due to limitations of experimental arrangement. As an example, the loss of energy due to radiation causes errors in calorimetric readings. Necessary corrections are required to be applied in such cases.

III. Gross Errors. These errors are due to one or more than one of the following:

(i) Improper setting of the instrument.
(ii) Recording observations wrongly.
(iii) Not to take into account the sources of error and precautions.
(iv) Using some wrong value in calculations.

These errors can be minimised only if the observer is very careful in his approach.

IV. Random Errors. It is a common experience that the repeated measurements of a quantity give values which are slightly different from each other. These errors have no set pattern. These take place in a random manner and are, therefore, called random errors. These errors depend on the error in the measuring process and also on the individual measuring person.

Since the random errors are governed by chance, therefore, it is possible to minimise these errors by repeating the measurements many times and taking the arithmetic mean of all measurements as the correct value of the measured quantity.

If $A_1, A_2, A_3, \ldots\ldots, A_n$ be the values obtained in several measurements, then the **best possible value** of the quantity is given as

$$*A_{mean} = \frac{A_1 + A_2 + A_3 + \ldots\ldots + A_4}{n}$$

$$= \frac{1}{n}(A1 + A2 + A3 + \ldots\ldots + An) = \frac{1}{n}\sum_{i=1}^{n} Ai$$

This method of minimising of random errors is based on the fact that it is reasonable to assume that individual measurements are as *likely to underestimate as to overestimate the value of the quantity.*

1.23 HOW TO EXPRESS AN ERROR?

There are three ways of expressing an error:

(i) absolute error (ii) relative error (iii) percentage error.

The absolute error of measurement is the magnitude of the difference between the value of the quantity and the individual measurement value.

Since we are not sure of the correct value of a quantity therefore we will accept the arithmetic mean A_{mean} as the correct or true value.

The absolute errors in our measurements are given by

$$\Delta A_1 - A_{mean} - A_1$$
$$\Delta A_2 = A_{mean} - A_2$$
$$\ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots$$
$$\Delta A_n = A_{mean} - A_n$$

If we take the arithmetic mean of all absolute errors, we get the f**inal absolute error** $\Delta A_{mean.}$ [*When arithmetic mean is taken, only the magnitude of the absolute errors are taken into account.*]

$$*A_{mean} = \frac{|\Delta A_1| + |\Delta A_2| + \ldots\ldots + |\Delta A_n|}{n} \frac{1}{n}\sum_{i=1}^{n} |\Delta Ai|$$

It follows from the above discussion that any single measurement of A has to be such that

$$A_{mean} - \Delta A_{mean} \leq A \leq A_{mean} + \Delta A_{mean}$$

Relative error is defined as the ratio of the mean absolute error and the value of the quantity being measured.

Relative error = $\frac{\Delta A_{means}}{A_{means}}$; Percentage error = $\frac{\Delta A_{means}}{A_{means}} \times 100$

It is the relative error or the percentage error and not the absolute error which is true index of the accuracy of a measurement. When the quantity to be measured is large, relative error or percentage error is less. When we have to measure small quantities, we have to devise special techniques of measurement to keep the errors low.

Suppose we determine the atmospheric pressure by using Boyle's law apparatus. Suppose it comes out to be 74.5 cm of mercury. Let the actual value, as determined by Fortin's Barometer, be 75.8 cm of mercury. In this case, the absolute error is **1.3 cm of mercury.** But the percentage error is

$$\frac{1.3}{75.8} \times 100 = \mathbf{1.7\%}$$

Consider the case of specific hent of copper. Suppose its experimental value is 0.38 J g^{-1} °C^{-1}. Its actual value is 0.39 J g^{-1} °C^{-1}. Now, absolute error is 0.01. But the percentage error is $\frac{0.01}{0.39} \times 100$ *i.e.,* **2.6%**

In the two examples mentioned above, the absolute error in the second case is less but percentage error is more as compared to first case. So, the first result is more accurately determined than the second case because its percentage error is less. Thus, it is the percentage error with the help of which we can judge the accuracy of the result.

1.24 PROPAGATION OF ERRORS

I. When the result involves the sum of two observed quantities

Consider two quantities A and B which have measured values ($A \pm \Delta A$) and ($B \pm \Delta B$) respectively. Here ΔA and ΔB are the absolute errors in A and B respectively. Let us now calculate the absolute error ΔZ in Z such that

$$Z = A + B$$

Now $Z \pm AZ = (A \pm \Delta A) + (B \pm \Delta B)$

or $Z \pm \Delta Z = A \pm \Delta A + B + \Delta B$ or $Z \pm \Delta Z = (A + B) \pm (\Delta A + \Delta B)$

or $\pm \Delta Z = \pm \Delta A \pm \Delta B$ $(\because Z = A + B)$

The four possible values of ΔZ are $(+ \Delta A + \Delta B)$; $(+\Delta A - \Delta B)$; $(-\Delta A + \Delta B)$ and $(- \Delta A - \Delta B)$.

The maximum absolute error in Z is given by $\Delta Z = \Delta A + \Delta B$

So, the maximum possible error in Z is the sum of the maximum absolute errors in the physical quantities A and B.

Sample Problem 1.6. Two rods are of lengths 3.161 ± 0.3 cm and 1.121 ± 0.1 cm. What is their combined length?

Solution. Combined length

$= (3.161 \pm 0.3 \text{ cm}) + (1.121 \pm 0.1 \text{ cm})$

= 4.289 ± 0.4 cm

II When the result involves the difference of two observed quantities

Let us now determine the absolute error ΔZ in Z such that

$$Z = A - B$$

Now, $Z \pm \Delta Z = (A \pm \Delta A) - (B \pm \Delta B)$

or $Z \pm \Delta Z = (A - B) \pm \Delta A \mp \Delta B$

or $\pm \Delta Z = \pm \Delta A \mp \Delta B$

The four possible values of ΔZ are $(+ \Delta A - \Delta B)$, $(+\Delta A + \Delta B)$; $(-\Delta A - \Delta B)$ and $(- \Delta A + \Delta B)$.

∴ The maximum absolute error in Z is given by

$$\Delta Z - \Delta A + \Delta B$$

So, the maximum possible error is again the sum of the maximum absolute errors in the physical quantities.

Thus, we arrive at the following general rule:

When we add or subtract two quantities, the maximum absolute error in the final result is equal to the sum of the maximum absolute errors in the quantities.

Sample Problem 1.7. What is the difference between the following lengths?

11.5 ± 0.1 cm and 7.8 ± 0.1 cm.

Solution. Difference of lengths

$$= (11.5 \pm 0.1 \text{ cm}) - (7.8 \pm 0.1 \text{ cm})$$

$$\mathbf{= 3.7 \pm 0.2 \text{ cm}}$$

III. When the result involves the product of two observed quantities

If Z = AB, then

$$Z \pm \Delta Z = (A \pm \Delta A)(B \pm \Delta B)$$

or $$Z \pm \Delta Z = AB \pm B\Delta A \pm A\Delta B \pm \Delta A\Delta B$$

or $$\pm \Delta Z = \pm B\,\Delta A \pm A\,\Delta B \pm \Delta A\,\Delta B \qquad (\therefore\ Z = AB)$$

Dividing L.H.S. by Z and R.H.S. by AB, we get

$$\pm \frac{\Delta Z}{Z} = \pm \frac{\Delta A}{A} \pm \frac{\Delta B}{B} \pm \frac{\Delta A}{A}\,\frac{\Delta B}{B}$$

Each of the two terms $\frac{\Delta A}{A}$ and $\frac{\Delta B}{B}$ is considerably smaller than one. So, the product $\frac{\Delta A}{A} \times \frac{\Delta B}{B}$ will be very very small and can, therefore, be neglected.

$$\pm \frac{\Delta Z}{Z} = \pm \frac{\Delta A}{A} \pm \frac{\Delta B}{B}$$

Four possible values of $\frac{\Delta Z}{Z}$ are:

$$\left(+\frac{\Delta A}{A} + \frac{\Delta B}{B}\right);\ \left(+\frac{\Delta A}{A} - \frac{\Delta B}{B}\right);\ \left(-\frac{\Delta A}{A} + \frac{\Delta B}{B}\right) \text{ and } \left(-\frac{\Delta A}{A} - \frac{\Delta B}{B}\right)$$

So, the possible relative error in Z is given by

$$\frac{\Delta Z}{Z} = +\frac{\Delta A}{A} + \frac{\Delta B}{B}$$

$\frac{\Delta Z}{Z}$, $\frac{\Delta A}{A}$ and $\frac{\Delta B}{B}$ represent the maximum relative (or fractional) error in the measurement of Z, A and B respectively.

Thus, we conclude that *when two quantities are multiplied, the maximum relative error in the result is the sum of the maximum relative error in the quantities.*

In term of percentage error, $\frac{\Delta Z}{Z} \times 100 = \frac{\Delta A}{A} \times 100 + \frac{\Delta B}{B} \times 100$

So, the maximum percentage error in the result is the sum of the maximum percentage error in the quantities.

Aliter. We can arrive at the above result by another method described below:

$$Z = AB$$

$$\text{Log } Z = \log (AB) = \log A + \log B$$

On differentiation, $\frac{dZ}{Z} = \frac{dA}{A} + \frac{dB}{B}$

So the maximum relative error in Z is the sum of the maximum relative error is A and B.

Simple Problem 1.8. A capacitor of capacitance C = 2.0 ± 0.1 μF is charged to a voltage V = 20 ± 0.2 volt. What will the charge Q on the capacitor? Use Q = CV.

Solution. If we omit all error, then

$$Q = CV = 2.0 \times 10^{-6} \times 20 \text{ coulomb}$$

$$= 40 \times 10^{-6} \text{ coulomb}$$

Error in C = 0.1 part in 2 = 1 part in 20 = 5%

Error in V = 0.2 part in 20 = 2 part in 200

= 1 part in 100 = 1%

Error in Q = 5% + 1% = 6%

∴ Charge, Q = $40 \times 10^{-6} \mp 6\%$ coulomb

= $40 \pm 2.4 \times 10^{-6}$ coulomb

IV When the result involves the quotient of two observed quantities

Let $Z = \frac{A}{B}$

If ΔA, ΔB and ΔZ are the absolute error in A, B and Z respectively, then

$$Z \pm Z = \frac{A \pm \Delta A}{B \pm \Delta B}$$

or $\quad Z \pm \Delta Z = (A \pm \Delta A)\ (B \pm \Delta B)^{-1} = A\left(1 \pm \frac{\Delta A}{A}\right) B^{-1}\left(1 \pm \frac{\Delta B}{B}\right)^{-1}$

$$= AB\text{-}1\left(1 \pm \frac{A}{A}\right)\left(1 \mp \frac{\Delta B}{B} + \text{terms containing higher power of } \frac{\Delta B}{B}\right)$$

But $\frac{\Delta B}{B}$ is so small that terms containing higher powers of $\frac{\Delta B}{B}$ can be neglected.

$\therefore \quad Z \pm \Delta Z = \frac{A}{B}\left(1 \pm \frac{\Delta A}{A}\right)\left(1 \mp \frac{\Delta B}{B}\right)$

or $\quad Z \pm \Delta Z = Z\left[1 \pm \frac{\Delta A}{A} \mp \frac{\Delta B}{B} - \frac{\Delta A\,\Delta B}{AB}\right]$

Neglecting $\frac{\Delta A\,\Delta B}{AB}$, we get $Z \pm \Delta Z = Z\left[1 \pm \frac{\Delta A}{A} \mp \frac{\Delta B}{B}\right]$

$$1 \pm \frac{\Delta Z}{Z} = 1 \pm \frac{\Delta A}{A} \mp \frac{\Delta B}{B} \text{ or } \pm\frac{\Delta Z}{Z} = \pm\frac{\Delta A}{A} \mp \frac{\Delta B}{B}$$

Four possible value of $\frac{\Delta Z}{Z}$ are:

$$\left(+\frac{\Delta A}{A} - \frac{\Delta B}{B}\right); \left(+\frac{\Delta A}{A} + \frac{\Delta B}{B}\right); \left(-\frac{\Delta A}{A} - \frac{\Delta B}{B}\right) \text{ and } \left(-\frac{\Delta A}{A} + \frac{\Delta B}{B}\right)$$

The maximum possible relative error in Z in given by

$$\frac{\Delta Z}{Z} \times 100 = \frac{\Delta A}{A} \times 100 + \frac{\Delta B}{B} \times 100$$

So, the maximum percentage error in Z is the sum of the maximum percentage errors in A and B.

Simple Problem 1.9. The resistance R = V/I, where V = (100 $\pm$ 5) V and I = (10 $\pm$ 0.2) A. Find the percentage error is R.

Solution. The percentage error in V is 5% and in I it is 2%. The total error in R would therefore be 5% + 2% = 7%

V. When the result involves the product of some powers of the measured values

Let $\quad Z = A^n B^m$

Now, $\quad Z \pm \Delta Z = (A \pm \Delta A)^n (B \pm \Delta B)^m$

or $\quad Z \pm \Delta Z = A^n B^m \left(1 \pm \frac{\Delta A}{A}\right)^n \left(1 \pm \frac{\Delta B}{B}\right)^m$

or $\quad Z \pm \Delta Z = Z\left(1 \pm \frac{\Delta A}{A}\right)^n \left(1 \pm \frac{\Delta B}{B}\right)^m$

Applying Binomial theorem and neglecting squares and higher powers, we get

$$Z \pm \Delta Z = Z \left(1 \pm n \frac{\Delta A}{A}\right)\left(1 \pm m \frac{\Delta B}{B}\right)$$

or $$1 \pm \frac{\Delta Z}{Z} = \left(1 \pm n \frac{\Delta A}{A}\right)\left(1 \pm m \frac{\Delta B}{B}\right)$$

or $$1 \pm \frac{\Delta Z}{Z} = \left(1 \pm n \frac{\Delta A}{A} + m \frac{\Delta B}{B} + nm \frac{\Delta A}{A} \frac{\Delta B}{B}\right)$$

Neglecting the term $nm \frac{\Delta A}{A} \frac{\Delta B}{B}$, we get $1 \pm \frac{\Delta Z}{Z} = 1 \pm n \frac{\Delta A}{A} \pm m \frac{\Delta B}{B}$

or $$\pm \frac{\Delta Z}{Z} = \pm n \frac{\Delta A}{A} \pm m \frac{\Delta B}{B}$$

The maximum possible relative error in Z is given by $\frac{\Delta Z}{Z} = n \frac{\Delta A}{A} + m \frac{\Delta B}{B}$

We may conclude from the above that wen a quantity appears with a power n greater than one in an expression, then its error contribution to the final result increase n time. So, such quantities should be measured with a high degree of accuracy. But if a quantity appears with a power n less than one on an expression, then the error is reduced in the final result.

VI. When the result involves the division of two quantities (using the method of differentiation)

$$Z = \frac{A^n}{B^m}$$

Taking log of both of sides,

$$\log Z = \log A^n - \log B^m$$

or $$\log Z = n \log A - m \log B$$

Differentiating both the sides, we get $\frac{dZ}{Z} = n\frac{dA}{A} - m\frac{dB}{B}$

In terms of fractional error, this equation may be rewritten as

$$\pm \frac{\Delta Z}{Z} = \pm n \frac{\Delta A}{A} \mp m \frac{\Delta B}{B}$$

The maximum possible relative error in Z in given by $\frac{\Delta Z}{Z} = n \frac{\Delta A}{A} + m \frac{\Delta B}{B}$

VII. When the result involves the product of three quantities (Using calculus method)

$$u = x^m y^n z^p$$

Taking log of both the sides, we get

$$\text{Log } u = \log x^m + \log y^n + \log z^p$$

or $$\log u = m \log x + n \log y + p \log z$$

Differentiating both the sides, we get $\frac{du}{u} = m\frac{dx}{x} + n\frac{dy}{y} + p\frac{dz}{z}$

In terms of the fractional error, this equation may be rewritten as

$$\pm \frac{\Delta u}{u} = \pm m\frac{\Delta x}{x} \pm n\frac{\Delta y}{y} \pm p\frac{\Delta z}{z}$$

The maximum possible relative error in u is given by

$$\frac{\Delta u}{u} = m\frac{\Delta x}{x} + n\frac{\Delta y}{y} + p\frac{\Delta z}{z}$$

In terms of percentage error,

$$\frac{\Delta u}{u} \times 100 = m\frac{\Delta x}{x} \times 100 + n\frac{\Delta y}{y} \times 100 + p\frac{\Delta z}{z} \times 100.$$

Simple Problem 1.10. Find the relative error in Z if

$Z = A^4 B^{1/3}/CD/^{3/2}$.

Solution. $\frac{\Delta Z}{Z} = 4\frac{\Delta A}{A} + \frac{1}{3}\frac{\Delta B}{B} + \frac{\Delta C}{C} + \frac{3}{2}\frac{\Delta D}{D}$

EXERCISES

1. Add 28.7 ± 0.5 and 19.6 ± 0.3 [**Ans.** 48.3 ± 0.8]

2. Add 5.181 ± 0.3 and 17.81 ± 0.6. [**Ans.** 22.9 ± 0.9]

3. Subtract 16 ± 0.5 from 41 ± 0.1. [**Ans.** 25 ± 0.6]

4. Substract 283.4 ± 0.1 from 304.3 ± 0.1 [**Ans.** 20.9 ± 0.2]

SUMMARY

- The International system of Units (SI) based on seven base units is at present internationally accepted unit system. It is widely used throughout the world.
- The unit of a physical quantity is the reference standard used to measure the physical quantity.
- Metre, kilogram, second, ampere, kalvin, candela and mole are the seven fundamental SI units.
- Radian and steradian are the two supplementary SI units.
- When a physical quantity is equated with its dimensional formula, we get dimensional equation.
- In multiplication or division, the number of significant figure in the product or quotient is the same as the smallest number of significant figure in any of the factors.
- The difference between the measured and true values of a physical quantity is called error.
- In addition and subtraction, maximum absolute errors are added.

- In multiplication and division, and maximum relative and percentage errors are added up.

TEST YOURSELF

1. Define the seven base units of International System of Units.
2. What are the two supplementary units in International System of Units?
3. Give dimensions of the following physical quantities:
   (i) Force (ii) Power
   (iii) Kinetic energy (iv) Moment of inertia
   (v) Angular momentum (vi) Angular impulse
   (vii) Torque (viii) Rotational kinetic energy.
4. What is principle of homogeneity of dimensions?
5. How can you check the dimensional correctness of a given physical relation?
6. Using principle of homogeneity of dimensions, how can you derive simple physical relation?
7. What are the limitations of dimensional analysis?
8. Give rules for counting significant figures.
9. What are the rules for carrying out arithmetical operations with due regard for significant figures?
10. Discuss error propagation in
    (i) addition (ii) Subtraction
    (iii) multiplication (iv) division.

2 FORCE AND MOTION

LEARNING OBJECTIVES

- Mathematical tools.
- What are physical quantities?
- Scalars.
- Vectors.
- Differences between scalars and vectors.
- Representation of a vector.
- Important terms.
- Addition of composition of vectors.
- Rectangular components of a vector.
- Direction cosines of a vector.
- Dot product of vector.
- Cross product of vectors.
- Important terms.
- Speed.
- Velocity.
- Acceleration.
- Velocity-time graphs of accelerated motion.
- Derivation of $v = u + at$.
- Derivation of $S = ut + \frac{1}{2} at^2$
- Derivation of $v^2 - u^2 = 2as$.
- Derivation of $S_{nth} = u + \frac{a}{2}(2n - 1)$.
- Graphical method for derivation of equations of motion.
- Calculus method for derivation of equations of motion.
- Motion under gravity.
- Newton's first law of motion and derivation of definition of force.
- Units and dimension of force.
- Law of conservation of momentum.
- Newton's third law of motion and derivation of properties of force.
- Illustrations of Newton's third law of motion.
- Apparent weight of a person in an elevator/lift.

- Second law is the real law of motion.
- Connected motion.
- Composition of forces.
- Triangle law of forces.
- Parallelogram law of forces.
- Polygon law of forces.
- Resolution of forces.
- Resolution of a force into three rectangular components.
- Equilibrium of concurrent forces.
- Lami's theorem.
- Law of shines.
- What is a projectile?
- Two types of projectiles.
- Principle of physical independence of motions.
- Horizontal projection.
- Projection at an angle (Trajectory of oblique projectile)
- Resultant velocity of oblique projectile.
- Maximum height.
- Time of flight.
- Horizontal range.
- Maximum horizontal range.
- Two angles of projection for the same range.
- Friction.
- Cause of friction.
- Static friction.
- Limiting friction.
- Laws of limiting friction.
- Dynamic of kinetic friction.
- Laws of siding friction.
- Variation of frictional force with the applied force.
- Coefficient of static friction.
- Coefficient of kinetic friction.
- Angle of friction.
- Angle of siding or angle of repose.
- Body accelerating down an inclined plane.
- Rolling friction.
- Applications.
- Methods of reducing friction.
- Important terms and concepts of circular motion.
- Centripetal force and acceleration.
- Centrifugal force.

- Motion of car on circular level road.
- Application to baking of roads.

2.1 MATHEMATICAL TOOLS

(a) Fundamental formulae of differentiation

1. The differential coefficient of an isolated constant (c) is zero.

$$\therefore \qquad \frac{d}{dx}(c) = 0$$

2. $\frac{d}{dx}(x^n) = nx^{n-1}$ where n has some real value (positive or negative integer or a fraction).

3. $\frac{d}{dx}(u^n) = nu^{n-1}\frac{d}{dx}(u)$ where u is a function of x.

4. $\frac{d}{dx}(cu) = c\frac{d}{dx}(u)$

5. $\frac{d}{dx}(u \pm v \pm w \pm \ldots\ldots) = \frac{d}{dx}(u) \pm \frac{d}{dx}(v) \pm \frac{d}{dx}(w) \pm \ldots\ldots$

   where u, v and w.... are all functions of x.

6. Derivation of product of two functions

$$\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$$

7. derivation of a quotient $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$

Derivation of Trigonometrical Functions

8. $\frac{d}{dx}(\sin x) = \cos x$

9. $\frac{d}{dx}(\sin u) = \cos u\frac{d}{dx}(u)$

10. $\frac{d}{dx}(\cos x) = -\sin x$

11. $\frac{d}{dx}(\cos u) = -\sin u\frac{d}{dx}(u)$

12. $\frac{d}{dx}(\tan x) = \sec^2 x$

13. $\frac{d}{dx}(\tan u) = \sec^2 u\frac{d}{dx}(u)$

14. $\frac{d}{dx}(\sec x) = \sec x \tan x$

15. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

16. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

17. $\frac{dy}{dx} = \frac{dy}{du}, \frac{du}{dx}$

(b) Fundamental formulae of Integration

1. $\int x^n\, dx = \frac{x^n+1}{n+1}$ provided $n \neq -1$

2. $\int \sin x\, dx = -\cos x$

3. $\int \cos x \ dx = \sin x$

4. $\int \sec^2 x \ dx = \text{tax } x$

5. $\int \text{cosec}^2 x \ dx = - \cot x$

6. $\int \sec x \tan x \ dx = \sec x$

7. $\int \text{cosec } x \cot x \ dx = - \text{cosec } x$

8. $\int \sin (ax + b) \ dx = - \frac{\cos (ax + b)}{a}$

9. $\int \cos (ax + b) \ dx = - \frac{\sin (ax + b)}{a}$

10. $\int (u \pm v \pm w \pm \ldots\ldots) \ dx = \int u \, dx \pm \int v \, dx \pm \int w \, dx \pm \ldots\ldots$

11. $\int \frac{1}{x} dx = \log_e x$

12. $\int (ax + b)^n \ dx = \frac{(ax + b)^{n+1}}{n+1} \cdot \frac{1}{a}$, provided $n \neq -1$

13. Constant of intergration

We know that $\frac{d}{dx}(x) = 1$ ...(i)

$$\frac{d}{dx}(x + 1) = 1 \quad \text{...(ii)}$$

$$\frac{d}{dx}(x + 2) = 1 \quad \text{...(iii)}$$

$$\frac{d}{dx}(x + 3) = 1 \quad \text{...(iv)}$$

Since integration is the inverse of differentiation,

$\therefore \quad \int 1 \ dx = x$ [from (i)]

$$\int 1 \ dx = x + 1 \quad \text{[from (ii)]}$$

$$\int 1 \ dx = x + 2 \quad \text{[from (iii)]}$$

$$\int 1 \ dx = x + 3 \quad \text{(from (iv)]}$$

In general, we may write:

$$\int 1 \ dx = x + c$$

where c is a constant of integration.

In all indefinite integrals, constant of integration is supposed to be present even if it is not specificllly mentioned.

In case of definite integration, the value of integration constant can be evaluated with the help of upper and lower limit of function. Definite integral is being solved as follow:

$$\int_a^b f(x) \ dx = [\text{F}'(x)]_a^b = \text{F}'(x = b) - \text{F}'(x = a)$$

Step I. Integrate function first using formula of integration and carry forward upper and lower limit with integrated result.

Step II. Now substitute the upper limit in the integrated result, place negative sign and then substitute the lower limit.

(c) Some commonly used formulae of algebra

1. $(a + b) = a^2 + 2ab + b^2$

2. $(a - b) = a^2 - 2ab + b^2$

3. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

4. $(a + b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$

5. $(a - b)^3 = a^3 - b^3 + 3a^2b + 3ab^2$

(d) Quadratic equation

An algebraic equation of second degree having the form $ax^2 + bx + c = 0$ is called quadratic equation. Here 'a' is called the coefficient of x^2, 'b' is called the coefficient of x and c is a constant term.

General solution of the quadratic equation is given by

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Clearly, there are two roots of the equation:

$$x_1 = \frac{-b + \sqrt{b^2-4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2-4ac}}{2a}$$

$$x_1 + x_2 = -\frac{b}{a}$$

In general, sum of roots = $-\dfrac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$

Again, $x_1 x_2 = \dfrac{c}{a}$

In general, product of roots = $\dfrac{\text{Coefficient}}{\text{Coefficient of } x^2}$

(e) Binomial theorem

If $|x| < 1$ i.e., x lies between – 1 and + 1, then

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}\, x^2 \;\frac{n(n-1)(n-2)}{3!}\, x^3 + \ldots\ldots\ldots$$

where n is any number which may be positive, negative integer or a fraction.

Here, $2\,! = 2 \times 1$, $3\,! = 3 \times 2 \times 1$, ..., $n\,!$

$= n\,(n-1)\,(n-2)\,\ldots.. \times 3 \times 2 \times 1$

If n is a positive integer, then the expansion will have $(n + 1)$ terms.

If n is a negative integer or a fraction, then the number of terms in the expansion will be infinite.

When $|x| << 1$, then only the first two terms of the expansion are significant. The second and higher order terms can be neglected. In this case, the expansion shall reduce to the following simplified forms:

$$(1 + x)^n = 1 + nx$$

$$(1 + x)^{-n} = 1 - nx$$

$$(1 - x)^n = 1 - nx$$

$$(1 + x)^{-n} = 1 + nx$$

(f) Fundamental relation between the T-ratios of an angle

1. $\cot\theta = \frac{1}{\tan\theta}$ 2. $\sec\theta = \frac{1}{\cos\theta}$ 3. $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$

4. $\tan\theta = \frac{\sin\theta}{\cos\theta}$ 5. $\cot\theta = \frac{\cos\theta}{\sin\theta}$ 6. $\sin^2\theta + \cos^2\theta + \ = 1$

7. $\sin^2\theta = 1 + \tan^2\theta$ 8. $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$

(g) Aid to memory for value of T-ratio

T-ratio	0°	30°	45°	60°	90°
sine	$\frac{\sqrt{0}}{2}$ or 0	$\frac{\sqrt{1}}{2}$ or $\frac{1}{2}$	$\frac{\sqrt{2}}{2}$ or $\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$ or 1
*cosine	$\frac{\sqrt{4}}{2}$ or 1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$ or $\frac{1}{\sqrt{2}}$	$\frac{\sqrt{1}}{2}$ or $\frac{1}{2}$	$\frac{\sqrt{0}}{2}$ or 0
tangent	0	$\frac{1}{\sqrt{2}}$	1	$\sqrt{3}$	$\propto$

* since written backwards.

(h) Some important formulae of trigonometry

1. $\sin(A + B) = \sin A \cos B + \cos A \sin B$

2. $\cos(A + B) = \cos A \cos B - \sin A \sin B$

3. $\tan(A + B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

4. $\sin(A - B) = \sin A \cos B - \cos A \sin B$

5. $\cos(A - B) = \cos A \cos B + \sin A \sin B$

6. $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

7. $2\sin A \cos B = \sin(A + B) + \sin(A - B)$

8. 2 cos A sin B = sin (A + B) - sin (A – B)

9. 2 cos A cos B = cos (A + B) + cos (A – B)

10. 2 sin A sin B = cos (A - B) - sin (A + B)

11. sin C + sin D = 2 sin $\frac{C + D}{2}$ cos $\frac{C - D}{2}$

12. sin C - sin D = 2 cos $\frac{C + D}{2}$ cos $\frac{C - D}{2}$

13. cos C + cos D = 2 cos $\frac{C + D}{2}$ cos $\frac{C - D}{2}$

14. cos C - cos D = 2 sin $\frac{C + D}{2}$ sin $\frac{D - C}{2}$

15. sin 2A = 2 sin A cos A

16. cos 2 A = $\cos^2$ A – $\sin^2$ A

17. cos 2 A = 1 – 2 $\sin^2$ A

18. cos 2 A = 2 $\cos^2$ A – 1

19. tan 2A = $\frac{2 \tan B}{1 - \tan^2 B}$

SCALARS AND VECTORS

2.2 WHAT ARE PHYSICAL QUANTITIES?

All those quantities which can be measured are known as physical quantities. These quantities can be broadly classified into two categories-scalar quantities and vector quantities.

2.3 SCALARS

Scalar quantities are those physical quantities which are characterized by magnitude only.

These directionless quantities are briefly called scalars. These obey the ordinary laws of Algebra. A scalar quantity is completely specified by merely stating a number. A few examples of scalars are volume, mass, speed, density, number of moles, angular frequency, temperature, pressure, time, power, total path length, energy, gravitational potential, coefficient of friction, charge and specific heat.

2.4 VECTORS

Vector quantities are those physical quantities which are characterised by both magnitude and direction.

These quantities are briefly called vectors. A vector is specified not by merely stating a number but a direction as well. Since the concept of vectors involves the idea of direction, therefore, vectors do not follow the ordinary laws of Algebra. We shall formulate certain laws for the addition of vectors. These laws will be based on geometrical constructions. A few examples of vectors are : displacement, velocity, angular velocity, acceleration, impulse, force, angular momentum, linear momentum, electric filed, magnetic moment and magnetic field.

2.5 DIFFERENCES BETWEEN SCALARS AND VECTORS

The differences between scalars and vectors are given here in a tabular form.

S. No.	Scalars	Vectors
1.	These possess only magnitude.	These possess both magnitude and direction.
2.	These obey the ordinary laws of Algebra.	These do not obey the ordinary laws of Algebra.
3.	These change if magnitude changes.	These change is either magnitude of direction of both change.
4.	These are represented by ordinary letters.	these are represented by bold-faced letters or letters having arrow over them.

2.6 REPRESENTATION OF VECTOR

A vector is represented by a line with an arrow head. In Fig. 2.1, a vector is represented by a directed line PQ. The length of the line gives the magnitude of the vector. The magnitude of the vector is called the modulus of the vector. The direction of the arrow represents the direction of the vector. The point P from where the arrow starts is called the tail or initial point or origin of the vector. The point Q where the arrow ends is called the tip or head or terminal point or terminus of the vector. In books, vectors are sometimes represented by bold-faced letters. As an example, $\vec{a}$ may be written as $\boldsymbol{a}$.

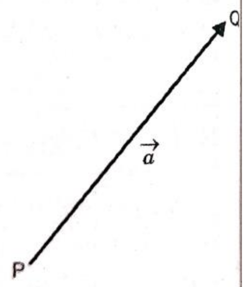


Fig. 2.1

2.7 IMPORTANT TERMS

(i) Collinear vectors are those vectors which act either along the same line or along parallel lines. These vectors may act either in the same direction or in opposite directions.

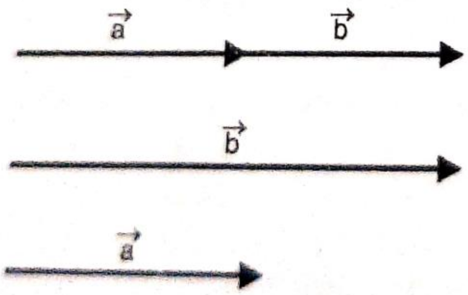


Fig 2.2

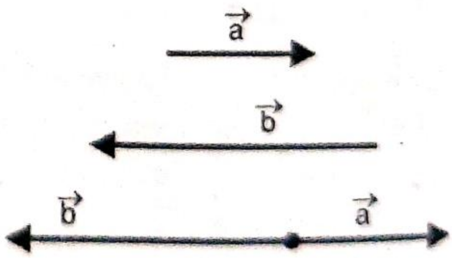


Fig 2.3

(a) Parallel Vectors. If two collinear vectors $\vec{a}$ and $\vec{b}$ act in the same direction, then the angle between then is 0°. This is a case of parallel vectors. When vectors act along the same direction, they are called parallel vector (Fig. 2.2)

(b) Anti-parallel vectors. If two collinear vectors act in opposite directions then the angle between then is 180° or π radian. This is case of anti-parallel vectors. Vectors are said to be anti-parallel if they act in opposite directions. (Fig. 2.3)

(ii) Unit vector is a vector having unit magnitude. It is used to denote the direction of a given vector.

Any vector $\vec{a}$ can be expressed in terms of its unit vector $\hat{a}$ as follows:

$$\vec{a} = a\,\hat{a}$$

Here a is in the direction $\vec{a}$. $\hat{a}$ is read as 'a hat' or 'a cap' or 'a caret'

$$\hat{a} = \frac{\vec{a}}{a} \text{ or } \frac{\vec{a}}{|\vec{a}|}$$

So, if a given vector is divided by its magnitude, we get a unit vector.

(iii) The **three rectangular unit vectors**. $\hat{\imath}$. $\hat{\jmath}$ and $\hat{k}$ are shown in Fig. 2.4. $\hat{\imath}$ denotes the direction of X-axis. $\hat{\jmath}$ denotes the direction of Y-axis and $\hat{k}$ denotes the direction of Z-axis. The three unit vectors $\hat{\imath}$. $\hat{\jmath}$ and $\hat{k}$ are collectively known as 'orthogonal triad of unit vectors'. These are also known as base vectors.

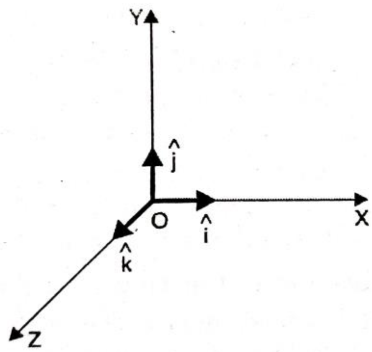


Fig. 2.4

(iv) Fixed vector is that vector whose initial point or tail is fixed. It is also known as localised vector.

Examples. (i) The initial point of a position vector is fixed at the origin of the co-ordinate axes. So, position vector is a fixed or localised vector.

(iii) The displacement vector is also a fixed vector.

(v) Free vector is that vector whose initial point or tail is not fixed. It is also known as non-localised vector.

Velocity vector of a particle moving along a straight line is a free vector.

(vi) Co-initial vectors are those vectors which have the same initial point.

Fig. 2.5 shows four co-initial vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$.

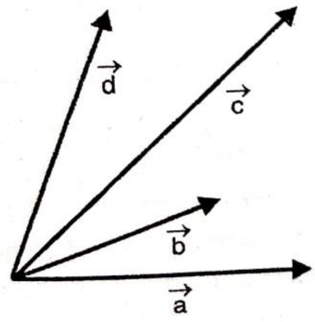


Fig. 2.5

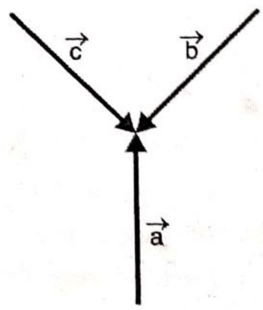


Fig. 2.6

(vii) Co-terminus vectors are those which have common terminal point.

Fig. 2.6 shows three co-terminus vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$

(viii) A vector is said to be negative of a given vector if its magnitude is the same as that of the given vector but direction is reversed.

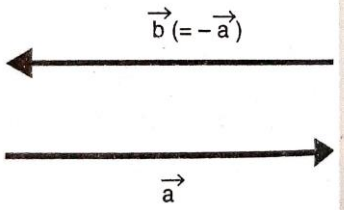


Fig. 2.7

The negative of a vector $\vec{a}$ is denoted by ‘ - $\vec{a}$’

In Fig. 2.7, $\vec{b}$ is the negative of

$$\vec{a} \;.\; \vec{b} - \vec{a}$$

Note. The negative sign reverse the direction of vectors. But the negative sign in the case of scalara has a different significance. As an example, an angle measured anticlockwise may be taken as + θ while an angle measured clockwise may be taken as - θ.

(ix) Position Vector. A vector which gives the position of a point with reference to the origin of the co-ordinate system is called **position vector.**

Consider a particle moving in a plane. To describe the position of this particle at any time *t*, we use a vector called position vector (Fig. 2.8). Suppose at any instant of time, the particle is at P. Then $\overrightarrow{OP}$ is the position vector which gives the position of the particle with reference to a point O in the plane of motion. This point O has been chosen as the origin.

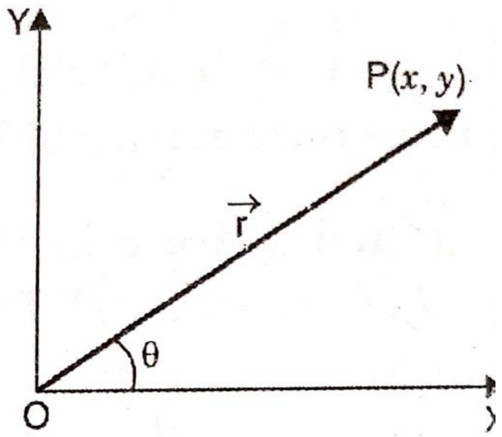


Fig 2.8

The magnitude of the position vector gives the distance of the particle from some arbitrarily chosen origin. In addition to this, the direction of the position vector gives us the direction in which P lies as viewed from 0.

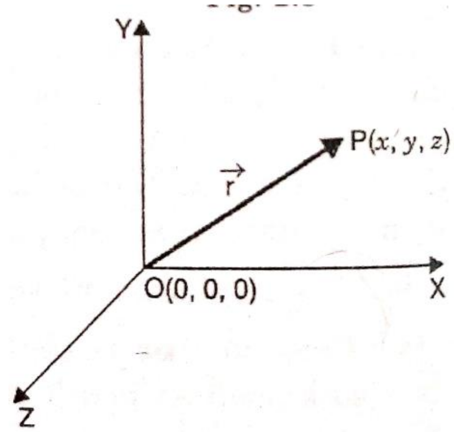


Fig. 2.9

The position vector $\vec{r}$ at any time t, in terms of co-ordinates x and y, is given by

$$\vec{r} = \vec{x} + \vec{y} \quad \text{or} \quad \vec{r} = x\,\hat{\imath} + y\,\hat{\jmath}$$

In magnitude, $|\vec{r}|$ or $r = \sqrt{x^2 + y^2}$

If the position of a point P is chosen with reference to the origin of the three-dimensional rectangular co-ordinate system as shown in Fig. 2.9, then the position vector is given by

$$\vec{r} = \vec{x} + \vec{y} + \vec{z} \quad \text{or} \quad \vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$$

The magnitude or modulus of $\vec{r}$ is given by r or $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

(x) Displacement vector. It is a vector which gives the position of a point with reference to a point other than the origin of the co-ordinate system.

Suppose a particle is at P at time t and moves to Q at time t'. The position vectors of P and Q with reference to the origin O are $\vec{r}_1$ and $\vec{r}_2$ respectively. The vector $\overrightarrow{PQ}$ (= Δr) with tail P and tip Q is the displacement vector corresponding to the motion from t to t'.

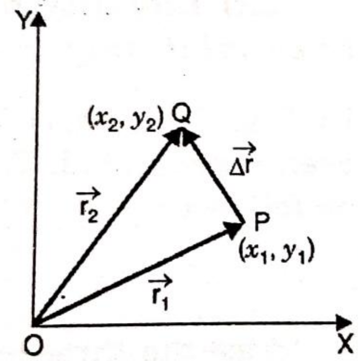


Fig. 2.10

Displacement vector $\overrightarrow{PQ}$ gives the position of Q with reference to P. So, it is a sort of position vector. The only difference is that while the position vector gives the position of a point with reference to the origin O, the displacement vector gives the position with reference to a point other than the origin.

Applying triangle law of vectors (discussed later), we get

$$\vec{r}_1 + \Delta r = \vec{r}_2 \quad \text{or} \quad \Delta r = \vec{r}_2 - \vec{r}_1$$

Thus, displacement vector is merely the difference of two position vectors. If $(x_1 y_1)$ and $(x_2 y_2)$ are the co-ordinates of P and Q respectively, then

$$\vec{r}_1 = x_1\ \hat{\imath} + \vec{y}_1\ \hat{\jmath} \text{ and } \vec{r}_1 = \vec{x}_2\ \hat{\imath} + \vec{y}_2\ \hat{\jmath}$$

$$\overrightarrow{\Delta r} = (x_2 - x_1)\ \hat{\imath} + (y_2 - y_1)\ \hat{\jmath}\ ;\ |\overrightarrow{\Delta r}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Generalising this result for three dimensions, we get

$$|\overrightarrow{\Delta r}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

This result is generally referred to as distance formula.

(xi) Scalar multiple of a vector. When a vector $\vec{a}$ is multiplied by a real number λ, we get another vector $\lambda\ \vec{a}$. The magnitude of $\lambda\ \vec{a}$ is λ times the magnitude of $\vec{a}$. If λ is positive, then the direction of $\lambda\ \vec{a}$ is the same as that of $\vec{a}$. If λ is negative, then the direction of $\lambda\ \vec{a}$ is opposite to that of $\vec{a}$.

If $\vec{a}$ is multiplied by zero, we get a vector whose magnitude is zero and whose direction is arbitrary. This vector is called zero vector or null vector.

If λ is a pure number and has no units, then the units of $\lambda\,\vec{a}$ are the same as those of $\vec{a}$. But generally λ is a scalar which has certain units. In these cases, the units of $\lambda\,\vec{a}$ will be obtained by multiplying the units of $\vec{a}$ by unit of λ.

Examples. (i) The multiplication of velocity vector by time gives us displacement. So, the units of displacement can be obtained by multiplying the units of Velocity by units of time.

(ii) The multiplication of velocity vector by mass gives us momentum. So, the unit of momentum can be obtained by multiplying the unit of velocity by unit of mass.

(xii) Equality of vectors. Two vectors are said to be equal if they have the same magnitude and direction.

In Fig. 2.11, three equal vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$. Have been represented. The equality of vectors is represented as follows:

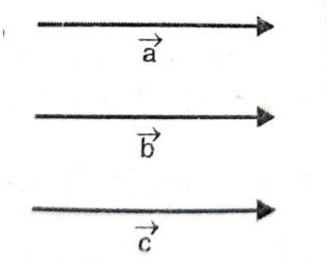


Fig. 2.11

$$\vec{a} = \vec{b} = \vec{c}$$

Since the three vectors are pointing in the same direction,

$$\therefore \qquad \hat{\vec{a}} = \vec{b} = \vec{c}$$

Also, since the three vectors have equal magnitudes,

$$\therefore \qquad |\vec{a}| = |\vec{b}| = |\vec{c}|$$

Equality of two vectors does not depend upon their location in space. So, two vectors can be equal even if they are differently located in space. A direct consequence of this physical fact is that a vector can be displaced parallel to itself.

Note. It would be meaningless to compare two vectors with different physical dimensions. As an example, there is no point in comparing velocity vector and force vector.

(xiii) Zero vector. Zero vector or null vector is a vector which has zero magnitude and an arbitrary direction. It is represented by $\vec{0}$.

What is the physical meaning of ?

The physical meaning of zero vector can be clearly understood from the following examples :

(i) The position vector of the origin of the co-ordinate axes is zero vector.

(ii) The displacement of a stationary particle from time t to time t' is zero.

(iii) The displacement of a ball thrown up and received back by the thrower is a zero vector.

(iv) The velocity vector of a stationary body is a zero vector.

(v) The acceleration vector of a body in uniform motion is a zero vector.

2.8 AUDITION OF COMPOSITION OF VECTORS

The process of adding two or more than two vectors is called 'addition or composition of vectors'.

When two or more than two vectors are added, we get a single vector called resultant vector.

The resultant of two or more than two vectors is a single vector which produces the same effect as the individual vectors together produce.

Following three laws have been evolved for the addition of vectors.

(i) Triangle law of vectors (for addition of two vectors)

If two vectors can be represented both in magnitude and direction by the two sides of a triangle taken in the same order, then the resultant is represented completely, both in magnitude and direction, by the third side of the triangle taken in the opposite order.

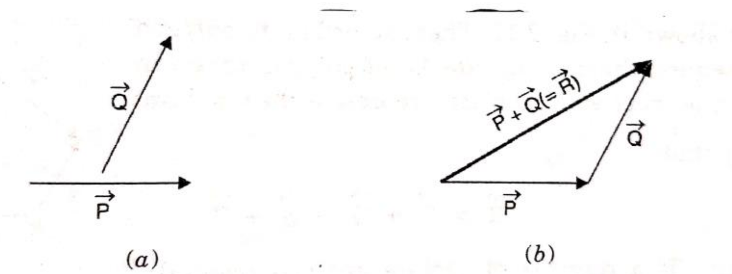


Fig. 2.12

Suppose we have to add two vectors $\vec{P}$ and $\vec{Q}$ as shown in Fig. 2.12 (a). Now, displace $\vec{Q}$ parallel to itself in such a way that the tail of $\vec{Q}$ touches the tip of $\vec{P}$. Complete the triangle to get a new vector $(\vec{P}+\vec{Q})$ running straight from the tail of $\vec{P}$ to the tip of $\vec{Q}$. According to triangle law of vectors, this new vector is the resultant $\vec{R}$ of the given vectors $\vec{P}$ and $\vec{Q}$ such that

$$\vec{R} = \vec{P} + \vec{Q}$$

Triangle law of vectors is applicable to triangle of any shape.

Corr. It follows from triangle law of vectors that if three vectors are represented by the three sides of a triangle taken in order, then their resultant is zero. Thus, if three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ can be represented completely by the three sides of a triangle taken in order, then their vector sum is zero.

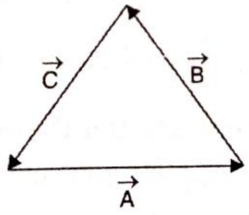


Fig. 2.13

$$\therefore \quad \vec{A} + \vec{B} + \vec{C} = \vec{0}$$

(ii) Parallelogram law of vectors (for addition of two vectors)

"If two vectors, acting simultaneously at a point, can be represented both in magnitude and direction by the two adjacent sides of a parallelogram drawn from a point, then the resultant is represented completely both in magnitude and direction by the diagonal of the parallelogram passing through that point."

In Fig. 2.14, two vectors $\vec{P}$ and $\vec{Q}$ are completely represented by the two sides OA and OD respectively of a parallelogram. Then, according to parallelogram law of vectors, the diagonal OC of the parallelogram will give the resultant $\vec{R}$ such that $\vec{R} = \vec{P} + \vec{Q}$.

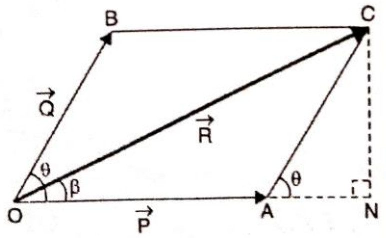


Fig. 2.14

(iii) Polygon law of vectors (for addition of more than two vectors).

"If a number of vectors can be represented both in magnitude and direction by the sides of an open convex polygon taken in the same order, then the resultant is represented completely in magnitude and direction by the closing side of the polygon, taken in the opposite order."

Suppose four vectors $\vec{P}$, $\vec{Q}$, $\vec{S}$ and $\vec{T}$ are represented completely by the four sides AB, BC, CD and DE respectively of a polygon, all taken in the same order as shown in Fig. 2.15. Then, according to polygon law of vectors, the closing side AE of polygon taken in the opposite order will completely represent the resultant $\vec{R}$ such that

$$\vec{R} = \vec{P} + \vec{Q} + \vec{S} + \vec{T}$$

Corr. If a number of vectors are represented by the sides of a closed polygon taken in order, then their resultant is zero.

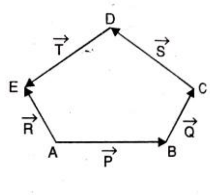


Fig. 2.15

2.9 RECTANGULAR COMPONENTS OF A VECTOR

If the components of a given vector are perpendicular to each other, the they are called **rectangular components.** These are the most important components of a vector.

Let a vector $\vec{A}$ be represented by $\overrightarrow{OP}$ as shown in Fig. 2.16. With Q as origin, construct a rectangular parallelopiped with three edges along the three rectangular axes which meet at O. $\vec{A}$ becomes the diagonal of the parallelopiped. $\overrightarrow{A_x}$, $\overrightarrow{A_y}$ and $\overrightarrow{A_z}$ are three vector intercepts along x, y and z axes respectively. These are the three rectangular components of $\vec{A}$.

Applying triangle law of vector,

$$\overrightarrow{OP} = \overrightarrow{OK} + \overrightarrow{KP}$$

Applying parallelogram law of vectors,

$$\overrightarrow{OK} = \overrightarrow{OT} + \overrightarrow{OQ}$$

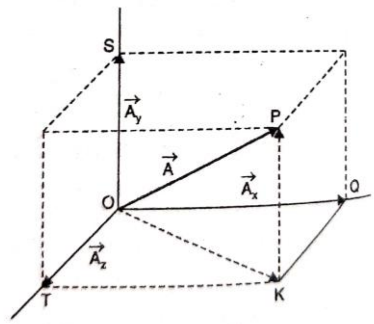


Fig. 2.16

$\therefore$ $$\overrightarrow{OP} = \overrightarrow{OT} + \overrightarrow{OQ} + \overrightarrow{KP}$$

But $$\overrightarrow{KP} = \overrightarrow{OS}$$

$\therefore$ $$\overrightarrow{OP} + \overrightarrow{OT} + \overrightarrow{OQ} + \overrightarrow{OS}$$

$$\vec{A} = \overrightarrow{A_z} + \overrightarrow{A_x} + \overrightarrow{A_y} \quad \text{or} \quad \boxed{\vec{A} = \overrightarrow{A_z} + \overrightarrow{A_y} + \overrightarrow{A_z}}$$

or $$\boxed{\vec{A} = A_x\,\hat{\imath} + A_y\,\hat{\jmath} + A_z\,\hat{k}}$$

Again $$OP^2 = OK^2 + KP^2$$

$$OP^2 = OQ^2 + QK^2 + KP^2 \quad \text{or} \quad OP^2 = OQ^2 + OT^2 + KP^2$$

[$\therefore$ QK = OT]

or $$A^2 = {A_x}^2 + {A_z}^2 + {A_y}^2$$

or $$A^2 = {A_x}^2 + {A_y}^2 + {A_z}^2 \qquad [\therefore KP = OS = A_y]$$

or $$A^2 = \sqrt{{A_x}^2 + {A_y}^2 + {A_z}^2}$$

This gives the magnitude of $\vec{A}$ in terms of the magnitude of components $\overrightarrow{A_x}$, $\overrightarrow{A_y}$ and $\overrightarrow{A_z}$.

2.10 DIRECTION COSINES OF A VECTOR

The direction cosines l, m and n of a vector are the cosines of the angles α, β and γ which a given vector makes with x-axis, y-axis and z-axis respectively.

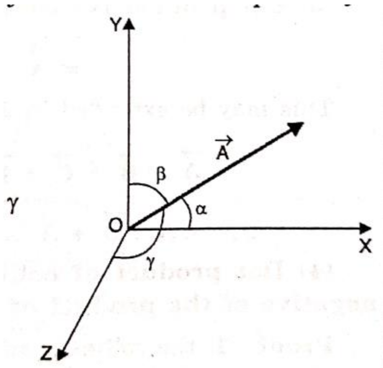


Fig. 2.17

$$\cos\alpha = \frac{A_x}{A} = l,\ \cos\beta = \frac{A_y}{A} = m$$

and $$\cos\gamma = \frac{A_z}{A} = n,$$

Squiring and adding $\cos^2\alpha + \cos^2\beta + \cos^2\gamma$

$$= \frac{{A_x}^2 + {A_y}^2 + {A_z}^2}{A}$$

or $$l^2 + m^2 + n^2 = \frac{A^2}{A^2} = 1$$

2.11 DOT PRODUCT OF VECTORS

It is defined as the product of the magnitudes of the two vectors and the cosine of the angle between them.

Since dot product of two vectors is a scalar, therefore, dot product is also known as scalar product.

Consider two vectors $\vec{A}$ and $\vec{B}$ inclined to each other at an angle θ as shown in Fig. 2.18. The dot or scalar product of these vectors is given by

$$\vec{A} \cdot \vec{B} = AB\cos\theta$$

where A and B are the magnitudes of $\vec{A}$ and $\vec{B}$ respectively.

Important proerpties of dot product are as under :

(1) Dot product is commutative i.e., $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$.

Proof. By definition of dot product,

$$\vec{A} \cdot \vec{B} = AB\cos\theta = BA\cos\theta = \vec{B} \cdot \vec{A}.$$

Applying the commutative property to unit vectors, we get

$$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{i},\ \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{j},\ \hat{k} \cdot \hat{i} = \hat{i} \cdot \hat{k}$$

(2) Dot product of perpendicular vectors is zero.

Proof. If $\vec{A}$ is perpendicular to $\vec{B}$ then θ = 90°.

∴ $\vec{A} \cdot \vec{B}$ = AB cos = 90° = 0 [∴ cos = 90° = 0]

This leads us to the following condition of perpendicularity of two vectors. "Two given non-zero vectors will be perpendicular to each other if and only if their dot product is zero"

Applying the result to unit vectors, we get

$$\hat{\imath} \cdot \hat{\jmath} = \hat{\jmath} \cdot 0,\ \hat{\jmath} \cdot \hat{k} = 0 \quad \text{and} \quad \hat{k} \cdot \hat{\imath} = 0$$

(3) Dot product is distributive i.e., $\vec{A} \cdot (\vec{B} + \vec{C})$

$$= \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$$

This may be extended to any number of vector

$$\vec{A} \cdot (\vec{B} + \vec{C} + \vec{D} + \ldots..)$$

$$= \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C} + \vec{A} \cdot \vec{D} + \ldots..$$

(4) Dot product of collinear vectors is equal to the positive or negative of the product of their magnitudes.

Proof. If the collinear vector point in the same direction, then θ = 0°.

∴ $\vec{A} \cdot \vec{B}$ = AB cos 0° = AB [∴ cos 0° = 1]

If the collinear vectors point in opposite direction then θ = 180°.

∴ $\vec{A} \cdot \vec{B}$ = AB cos 180° = AB [∴ cos 180° = - 1]

The dot product of $\vec{A}$ and $\vec{B}$ varies from AB to – AB

(5) Dot product of two equal vectors is equal to the square of the magnitude of either of the two vectors.

Proof. In this case $\vec{A} \cdot \vec{B} = \vec{A} \cdot \vec{A}$

= A × A cos 0°

[Angle between equal vectors is 0°.]

= A^2 [∴ cos 0° = 1]

Applying the result to unit vectors we get

$$\hat{\imath} \cdot \hat{\imath} = 1,\ \hat{\jmath} \cdot \hat{\jmath} = 1 \text{ and } \hat{k} \cdot \hat{k} = 1$$

(6) Dot product of $\vec{A}$ and $\vec{B}$ in terms of three rectangular components.

In terms of rectangular components. $\vec{A} = \vec{A_x} + \vec{A_y} + \vec{A_z}$

and $$\vec{B} = \overrightarrow{B_x} + \overrightarrow{B_y} + \overrightarrow{B_z}$$

Also $\vec{A} = \overrightarrow{A_x}\ \hat{\imath} + \overrightarrow{A_y}\ \hat{\jmath} + \overrightarrow{A_z}\ \hat{k}$ and $\vec{B} = \overrightarrow{B_x}\ \hat{\imath} + \overrightarrow{B_y}\ \hat{\jmath} + \overrightarrow{B_z}\ \hat{k}$

$\therefore \vec{A} \cdot \vec{B} = (A_x\ \hat{\imath} + A_y\ \hat{\jmath} + A_z\ \hat{k}) \cdot (B_x\ \hat{\imath} + A_y\ \hat{\jmath} + B_z\ \hat{\jmath})$

$= A_xB_x(\hat{\imath} \cdot \hat{\imath}) + A_xB_y(\hat{\imath} \cdot \hat{\jmath}) + A_xB_z\ (\hat{\imath} \cdot \hat{\jmath}) + A_yB_x\ (\hat{\jmath} \cdot \hat{\imath})$

$+ A_yB_y\ (\hat{\jmath} \cdot \hat{\jmath}) + A_yB_z\ (\hat{\jmath} \cdot \hat{k})$

$+ A_zB_x\ (\hat{k} \cdot \hat{\imath}) + A_zB_y\ (\hat{k} \cdot \hat{\jmath}) + A_zB_z\ (\hat{k} \cdot \hat{k})$

$\therefore \vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_yB_y$

If $\vec{A} = \vec{B}$, then $\vec{A} \cdot \vec{A} = A_xA_x + A_yA_y + A_zA_z$

or $A^2 = {A_x}^2 + {A_y}^2 + {A_z}^2$ or $A = \sqrt{{A_x}^2 + {A_y}^2 + {A_z}^2}$

Application of Dot Product

(i) Work done by constant force is the dot product of force and displacement.

$$W = \vec{F} \cdot \vec{S}$$

(ii) Power is the dot product of force and velocity.

Sample Problem 2.1. Prove that

$(\vec{A} + 2\,\vec{B}) \cdot (2\vec{A} - 3\,\vec{B}) = 2A^2 + AB\cos\theta - 6B^2$.

Solution. $(\vec{A} + 2\,\vec{B}) \cdot (2\vec{A} - 3\,\vec{B})$

$= 2\ \vec{A} \cdot \vec{A} - 3\ \vec{A} \cdot \vec{B} + 4\ \vec{B} \cdot \vec{A} \cdot - 6(\vec{B} \cdot \vec{B})$

$= 2\ (\vec{A} \cdot \vec{A}) - 3\ \vec{A} \cdot \vec{B} + 4\ \vec{B} \cdot \vec{A} \cdot - 6(\vec{B} \cdot \vec{B})$

$= 2\ A^2 + AB\cos\theta - 6B^2$

Which was to be proved

EXERCISES

1. Prove that the vector $\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ and $2\hat{\imath} - \hat{\jmath}$ are perpendicular to each other.

2. Given: $\vec{A} = \hat{\imath} - 2\hat{\jmath} - 3\hat{k}$ and $\vec{A} = 4\hat{\imath} - 2\hat{\jmath} + 6\hat{k}$. Calculate the angle made by $(\vec{A} + \vec{B})$ with x-axis?

[**Ans.** 45°]

2.12 CROSS PRODUCT OF VECTORS

Cross product of $\vec{A}$ and $\vec{B}$ inclined to each other at an angle θ is a vector whose magnitude is AB sin θ and direction is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$.

Since the cross product of two vectors is a vector therefore it is also known as vector product.

Consider two vectors $\vec{A}$ and $\vec{B}$ inclined to each other at an angle θ. Their cross product is given by

$$\vec{A} \times \vec{B} = AB \sin\theta\, \hat{n} = \hat{C}$$

Here A and B are the magnitude of $\vec{A}$ and $\vec{B}$ respectively, $\hat{n}$ is a unit vector which points in a direction perpendicular to the plan of $\vec{A}$ and $\vec{B}$.

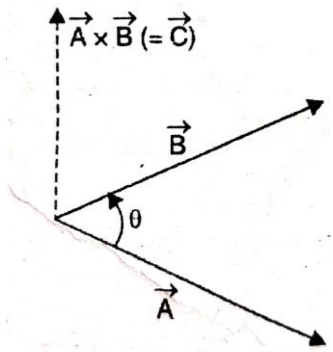


Fig. 2.19

There are **two rules** for determining the direction of the cross product.

Before applying these rules, it should be ensured that the vectors are either directed away from the point or towards the point.

(1) Right hand thumb rule. Curl the fingers of your right hand from $\vec{A}$ to $\vec{B}$. Then the direction of the erect thumb will point in the direction $\vec{A} \times \vec{B}$ (Fig. 2.20 and 2.22)

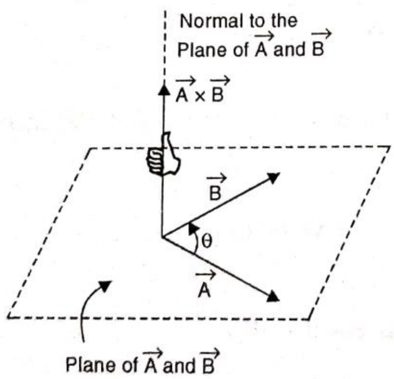


Fig. 2.20

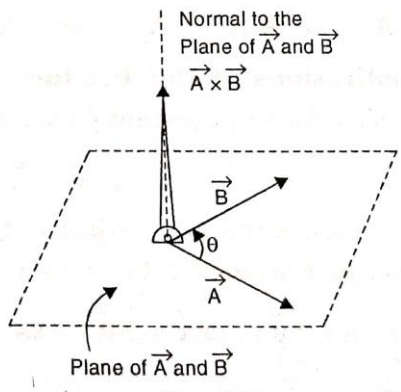


Fig. 2.21

(2) *Right hand screw rule. Hold a right hand screw with its axis perpendicular to the plane containing $\vec{A}$ and $\vec{B}$. Now, turn the screw from $\vec{A}$ to $\vec{B}$. The direction of advance of the screw gives the direction of $\vec{A} \times \vec{B}$ (Figs. 2.21 and 2.23).

Important **properties** of cross product are as under :

(1) Cross product is anti-commutative i.e., $\vec{A} \times \vec{B} \neq \vec{A} \times \vec{B}$

Proof. By definition of cross product, $\vec{A} \times \vec{B} = AB \sin \theta\ \hat{n}$

$\vec{B} \times \vec{A} = BA \sin (-\theta)\ \hat{n} = BA \sin \theta\ \hat{n}$ [$\therefore \sin (-\theta) = - \sin \theta$]

$= -AB \sin \theta\ \hat{n} = - \vec{A} \times \vec{B} \quad \therefore \quad \vec{B} \times \vec{A} \neq \vec{A} \times \vec{B}$

Applying this property to unit vectors, we get

$\hat{\imath} \times \hat{\jmath} = - \hat{\jmath} \times \hat{\imath},\ \hat{\jmath} \times \hat{k} = - \hat{k} \times \hat{\jmath},\ \hat{k} \times \hat{\imath} = - \hat{\imath} \times \hat{k}$

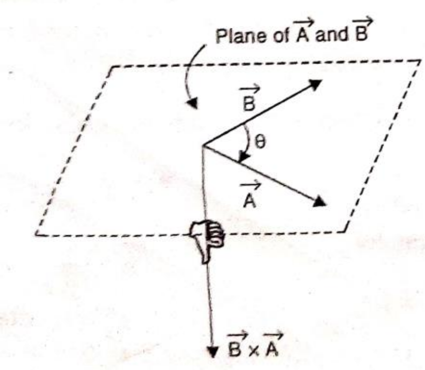


Fig. 2.22

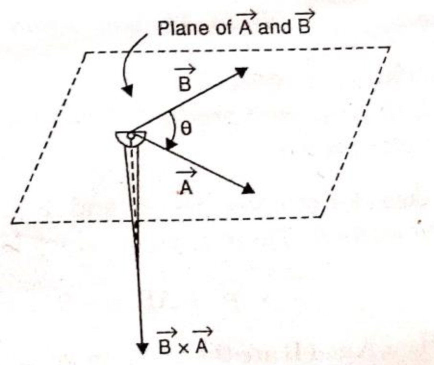


Fig. 2.23

(2) Cross product of two perpendicular vectors is equal to the product of the magnitudes of the given vectors, the direction being perpendicular to the plane of the given vectors.

Proof. If A is perpendicular to $\vec{B}$, then $\theta = \frac{\pi}{2}$

In that case, $\vec{A} \times \vec{B} = AB \sin \frac{\pi}{2}\ \hat{n} = AB\ \hat{n}$ [$\therefore \sin \frac{\pi}{2} =$]

Applying the result to unit vectors, we get

$\hat{\imath} \times \hat{\jmath} = \hat{k},\ \hat{\jmath} \times \hat{k} = \hat{\imath},\ \hat{k} \times \hat{\imath} = \hat{\jmath}$

Similarly, $\hat{\jmath} \times \hat{\imath} = - \hat{k},\ \hat{k} \times \hat{\jmath} = - \hat{\imath}$

and $\hat{\imath} \times \hat{k} = - \hat{\jmath}$

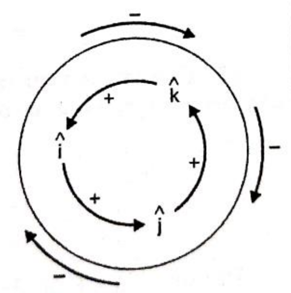


Fig. 2.24

It is easy to memorise those results with the help of Fig. 2.24

(3) Cross product is distributive i.e.,

$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + = \vec{A} \times \vec{C}$

(4) Cross product of two parallel vectors is zero.

Proof. In the case of parallel vectors, $\theta = 0°$

$\therefore\ \vec{A} \times \vec{B} = AB \sin 0°\ \hat{n} = 0$

This leads us to the following condition of parallelism of two vectors.

"Two non-zero vectors are parallel if and only if the magnitude of their cross product is zero."

Applying this result to unit vectors, we get $\hat{\imath} \times \hat{\imath} = 0$, $\hat{\jmath} \times \hat{\jmath} = 0$, $\hat{k} \times \hat{k} = 0$

(5) If two vectors are represented by the adjacent sides of a parallelogram, then the magnitude of their cross product will give the area of the parallelogram.

Proof. Two vectors $\vec{A}$ and $\vec{B}$ are represented by the two adjacent sides PQ and PS respectively in Fig. 2.25.

Now, $\vec{A} \times \vec{B} = AB \sin \theta\ \hat{n}$

$\left|\vec{A} \times \vec{B}\right| = AB \sin \theta$

$= A(B \sin \theta) = A(SN) = A \times h$

$=$ base $\times$ height of parallelogram

$=$ area of parallelogram

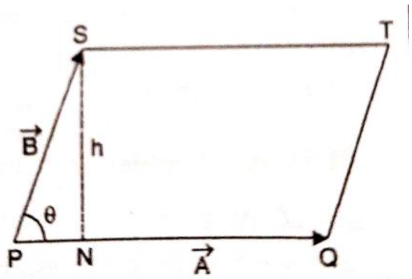


Fig. 2.25

(6) If two vectors are represented by the two slides of a triangle, then half the magnitude of their cross product will give the area of the triangle

Proof. If Fig. 2.26 two vectors $\vec{A}$ and $\vec{B}$ are represented by the two sides PQ and PS respectively of the triangle PQS.

Now, $\frac{1}{2}\left|\vec{A} \times \vec{B}\right| = \frac{1}{2} AB \sin \theta$

$= \frac{1}{2}(A)(B \sin \theta)$

$= \frac{1}{2} \times A \times h = \frac{1}{2} \times$ base $\times$ height

$=$ area of triangle

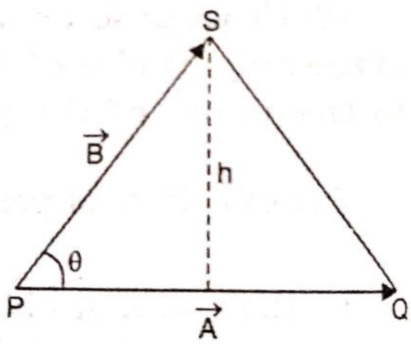


Fig 2.26

(7) The cross product of two vectors $\vec{A}$ and $\vec{B}$, in terms of rectangular components, is given by

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Proof. $\vec{A} \times \vec{B} = (A_x\,\hat{\imath} + A_y\,\hat{\jmath} + A_z\,\hat{k}) \times (B_x\,\hat{\imath} + B_y\,\hat{\jmath} + B_z\,\hat{k})$

$$= A_xB_x\,(\hat{\imath} \times \hat{\imath}) + A_xB_y\,(\hat{\imath} \times \hat{\jmath}) + A_xB_z\,(\hat{\imath} \times \hat{k})$$

$$+ A_yB_x\,(\hat{\jmath} \times \hat{\imath}) + A_yB_y\,(\hat{\jmath} \times \hat{\jmath}) + A_yB_z\,(\hat{\jmath} \times \hat{k})$$

$$+ A_zB_x\,(\hat{k} \times \hat{\imath}) + A_zB_y\,(\hat{k} \times \hat{\jmath}) + A_zB_z\,(\hat{k} \times \hat{k})$$

$$= A_xB_y\,\hat{k} - A_xB_z\,\hat{\jmath} - A_yB_x\,\hat{k} + A_yB_z\,\hat{\imath} + A_zB_x\,\hat{\jmath} - A_zB_y\,\hat{\imath}$$

$$= \hat{\imath}\,(A_yB_z - A_zB_y) + \hat{\jmath}(A_zB_x - A_xB_z) + \hat{k}(A_xB_y - A_yB_x) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Applications of Cross Product

(i) The linear velocity ő is given by $\vec{v} = \vec{\omega} \times \vec{r}$.

Where $\vec{\omega}$ is the angular velocity and $\vec{r}$ is the position or radius vector.

(ii) The tangential acceleration $\vec{a}_t$ is given by $\vec{a}_t = \vec{a} \times \vec{r}$ where $\vec{a}$ is the angular acceleration.

(iii) The centripetal acceleration $\vec{a}_c$ is given by $\vec{a}_c = \vec{\omega} \times \vec{v}$.

(iv) The angular momentum $\vec{L}$ is given by $\vec{L} = \vec{r} \times \vec{p}$ where $\vec{p}$ is the linear momentum.

(v) The torque $\vec{\tau}$ is given by $\vec{\tau} = \vec{r} \times \vec{F}$ is the applied force.

Sample Problem 2.2. Prove that the vectors $2\,\hat{\imath} - 3\,\hat{\jmath} - \hat{k}$ and $-6\,\hat{\imath} + 9\,\hat{\jmath} + 3\,\hat{k}$ are parallel.

Solution. The given vectors will be parallel if their cress product is zero.

Now, $(2\hat{\imath} - 3\hat{\jmath} - \hat{k}) \times (-6\hat{\imath} + 9\hat{\jmath} + 3\hat{k})$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -3 & -1 \\ -6 & 9 & 3 \end{vmatrix}$$

$$= (-9 + 9)\,\hat{\imath} + (6 - 6)\,\hat{\jmath} + (18 - 18)\,\hat{k} = 0$$

So, the given vectors are parallel to each other

2.13 IMPORTANCE TERMS

(i) Particle kinematics. It is that branch of Mechanics which deals with the study of motion without taking into account the cause of the motion.

(ii) Particle dynamics. It is that branch of Mechanics in which we study not only the motion of the particle but also the cause of motion.

(iii) Objects in Motion. An object is said to be in motion if its position changes with respect to the surroundings in the course of time.

(iv) Concept of a point object. An object is said to be a point object if it changes its position by distances which are much greater than its size.

Examples. (i) Consider the revolution of Earth around the Sun. The diameter of the Earth is very small as compared to the length of its orbit around the Sun. So, Earth can be regarded as a point object.

(ii) A car travelling a few hundred kilometre may be regarded as a point object.

(ii) A kite flying in the sky is an example of a point object.

(iv) A satellite circling around the Earth is a point object,

(v) Motion in one dimension. One-dimensional motion is the motion of a particle moving along a straight line.

One-dimensional motion is sometimes known as rectilinear or linear motion.

Examples. (i) If we ignore the width and side-ways motion of a train moving on a straight railway track, then the motion of the train can be said to be one-dimensional.

(ii) A ball thrown vertically up or a ball dropped from a certain height above the ground moves along a straight line. So, motion of ball is one-dimensional motion.

(iii) A mass suspended from a vertical spring can oscillate along a straight line so, its oscillatory motion will be one-dimensional motion

(vi) Motion in two dimensions. A particle moving along a curved path in a plane has two- dimensional motion

In this type of motion, two of the three rectangular co-ordinates change with time. In order to describe this motion, we require two rectangular co-ordinate axes

Any object travelling a certain distance on the surface of Earth possesses two-dimensional motion provided the distance travelled is very small as compared to the circumference of Earth.

Examples. (i) An insect crawling on a globe.

(ii) À planet revolving around its star.

(iii) A carrom coin rebounding smoothly from the side of the board.

(vii) Motion in three dimensions. A particle moving in space has three dimensional motion,

In this type of motion, all the three rectangular co-ordinates change with time. In order to describe this motion, we require three rectangular co-ordinate axes

Three-dimensional motion is the most general type of motion. A kite flying on a wind day is an example of three-dimensional motion

(vili) Displacement and distance. Displacement of a particle is the change in the position of the particle in a particular direction

The path-length between the initial and final positions of the particle gives the distance covered by the particle

2.14 SPEED

(i) The time rate of change of position of an object is called speed. It tells us how fast the object is moving, i.e., it tells us how much distance is covered by the moving particle in each second. But it does not indicate the direction of motion. It is a scalar quantity. Speed is either positive or zero. It is never negative because the concept of speed is independent of the direction of motion.

The cgs and SI units of speed are cm s^{-1} and m s^{-1} respectively. Its dimension formula is ($M^{o}LT^{-1}$).

(ii) Uniform speed. An object is said to move with uniform speed if covers equal distances in equal intervals of time, howsoever small these interval of time may be.

(iii) Variable speed. An object is said to move with variable speed if it covers equal distances in unequal intervals of time or unequal distances in equal intervals of time, howsoever small these intervals of time may be.

(iv) Average speed. The average speed of an object is that constant speed with which the object covers the same distance in a given time as it does while moving with variable speed during the given time.

It may also be defined as under :

Average speed is defined as the ratio of the path length travelled and the corresponding time interval.

$$\text{Average speed} = \frac{\text{Path length}}{\text{Corresponding time interval}}$$

If Δx is the distance travelled in time Δt, then the average speed is given by :

$$v_{av} \quad \text{or} \quad \bar{v} = \frac{\Delta x}{\Delta t}$$

Average speed has obviously the same unit (m s^{-1}) as that of speed. Average speed does not tell us in what direction an object is moving. Thus, it is always positive.

(v) Instantaneous speed. The speed of an object at an instant of time is called instantaneous speed.

It may also be defined us under:

Instantaneous speed is the limit of the average speed as the time interval becomes infinitesimally small.

If Δx is the distance travelled in time Δt, then

$$\text{Instantaneous speed } v = \lim \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

2.15 VELOCITY

(i) Velocity. The time rate of change of position of a particle in a particular direction is called velocity. It may also be defined as the time rate of change of displacement. In simple words, speed in a particular direction is called velocity. It is a vector quantity, i.e., it possesses both magnitude and direction. It is for this reason that velocity can have both positive and negative values including zero. If the particle does not change its direction of motion, then the magnitude of velocity $\vec{v}$ is known as speed v.

$$v = |\vec{v}|$$

(ii) Uniform Velocity. An object is said to move with uniform velocity if it covers equal displacements in equal intervals of time, howsoever small these intervals of time may be.

(iii) Variable Velocity. An object is said to move with variable velocity if it covers equal displacements in unequal intervals of time or unequal displacements in equal intervals of time.

(iv) Average Velocity. The average velocity of an object is that uniform velocity with which the object undergoes the same displacement in a given time as it undergoes while moving with variable velocity during the given time.

It may also be defined as under :

Average velocity is defined as the ratio of the displacement and the corresponding time interval.

$$\text{Average velocity} = \frac{\text{Displacement}}{\text{Corresponding time interval}}$$

If $\overrightarrow{\Delta x}$ is the distance travelled in time Δt, then the average velocity is given by :

$$\overrightarrow{v_{av}} = \frac{\overrightarrow{\Delta x}}{\Delta t}$$

(v) Instantaneous Velocity. When the velocity of a particle is variable, we are generally interested in instantaneous velocity.

Instantaneous velocity is the liming value of the average velocity $\frac{\overrightarrow{\Delta x}}{\Delta t}$ as Δt approaches zero. It is denoted by $\bar{v}$

$$\text{Instantaneous speed } \bar{v} = \text{Lt. } \frac{\overrightarrow{\Delta x}}{\Delta t} = \frac{\overrightarrow{dx}}{\mathrm{d}t}$$

The magnitude of the instantaneous velocity is known as instantaneous speed. It is given by

$$v = \frac{dx}{dt}$$

The speedometer of an automobile measures the instantaneous speed the automobile.

2.16 ACCELERATION

(i) The acceleration of a particle is the time rate of change of its velocity. It may also be defined as the change in velocity in unit time.

It is a vector quantity which is denoted by $\vec{a}$. It is characterized by both magnitude and direction.

In cgs system, it is measured in cm s^{-2}. while in SI, it is measured in m $s^{-2.}$

$$\text{Acceleration} = \frac{\text{Change in velocity}}{\text{Time taken}}$$

(ii) The average acceleration is the ratio of the change in velocity to the time taken to undergo this change.

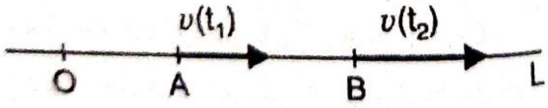


Fig. 2.27

$$\text{Average acceleration} = \frac{\text{Total change in velocity}}{\text{Total time taken}}$$

Consider a particle moving along a straight line L. Let the particle be at the origin O at $t = 0$. Let A and B be the positions of the particle at times t_1 and t_2 respectively. Let $v(t_1)$ and $v(t_2)$ be the magnitudes of the respective velocities. Then, the magnitude of average acceleration is given by

$$a_{av} = \frac{v(t_2)-v(t_1)}{t_2-t_1} = \frac{\Delta v}{\Delta t}$$

In vector notation, $\overrightarrow{a_{av}} = \frac{\overrightarrow{\Delta v}}{\Delta t}$

(iii) Instantaneous acceleration is the acceleration at any instant of time or at any given point.

The limiting value of the average acceleration as Δt tends to zero is called instantaneous acceleration $\vec{a}$

$$\vec{a} = \text{Lt.} \frac{\overrightarrow{\Delta v}}{\Delta t} = \frac{\overrightarrow{dv}}{dt}$$

(iv) The acceleration of a particle is said to be uniform if its velocity increases by equal amounts in equal intervals of time, howsoever small these intervals may be.

(v) A particle is said to move with **variable acceleration** if it's velocity changes by unequal amounts in equal intervals of time.

2.17 VELOCITY – TIME GRAPHS OF ACCELERATED MOTION

In a velocity-time graph, the velocity is plotted against y-axis and time against x-axis. This is because velocity is a dependent variable and time is an independent variable.

The slope of velocity-time graph gives acceleration.

The area under velocity-time graph gives displacement.

The area under speed-time graph gives distance.

(a) Zero Acceleration

In this case, the velocity-time graph is parallel to time-axis. (Fig. 2.28)

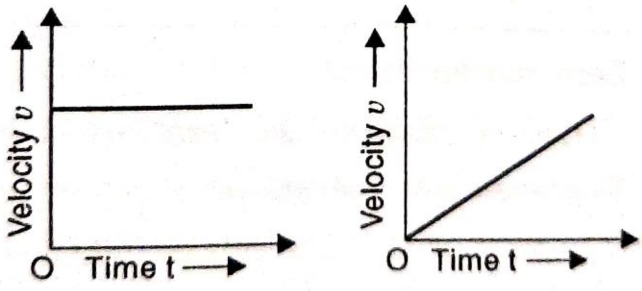


Fig. 2.28 **Fig. 2.29**

(b) Constant Positive Acceleration and Zero Initial Velocity

The velocity-time graph passes through the origin and is inclined to the time-axis such that the angle of inclination is greater than 0° and less than 90°. (Fig. 2.29)

(c) Constant Positive Acceleration and Non-zero Initial Velocity

In this case, the velocity-time graph is a straight line sloping upwards. It does not pass through the origin. (Fig. 2.30)

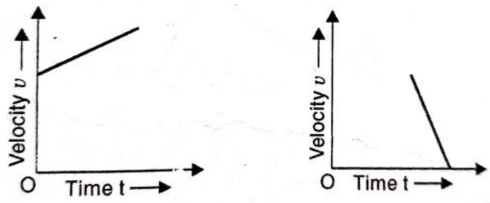


Fig. 2.30 **Fig. 2.31**

(d) Constant Negative Acceleration

In this case, the velocity-time graph is a straight line having negative slope. (Fig. 2.31)

(e) Increasing Acceleration

In this case, the velocity-time graph is a curve whose slope is increasing as we go farther from the origin O. (fig. 2.32)

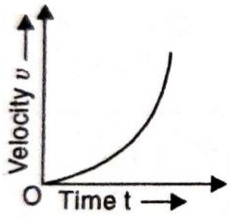


Fig. 2.32

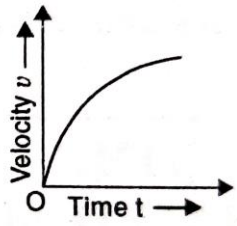


Fig. 2.33

(f) Decreasing Acceleration

In this case, the velocity-time graph is a curve whose slope is decreasing as we go farther from O. (Fig. 2.33)

If the velocity were to remain constant both in magnitude and direction then the acceleration would be zero.

All the velocity-time graphs discussed above have been combined in Fig. 2.3

Knowledge Plus

I. Zero acceleration
II. Constant positive acceleration, u = 0
III. Decreasing acceleration
IV. Constant positive acceleration, u = 0
V. Increasing acceleration
VI. Constant negative acceleration.
VII, Infinite acceleration

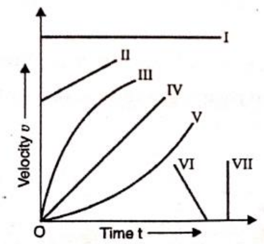


Fig. 2.34

EQUATIONS OF MOTION

2.18 DERIVATION OF *v* = *ut* + *at*

The equation $v = u + at$ is known as the first equation of motion. This equation helps us to find the final velocity v in terms of initial velocity u uniform acceleration a and time t.

The first equation of motion $v = u + at$ is derived as under :

Let, u = initial velocity of the body

v = final velocity of the body

t = time during which the velocity of the body changes from u to v

a = uniform acceleration of the body

Using the definition of acceleration,

$$\text{acceleration} = \frac{\text{Change in velocity}}{\text{Time taken}}$$

or $$\text{acceleration} = \frac{\text{Final velocity} - \text{Initial velocity}}{\text{Time taken}}$$

So, $$a = \frac{v-u}{t}$$

or $$at = v - u$$

or $$v - u = at$$

or $$v = u + at$$

This is the first equation of motion.

In words, Final velocity = Initial velocity + (Acceleration × Time)

Note 1. The term 'at' represents the change in velocity of the body.

Note 2. The first equation of motion contains four variables. If any of the three variables are known, the fourth can be determined. This equation can also be used to solve problems to retardation. But, in such problems, we have to use negative sign with a

2.19 DEVIATION OF S = $ut + \frac{1}{2}at^2$

The equation $S = ut + \frac{1}{2}at^2$ is known as the second equation of motion.

This equation helps us to find the distance S travelled by a body in terms of initial velocity u, time t and uniform acceleration a.

The second equation of motion $S = ut + \frac{1}{2}at^2$ is derived as under:

Let, u = initial velocity of the body

v = final velocity of the body

t = time during which the velocity of the body changes from u to v

S = distance travelled by the body in time t

a = uniform acceleration of the body

we know that Average velocity = $\frac{\text{Initial velocity} - \text{Final velocity}}{2}$

or Average velocity = $\frac{u+v}{2}$

Also, Distance travelled = Average velocity × Time

$$S = \left(\frac{u+v}{2}\right)t$$

Using the first equation of motion $v = u + at$, we get

$$S = \frac{(u+u+at)t}{2}$$

or $S = \frac{(2u+at)t}{2}$ or $S = \frac{2u+at^2}{2}$

or $S = ut + \frac{1}{2}at^2$

In words, distance travelled

= (initial velocity × time) + $\frac{1}{2}$ (acceleration × time^2)

Note 1. The term 'ut' represents the distance which the body would cover in time t while moving with constant velocity u

Note 2. The term $\frac{1}{2}at^2$ represents the distance which the body would cover, starting from rest, when accelerated with acceleration a for time t.

Note 3. The second equation of motion involves four variables. If any three of the four variables are known, then the fourth can be calculated. The problems of retardation can be solved by using negative sign with a.

2.20 DEVIATION OF $v^2 - u^2 = 2as$

The equation $v^2 - u^2 = 2as$ is known as the third equation of motion. This equation helps us to find the final velocity v in terms of initial velocity u, uniform acceleration a and distance S.

The third equation of motion $v^2 - u^2 = 2as$ is derived as under :

Let, u = initial velocity of the body

v = final velocity of the body

t = time during which the velocity of the body changes from u to v

a = uniform acceleration of the body

S = distance travelled by the body in time t

From the second equation of motion, $S = ut + \frac{1}{2}at^2$ …(i)

From the first equation of motion, $v = u + at$

Rearranging, $v - u = at$

or $$at = v - u$$

or $$t = \frac{v+u}{a}$$

Putting the value of t in equation (1), we get

$$S = u\left(\frac{v+u}{a}\right) + \frac{1}{2}a\left(\frac{v+u}{a}\right)^2$$

or $$S = \frac{uv+u^2}{a} + \frac{(v+u)^2}{2a}$$

or $$S = \frac{2uv+2u^2+v^2+u^2+2uv}{2a}$$

or $$S = \frac{v^2-u^2}{2a}$$

or $$v^2 - u^2 = 2aS$$

In words, (Final velocity)^2 – (Initial velocity)^2 = 2(acceleration) (distance)

Note. The third equation of motion contains four variables. If any of the three variables are known, the fourth can be determined.

2.21 derivation of $S_{nth} = u + \frac{a}{2}(2n-1)$

Let S_n be the distance covered in n second.

Let S_{n-1} be the distance covered in $(n-1)$ second.

Then, $S_n = un + \frac{1}{2}an^2$

$$S_{n-1} = u(n-1) + \frac{1}{2}a(n-1)^2$$

The distance covered during nth second of motion is given by

$$S_{nth} = S_n - S_{n-1}$$

$$S_{nth} = (un + \frac{1}{2}an^2) - u(n-1) - \frac{1}{2}a(n-1)^2$$

or $$S_{nth} = un + \frac{1}{2}an^2 - un + u - \frac{1}{2}a(n^2+1-2n) = u - \frac{1}{2}a + an$$

or $$S_{nth} = u + \frac{a}{2}(2n-1)$$

All the kinematic equations discussed above hold good only for uniformly accelerated motion.

They are NOT to be used in the case of variable acceleration.

2.22 GRAPHICAL METHOD FOR DERIVATION OF EQUATIONS OF MOTION

The equations of motion can be easily derived with the help of velocity-time graphs. Time is an independent quantity'. So, it is plotted against X-axis. Velocity depends upon time. So, it is a 'dependent quantity'. Therefore it is plotted against Y-axis.

(i) Derivation of $v = u + at$

Consider a body moving with uniform acceleration a. Let its velocity change from u to v time t. The velocity-time graph of the body is a straight line AB sloping upwards as shown in Fig. 2.35.

Slope of AB = *tan θ = $\frac{\text{BD}}{\text{AD}} = \frac{\text{CB}-\text{CD}}{\text{AD}} = \frac{\text{CB}-\text{OA}}{\text{AD}}$

or Slope of AB = $\frac{v-u}{t}$

But slope of velocity – time graph gives acceleration.

∴ $a = \frac{v-u}{t}$

or $v - u = at$

$v = u + at$

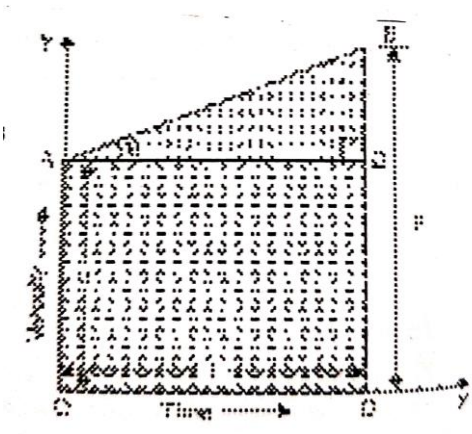


Fig. 2.35

(ii) Deviation Of S = $ut + \frac{1}{2}at^2$

Consider a body moving with uniform acceleration a. Let its velocity show from u to v over distance S. The velocity-time graph of this body is a straight line AB sloping upwards as shown in Fig. 2.35.

We know that the area under the velocity-time graph gives the distance S travelled by the body.

∴ Distance, S = Area OABC

= Area of rectangle OADC + Area of triangle ADB

= OA × OC + $\frac{1}{2}$ AD × BD …(i)

Now OA = u, OC = t, AD = t

Also, a = slope of AB = $\frac{\text{BD}}{\text{AD}} = \frac{\text{BD}}{t}$

or BD = at

Substituting values in equation (i), we get

$$S = ut + \frac{1}{2}\, t \times at$$

$$S = ut + \frac{1}{2} at^2$$

(iii) Derivation of v^2 - u^2= 2as

Consider a body moving with uniform acceleration a. Let its velocity from u to v over distance S. The velocity-time graph of this body is a stranght line AB sloping upwards as shown in Fig. 2.35.

We know that the area under the velocity-time graph gives the distance S travelled by the body.

∴ Distance, S = Area under velocity-time graph

= *Area of trapezium OABC = $\frac{1}{2}$ (OA+CB) AD

Now, OA = u, CB = v

Also, a = slope of AB = $\frac{\text{BD}}{\text{AD}} = \frac{v-u}{\text{AD}}$ or AD = $\frac{v-u}{a}$

Substituting value in equation (i), we get

$$S = \frac{1}{2}(u + v)\left(\frac{v-u}{a}\right)$$

or $$S = \frac{(v+u)(v-u)}{2a}$$

or $$S = \frac{v^2-u^2}{2a}$$

or $$v^2 - u^2 = 2as$$

2.23 CALCULUS METHOD FOR DERIVATION OF EQUATIONS OF MOTION

(i) $v = u + at$

We know that $$a = \frac{dv}{dt}$$

or $$dv = a\, dt$$

Integrating both sides,

$$\int_u^v dv = \int_0^t a\, dt$$

or $\left|v\right|_u^v = a \left|t\right|_u^v$ or $v - u = a\,(t - 0)$ or $v = u + at$

(ii) S = $ut + \frac{1}{2} at^2$

$v = \frac{dS}{dt}$ or $dS = vdt$

or $\int_0^S dS = \int_0^t v\,dt = \int_0^t (u + at)dt = \int_0^t udt + \int_0^t at\,dt$

or $\left|S\right|_0^s = u\left|t\right|_0^t + a\left|\frac{t^2}{2}\right|_0^t$

or $S - 0 = u(t - 0) + \frac{1}{2}a(t^2 - 0)$ **or** $S = ut + \frac{1}{2}at^2$

2.24 MOTION UNDER GRAVITY

While studying motion under gravity, the acceleration due to gravity is regarded as constant. It is also to be noted that the acceleration due to gravity g is always directed vertically downward. Some important relations for solving problems of motion under gravity are given below :

(1) Consider a body falling freely from rest through height h, Let the body acquire velocity v after falling through height h. Let t be the time taken to fall through height h.

Now, (a) $t = \frac{v}{g}$ (b) $t = \sqrt{\frac{2h}{g}}$ (c) $v = \sqrt{2gh}$

(ii) Let a body be projected vertically upward with velocity u. Let h be the maximum height attained by the body. Let t be the time taken,

Now, (a) $t = \frac{v}{g}$ (b) $t = \sqrt{\frac{2h}{g}}$ (c) $v = \sqrt{2gh}$

(iii) The speed with which a body is projected is equal to the speed with which it returns to the point of projection. It is based on the assumption that the motion takes place in vacuum.

(iv) Time taken by body to go up is equal to the time taken by body to fall through the same height.

(v) Consider a body falling freely from rest. Height covered during nth second is given by $h_{nth} = (2n - 1)\frac{g}{2}$

Since $\frac{g}{2}$ is constant, $\therefore h_{nth} \propto (2n - 1)$

So, the heights through which a body falls in 1st, 2nd, 3rd second etc, are in the ratio of 1:3 : 5 etc. i.e., in the ratio of odd integers.

(vi) Consider a body falling freely from rest.

$$h = \frac{1}{2}gt^2$$

since $\frac{1}{2}g$ is constant, $h \propto t^2$

So, the heights through which a body falls in times t, $2t$, $3t$ etc. are in the ratio of

$1^2 : 2^2 : 3^2$ etc. 1 : 4 : 9 etc. [Note the ratio involves the square of the integers.]

Sample Problem 2.3. A cyclist accelerates uniformly from rent at point A and reaches point Y at a velocity of 9 m $s^{-1'}$ as shown in 2.36. The cyclist covers the last 10 m of her ride in 2 s. Determine the acceleration of the cyclist.

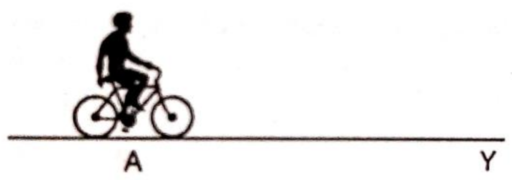


Fig. 2.36

Solution. We summarise the given information for the last 10 m before reaching Y.

u	v	S	a	t
'u'	9 m s^{-1}	10 m	'a'	2 s

$$v = u + at$$

$\Rightarrow \quad 9 = u + 2a \quad \Rightarrow \quad 2a = 9 - u \qquad \ldots$(i)

Now, $S = ut + \frac{1}{2} \times gt^2$

$$10 = u \times 2 + \frac{1}{2} \times a \times 2^2$$

$\Rightarrow \quad 10 = 2u + 2a \quad \Rightarrow \quad 2a = 10 - 2u \qquad \ldots$(ii)

From equations (i) and (ii)

$$10 - 2u = 9 - u$$

$\Rightarrow \quad u = 1\text{m s}^{-1} \quad \Rightarrow \quad 2a = 9 - 1 \quad \Rightarrow \quad a = 4\text{m s}^{-2}$

Sample Problem 2.4. You can measure your reaction time by a simple experiment. Take a ruler and ask your friend to drop it vertically through the gap between your thumb and forefinger. As soon as it is dropped, note the time elapsed before you catch it and the distance travelled by the ruler. In a particular case, the distance was found to be 21.0 cm. Estimate the reaction time.

Solution. The ruler drops under free fall.

$\therefore u = 0,\ a = g = 9.8 \text{ m s}^{-2}$

If is the distance travelled and t, is the reaction time, then using

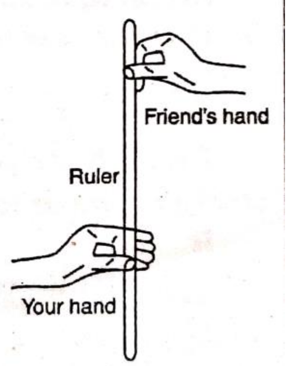


Fig. 2.37

$$S = ut + \frac{1}{2} at^2$$

We get $\quad d = 0 \times t_r + \frac{1}{2}\, a\, t_r^{\,2}$

or $\quad t_r = \sqrt{\frac{2d}{g}}$

$$= \sqrt{\frac{2 \times 0.21}{9.8}} \text{ s} = \mathbf{0.207\ s}$$

EXERCISES

1. An object is thrown vertically upwards with a velocity of 19.6 m s^{-1} Calculate the distance and displacement of the object after 3 second?

[**Ans.** 24.5 m, 14.7 m]

2. (a) A stone is dropped from a balloon moving upwards with a velocity of 4.5 m s^{-1} The stone reaches the ground in 5 second. Calculate the height of the balloon when the stone was dropped.

[**Ans.** 100 metre]

(b) A stone falls freely under gravity, starting from rest. Calculate the ratio of distance travelled by the stone during the first half of any interval of time to the distance travelled during the second half of the same interval.

[**Ans.** 1 : 3]

3. A stone dropped from a balloon at an altitude of 300 m. How long will the stone take to reach the ground if (*a*) the balloon is ascending with a velocity of 5 m s^{-1} (b) the balloon is descending with a velocity of 5 m s^{-1} (c) the ballon stationary?

[**Ans.**(a) 8.35 s (b) 7.33 s (c) 7.8 s]

4. A ball roll down an inclined plane 2 m long in 4 second. Find (i) its acceleration (ii) the time taken to ever the second metre of the track (iii) the speed of the ball at the bottom of the track.

[**Ans.** (i) 0.25 m s^{-1} (ii) 1.172 s (iii) 1 m s^{-1}]

5. A body travels 20 cm in the first 0.2 second and 22 cm in the next 0.4 second. What will be the velocity at the end of the 0.7th second from start ?

[**Ans.** 10m s^{-1}]

6. A stone falls from a cliff and travel 34.3 m in the last second before it reaches the round. Calculate the height of the cliff.

[**Ans**. 78.4 m]

7. During the last second of its free fall, a body covers half of the total distance travelled. Calculate (i) the approximate height from which the body falls (ii) the duration of the fall.

[**Ans.** (i) 57 m (ii) 3.41 s]

8. A car starts from rest and accelerates uniformly for 10 s to a velocity of 8 m s^{-1}. It then runs at a constant velocity and is finally brought to rest in 64 m with a constant retardation. The total distance covered by the car is 584 m. Find the values of acceleration, retardation and total time taken.

[**Ans.** 0.8 m s^{-1}, 0.5 m s^{-1}, 86 s]

NEWTON'S LAWS OF MOTION

2.25 NEWTON'S FIRST LAW OF MOTION AND DERIVATION OF DEFINITION OF FORCE

Statement. Every body continues in its state of rest or uniform motion in a straight line unless compelled by an external force to change that state.

OR

Every body persists in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces impressed on it.

Discussion. When a body is in a state of rest or in uniform motion along a straight line, it implies that body has zero acceleration. So, according to the first law, if the force F is zero, then the acceleration a is also zero.

The statement of Newton's first law of motion may be divided into two parts:

(i) Every body continues in its state of rest unless some external force compels it to change its state of rest.

This part of the law is self-evident. We come across many illustrations of this part in our daily life. A book lying on a table continues to be there unless somebody lifts it.

(ii) Every body continues in its state of uniform motion in a straight line unless external force compels it to change that state.

This part of the law implies that a body set in motion along a straight line shall continue to be in this state unless some external force acts. But this does not seem to be true in everyday life. As an example, ball set rolling on the ground comes of rest after covering some distance. It appears to violate the first law of motion. But this is not the case. The bill come to rest because of the two external forces – force of friction and air resistance.

Definition of force. It follows from Newton's first law of motion that an external force must act on a body to bring about a change in the state of rest or of uniform motion in a straight line. This leads us to the following definition of force.

Force is that push or pull which changes or tends to change the state of rest or of uniform motion in a straight line.

Force possesses both magnitude and direction. So, it is vector quantity.

2.26 MOMENTUM

The total quantity of motion possessed by a moving body is known as the momentum of the body. It is the product of the mass and velocity of a body. It is denoted by $\vec{P}$. $\vec{p} = m\vec{v}$

Since mass m is always positive therefore the direction of $\vec{p}$ is the same as that of $\vec{v}$

In magnitude, $|\vec{p}| = m\,|\vec{v}\,|$ or $p = mv$

Since velocity is a vector and mass is a scalar therefore momentum is a vector, Again, $\vec{p}$ has same direction as that of $\vec{v}$ because m is always positive.

The age and SI units of momentum are g cm s^{-1} and kg m s^{-1} respectively.

The dimensional formula of momentum is (MLT^{-1})

2.27 NEWTON'S SECOND LAW OF MOTION AND DERIVATION OF FORMULA OF FORCE

(1) Statement. The time rate of change of momentum of a body is directly proportional to the impressed force and takes place in the direction of the force.

(ii) Discussion. The statement can be divided into the following two parts:

(a) The time rate of change of momentum of a body in proportional to the impressed force.

A force acting on a body produces a certain change in the momentum of the body. When the given force is doubled, the 'change in momentum' of the body is also doubled. So, as the applied force in increased, the rate of change of momentum of the body is also increased.

(b) The change of momentum takes place in the direction of the force.

Consider a body to be at rest. When a force is applied on this body, the body will begin to move in the direction of the force. If a force is applied on a moving body in the direction of motion of the body, then there is an increase in the momentum of the body. However, if the force is applied on a moving body in a direction opposite to the direction of motion of the body, then there is a decrease in the momentum of the body.

(iii) Formula for force. Let a constant external force $\vec{F}$ acting on a body change its momentum from $\vec{p} + \vec{p} + \vec{d}p$ in time interval dt. Then, the time rate of change of linear momentum is $\frac{\vec{d}p}{dt}$

According to Newton's second law of motion,

$$\frac{d\vec{p}}{dt} \propto \vec{F} \qquad \text{or} \qquad \vec{F} \propto \frac{d\vec{p}}{dt} \qquad \text{or} \qquad \vec{F} = k\frac{d\vec{p}}{dt}$$

Here k is a constant of proportionality. The value of a depends upon the units selected for the measurement of force. In both SI and cgs system, the unit of force is so chosen that $k = 1$.

$$\therefore \qquad \vec{F} = \frac{d\vec{p}}{dt}$$

If m be the mass of a body and $\vec{v}$ its velocity, then $\vec{p} = m\vec{v}$

$$\therefore \qquad \vec{F} = \frac{d\vec{p}}{dt}(m\vec{v})$$

In Newtonian mechanics, mass of a body is taken as a constant.

$$\therefore \qquad \vec{F} = m\frac{d\vec{p}}{dt}(\vec{v})$$

But $\frac{d\vec{p}}{dt}(\vec{v})$ - $\vec{a}$. the acceleration of the body.

(iv) For a system of particles, $\vec{F}$ refers to the total external force on the system and $\vec{a}$ refers to the acceleration of the centre of mass of the system. Any internal forces in the system are not to be included in $\vec{F}$.

2.28 UNITS AND DIMENSIONS OF FORCE

(1) Absolute units of force

The absolute unit of force in cgs system is dyne. Its symbol is dyn.

One dyne of force is that much force which produces an acceleration of 1 cm s^{-2} in a body of mass 1 gram.

1 dyne = 1 gram × 1 cm s^{-2} = lg cm s^{-2}

[∴ F = ma]

In SI, the absolute unit of force is newton. Its symbol is N.

One newton of force is that much force which produces an acceleration of 1 m s^{-2} in a body of mass 1 kilogram.

1 newton = 1 kilogram × 1 m s^{-2} = 1 kg m s^{-2}

Relation between newton and dyne

1 N = 1 kg × 1 m s^{-2} = 10^3 g × 10^2 cm s^{-2}

= 10^5 g cm s^{-2} or **10^5 dyne**

1 N = 10^5 dyne   or   1 dyne = 10^{-5} N

(ii) Gravitational units of force

A gravitational unit of force is that force with which the Earth attracts a body of unit mass towards the centre.

A gravitational unit of force may also be defined as that much force which produces an acceleration equal to g (acceleration due to gravity) in a body of unit mass.

It may also be defined as the weight of a body of unit mass.

In **cgs system**, the gravitational unit of force is gram weight or gram forces. It may be defined in any of the following two ways.

One gram weight of force is the force with which a body of mass 1 g is attracted towards the centre of the earth.

One gram weight of force is that much force which produces an acceleration of 981 cm s^{-2} in a body of mass 1 gram.

$\therefore \quad 1 \text{ g wt} = 1 \text{ gram} \times 981 \text{ cm s}^{-2}$

or $\quad \mathbf{1 \text{ g wt} = 981 \text{ g cm s}^{-2}}$

In general, x g wt $= x \times$ g dyne

where is the value of acceleration due to gravity in cgs units

In SI the gravitational unit of force is kilogram weight or kilogram force. It may also be defined in any of the following two ways.

One kilogram weight of force is the force with which a body of mass 1 kg attracted towards the centre of the Earth.

One kilogram weight of force is that much force which produce an acceleration of 9.8 m s^{-2} in a body of mass 1 kilogram.

1 kg wt - 1 kg × 9.8 m s^{-2} = 9.8 kg m s^{-2} = 9.8 N

In general $\quad x$ kg wt $= x \times g$ newton

where g is the value of acceleration due to gravity in SI units

(iii) Difference between absolute units and gravitational units of force

The absolute units of force remain the same throughout the universe. On the other hand, gravitational unit of force are not constant. This is because the gravitational unit of force depend upon the value of g which is different at different places.

(iv) Dimensional formula of force

(Force) = [Mass] [Acceleration] = [MLT^{-2}]

2.29 LAW OF CONSERVATION OF MOMENTUM

Statement. If the vector sum of the external forces acting on a system is zero, then the total momentum of the system is conserved i.e., remain constant.

Derivation of the law of conservation of momentum from Newton' second law of motion.

According to Newton's second law of motion, the time rate of change momentum is equal to the applied force.

If the system is isolated, then $\vec{F} = 0$

In that case, $\frac{d}{dt}(\vec{P}) = 0$

$\therefore \vec{P}$ = constant [Differential coefficient of an isolated constant is zero.]

This leads us to the following statement of the law of conservation of momentum.

"In the absence of external forces, the total momentum of the system is conserved".

2.30 NEWTON'S THIRD LAW OF MOTION AND DERIVATION OF PROPERTIES OF FORCE

Forces acting on a body originate in other bodies that make up its environment, This property of forces was first stated by Newton in his third law of motion:

"To every action, there is always an equal (in magnitude) and opposite (in direction) reaction."

This law may also be stated as under :

"Action and reaction are equal in magnitude, opposite in direction and act on different bodies."

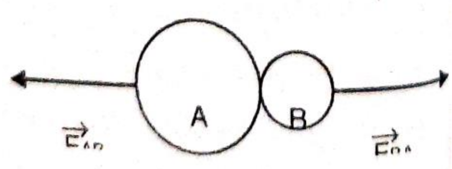


Fig. 2.38

Consider interaction (action and reaction) between two bodies A and B. Let $\overrightarrow{F_{BA}}$ be the force exerted by A on B and $\overrightarrow{F_{AB}}$ the force exerted by B on A (Fig. 2.38). Then, according to Newton's third law of motion,

$$\overrightarrow{F_{BA}} = -\overrightarrow{F_{AB}}$$

It is clear from this equation that the two forces are equal in magnitude but opposite in direction. These forces of action and reaction act along the line joining the centres of two bodies.

One of the two forces involved in the interaction between two bodies two bodies may be called 'action' force. The other force will be called 'reaction' force. The force of action and reaction constitute a mutual simultaneous interaction. It cannot be said that action is the cause of reaction or reaction is the effect of action.

Newton's third law of motion leads us to a very interesting fact about forces. It is that the forces always exist in pairs. They never exist singly. A "single isolated force is an impossibility.

Since the forces of action and reaction always act on different bodies therefore they cannot balance each other. [Two forces, equal in magnitude and opposite in direction, acting on the same body balance each other.)

Newton's third law of motion is valid when bodies are at rest or in motion or in contact or at a distance from each other.

2.31 ILLUSTRATIONS OF NEWTON'S THIRD LAW OF MOTION

(i) Consider a body of weight W resting on a horizontal surface (Fig. 2.39). The body exerts a **force (action) equal to weight W on the surface. The surface exerts a reaction R on the body in the upward direction such that

$$W = R$$

In vector notation, $\vec{W} = -\vec{R}$ or $\vec{R} = -\vec{W}$

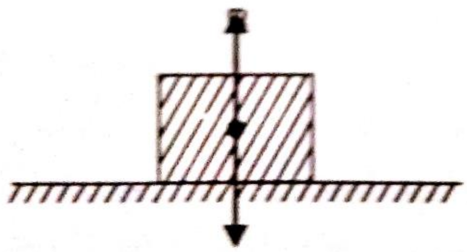


Fig. 2.39

(ii) When a man walks, he pushes the ground slantingly backwards with a certain force F. The ground offers a reaction R in the opposite direction such that

$$\vec{R} = -\vec{F} \qquad \text{(Newton's third law)}$$

The reaction R can be resolved into two rectangular components : horizontal component H and the vertical component V. The vertical component V supports the weight of the man while the horizontal component H helps the man to walk forward.

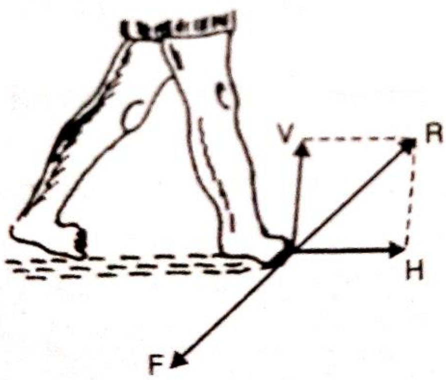


Fig. 2.40

It is difficult to walk on slippery ground or sand because we are unable to push the ground sufficiently hard. Consequently, reaction is not sufficient to help us walk. It is for the same reason that we cannot drive a nail into a wooden block without supporting the block.

(iii) In a lawn sprinkler (Fig. 2.41), when water issues out of the curved nozzles, a backward force is experienced by the sprinkler. Consequently, the sprinkler starts rotating and sprinkles water in all directions.

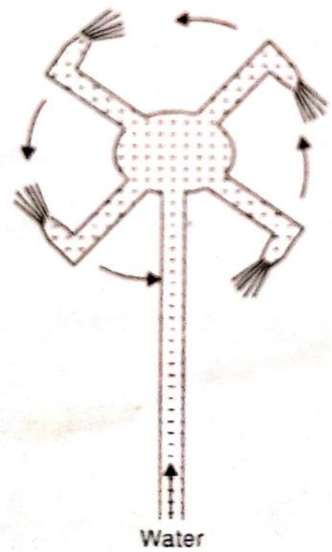


Fig. 2.41

(iv) Horse and Cart Problem

A horse pulls the cart. According to Newton's third law of motion, the cart pulls the horse with equal (in magnitude) but opposite force. Then, how does the cart move?

The Horse and Cart problem may also be expressed as under:

A horse is urged to pull a cart. The horse refuses to try, citing Newton's third law as his defence. "The pull of the horse on the cart is equal but opposite to the pull of the cart on horse. If I can never exert a greater force on the cart than it exerts on me, how can I ever set the cart moving ?" asks the horse. How would you reply?

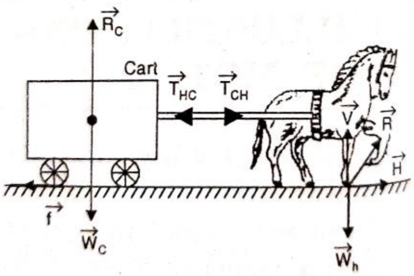


Fig. 2.42

Fig. 2.42 shows the horse-cart system.

Let, $\overrightarrow{T_{CH}}$ = forward pull of the horse on the cart

$\overrightarrow{T_{CH}}$ = reaction pull of the cart on the horse. Both these forces are the internal forces due to interaction (action and reaction) within the system. So, these cannot contribute to the motion of the system as a whole.

The weight $\overrightarrow{W_C}$ of the cart is balanced by the normal reaction $\overrightarrow{R_C}$ of the ground on the cart. Similarly, the weight $\overrightarrow{W_H}$ of the horse is balanced by the vertical component $\vec{V}$ of the reaction $\vec{R}$. Now following are the only two external forces.

(i) Horizontal component of the reaction.

(ii) Force of friction $\vec{f}$ between the ground and the wheels of the cart.

IF $|\vec{H}| > |\vec{f}|$, then the system moves forward with an acceleration $\vec{a}$ under the influence of the net external force H - f.

Applying Newton's second law of motion

$$H - f = (M_c + M_h)\, a \text{ or } a = \frac{H - f}{M_c + M_h}$$

where M_c and M_h represent the masses of cart and horse respectively. (Mass of the rope has been neglected.)

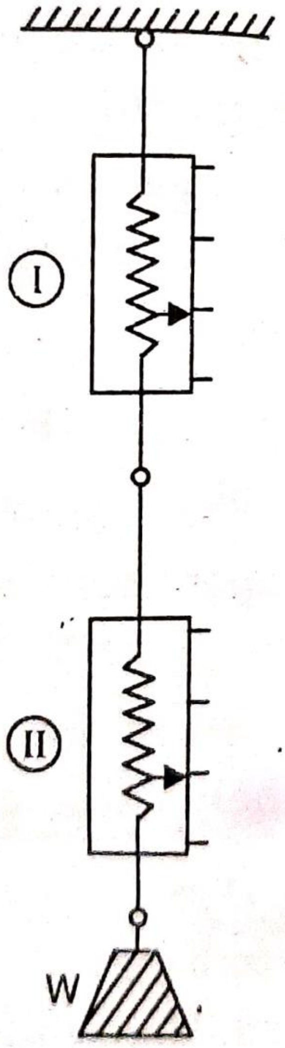


Fig. 2.43

(v) Consider two balances I and II connected to each other as shown in Fig. 2.43. A weight is suspended from balance II. The forces between the two balances are the forces of action and reaction. So, both the balances give the same reading.

2.32 APPARENT WEIGHT OF A PERSON IN AN ELEVATOR/LIFE

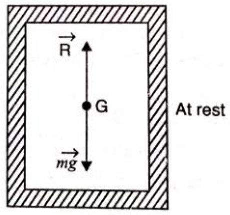


Fig. 2.44

Consider a person of mass *m* stand, on a weighting machine in an elevator. Let G represent the centre of gravity of the person. The true/actual weight of the person is *mg*. It acts vertically downwards through the centre of gravity of the person. It acts on the weighing machine which offers a reaction R.

R is the reading of the weighing machine. R is the weight experienced by the person. R is the appear weight of the person.

Let us discuss the relation between R and mg in the following different situations.

Case I. When the elevator is at rest (Fig. 2.44)

In this case, the acceleration of the person is zero. So, the net force on the person is also zero.

$\therefore \quad \boxed{R = mg}$

So, the apparent weight of the person is equal to the true weight of the person.

Again, $mg' = mg$ or $\boxed{g' = g}$

So, the effective value of acceleration due to gravity is equal to g.

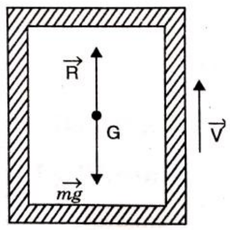


Fig. 2.45

Case II. When the elevator is moving vertically upward with a uniform velocity $\vec{V}$ (Fig. 2.45)

This case is similar to case I.

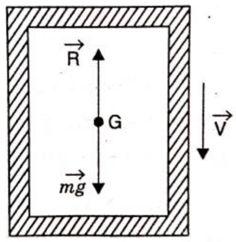


Fig. 2.46

Case III. When the elevator is moving vertically downward with uniform velocity $\vec{V}$ (Fig. 2.46)

This case is also similar to case I.

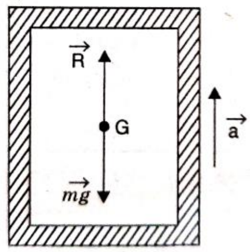


Fig. 2.47

Case IV. When the elevator is moving vertically upward with uniform acceleration $\vec{a}$ (Fig. 2.47)

In this case, the resultant force F acts in the upward direction.

$$F = R - mg$$

But $\quad F = ma$

$\therefore \quad ma = R - mg$

or $\quad R = mg + ma$

Since the reaction has increased therefore the passenger shall feel that his weight has increased. So, the apparent weight of the person becomes more than the actual weight of the person.

Again, $\quad mg' = mg + ma$

or $\quad g' = g + a$

So, the effective value of acceleration due to gravity is more than g.

Case V. When the elevator is moving vertically downward with uniform acceleration $\vec{a}$ (Fig. 2.48)

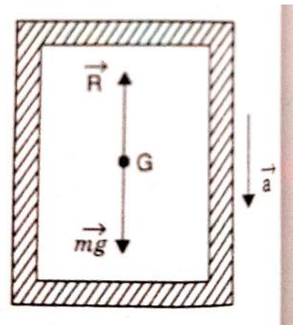


Fig. 2.48

In this case, the resultant force F acts in the downward direction and is given by

$$F = mg - R$$

But $\quad F = ma$

$\therefore \quad ma = mg - R \quad$ or $\quad \boxed{R = mg - ma}$

In this case, the reaction is decreased. So, the passenger shall feel that his weight has decreased. Thus, the apparent weight of the person becomes more than the actual weight of the person.

Again, $\quad mg' = mg - ma$

or $\quad mg' = m(g - a) \quad$ or $\quad \boxed{g' = g - a}$

So, the effective value of acceleration due to gravity is less than g.

Case VI. Freely falling elevator (Fig. 2.49)

Suppose the cable supporting the elevator breaks. Now, the elevator shall begin to fall freely. Its acceleration will be equal to g. In this case,

$R = m(g - g)$ $R = 0.$

The floor will not exert any reaction. The apparent weight of the person will be zero. The passenger will be in a state of weightlessness.

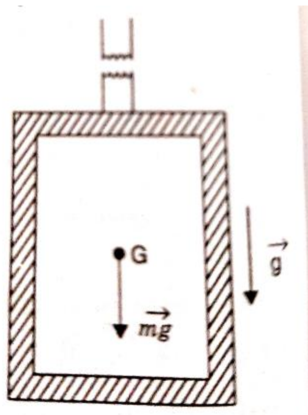


Fig. 2.49

Case VII. Elevator moving downward with an acceleration greater then g.

If the elevator is made to move down vertically with an acceleration a which is greater than g, then the person would experience negative reaction. This would be a case of negative apparent weight.

2.33 SECOND LAW IS THE REAL LAW OF MOTION

(i) First Law is contained in the Second Law

Newton's second law of motion gives us the following formula for the measurement of force.

Here, $\vec{F}$ is the force applied on a body of mass m and $\vec{a}$ is the resulting acceleration.

If $\vec{F} = 0$, i.e., if no external force action the body, then

$$\vec{a} = 0 \qquad [\therefore m \neq 0]$$

This implies that a body at rest shall remain at rest and a body moving with uniform motion in a straight line shall continue to do so.

This is precisely what the first law states. So, we conclude that first law is contained in the second law.

(ii) Third Law is contained in the Second Law

Consider an *isolated system of two bodies A and B. An isolated system is that on which no external force acts.

Suppose mutual interaction is present between A and B. Let $\overrightarrow{F_{BA}}$ be the force (action) exerted by A on B. Let $\frac{\overrightarrow{dp_B}}{dt}$ be the resulting rate of change of momentum of B. Let $\overrightarrow{F_{AB}}$ be the force (reaction) exerted by B on A. Let $\frac{\overrightarrow{dp_A}}{dt}$ be the resulting rate of change of momentum of A.

Now, $$\overrightarrow{F_{BA}} = \frac{\overrightarrow{dp_B}}{dt},\ \overrightarrow{F_{AB}} = \frac{\overrightarrow{dp_A}}{dt}$$

Adding, $$\overrightarrow{F_{BA}} + \overrightarrow{F_{AB}} = \frac{\overrightarrow{dp_B}}{dt} + \frac{\overrightarrow{dp_A}}{dt} = \frac{d}{dt}(\vec{p}_B + \vec{p}_A)$$

According to Newton's second law of motion, the rate of change of momentum is directly proportional to force. So, if no external force is applied on the system, then rate of change of momentum, i.e., $\frac{d}{dt}(\vec{p}_B + \vec{p}_A)$ should be zero.

$$\overrightarrow{F_{BA}} + \overrightarrow{F_{AB}} = 0 \qquad \text{or} \qquad \overrightarrow{F_{AB}} = -\overrightarrow{F_{BA}}$$

which is the mathematical statement of Newton's third law of motion. So, third law is contained in second law.

Since both the third and the first laws are contained in the second law therefore we can conclude that Newton's second law of motion is the real law of motion.

2.34 CONNECTED MOTION

Consider two masses m_1 and m_2 ($<m_1$) connected to the two ends of an inextensible string passing over a smooth frictionless pulley, Let a be the acceleration produced in the masses. While the heavier mass m_1 moves down with acceleration a, the lighter m_2 moves up with acceleration a. let T be the tension in the string.

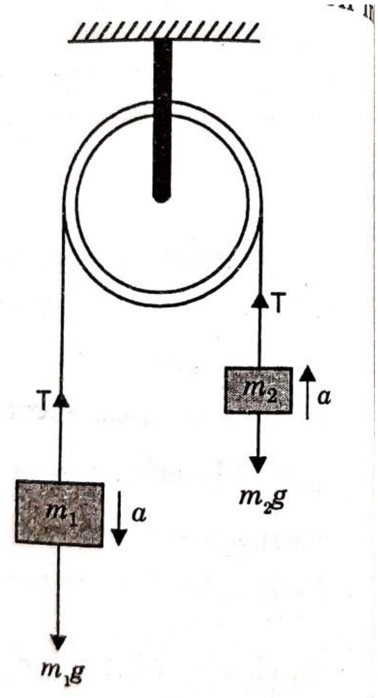


Fig. 2.50

Resultant downward force on mass m_1

$$= m_1g - \mathrm{T}$$

Using Newton's second law of motion

$$m_1a = m_1g - \mathrm{T} \quad \ldots(i)$$

Resultant upward force on mass m_2

$$= \mathrm{T} - m_2g$$

or $$m_2a = \mathrm{T} - m_2g \quad \ldots(ii)$$

adding (i) and (ii), we get

$$m_1a + m_2a = m_1g - m_2g$$

or $$a(m_1 + m_2) = (m_1 - m_2)g$$

or $$a = \frac{m_1 - m_2}{m_1 + m_2}\, g$$

Diving (i) by (ii), we get $\frac{m_1a}{m_2a} = \frac{m_1g - \mathrm{T}}{\mathrm{T} - m_2g}$

or $$\frac{m_1}{m_2} = \frac{m_1g - \mathrm{T}}{\mathrm{T} - m_2g} \quad \text{or} \quad m_1(\mathrm{T} - m_2g) = m_2(m_1g - \mathrm{T})$$

or $$m_1\mathrm{T} + m_2\mathrm{T} = 2m_1m_2g \quad \text{or} \quad \mathrm{T}(m_1 + m_2) = 2m_1m_2g$$

or $$\mathrm{T} = \frac{2m_1m_2}{m_1 + m_2}\, g$$

Simple Problem 2.5 Fig. 2.51 shows four penguins that are being playfully pulled along very slippery (frictionless) ice by a curator. The masses of three penguins and the tension in two of the cords are given. Calculate the penguin mass which is not given.

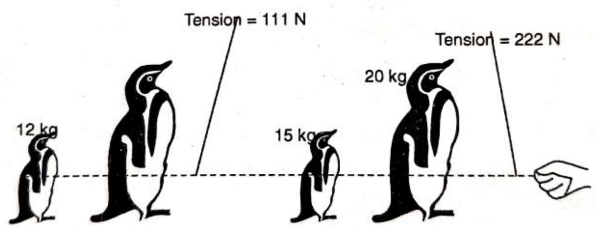


Fig. 2.51

Solution. $a = \frac{222}{12 + m + 15 + 20} = \frac{222}{m + 47}$

Again, $111 + (m + 12)a = (m + 12)\frac{222}{m+47}$

or $m + 47 = 2m + 24$ or $m =$ **23 kg.**

Sample Problem 2.6. Refer to the system shown in Fig. 2.52. Calculate acceleration a and tension F.

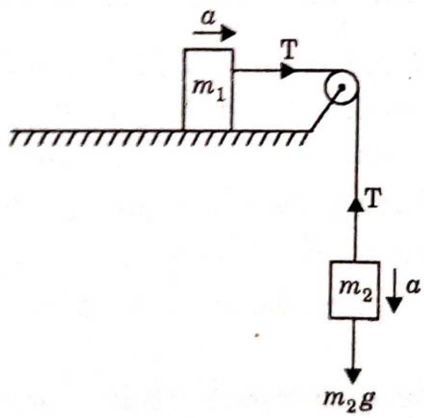


Fig. 2.52

Solution. For body of mass m_2

$$m_2g - T = m_2a \quad \ldots(i)$$

For body of mass m_1, $T = m_1a$ ...(ii)

Adding (i) and (ii), we get

$$m_2g = (m_1 + m_2)a \quad \text{or} \quad a = \frac{m_2}{m_1 + m_2}g$$

From equation (ii), $\mathbf{T} = \frac{m_1m_2}{m_1 + m_2}g$

EXERCISES

1. The engines of an airliner exert a force of 120 kN during take-off. The mass of the airliner is 40 tonnes (1 tonne - 1000 kg). Calculate

Fig. 2.53

(i) the acceleration produced by the engines.

(ii) the minimum length of runway needed if the speed required for take-off is 60 m s^{-1}.

[**Ans.** (i) 3 m s-2 (ii) 600 m]

2. A motor car of mass 20 quintal moving with a velocity of 60 km h^{-1}, by the application of brakes, is brought to rest in a distance of 3 km. Find the average force of resistance in newton.

[**Ans.** 92.6 N]

3. Two bodies of masses 4 kg and 3 kg respectively are connected by a light string passing over a smooth frictionless pulley. Calculate the acceleration of the masses and the tension in the string.

[**Ans.** 1.4 m s^{-2}, 33.6 N]

4. A person weighing 56 kg is standing in a lift. Find the weight as recorded by the weighing machine when

(a) the lift is stationary,

(b) the lift moves upwards with a uniform velocity of 2.1 m s^{-1},

(c) the lift moves downwards with a uniform velocity of 2.1 m s^{-1},

(d) the lift moves upwards with a uniform acceleration of 2.1 m s^{-2},

(e) the lift moves downwards with a uniform acceleration of 2.1 m s^{-1}?

[**Ans.** (a) 56 kg wt (b) 56 kg wt (c) 56 kg wt (d) 68 kg wt (e) 44 kg wt]

2.35 COMPOSITION OF FORCES

Scalars can be added algebraically. But forces do not obey the ordinary laws of Algebra. This is because they are vectors and possess both magnitude and direction. Forces are added geometrically.

The process of adding two or more than two forces is called addition or composition of forces.

When two or more than two forces are added, we get a single force called resultant force.

The resultant of two or more than two forces is a single force which produces the same effect as the individual forces together produces.

Following three lava have been evolved for the addition of forces.

(i) Triangle law of forces (for addition of two forces)

(ii) Parallelogram law of forces (for addition of two forces)

(iii) Polygon law of forces (for addition of more than two forces).

2.36 TRIANGLE LAW OF FORCES

Statement. If two forces can be represented both in magnitude and direction by the two sides of a triangle taken in the same order, then the resultant is represented completely, both in magnitude and direction by the third side of the triangle taken in the opposite order.

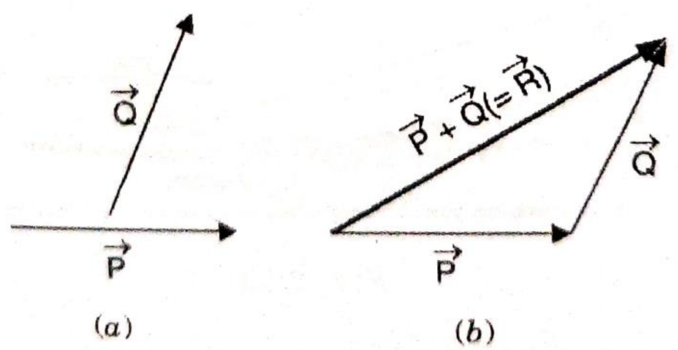


Fig. 2.54

Suppose we have to add two forces $\vec{P}$ and $\vec{Q}$ as shown in Fig. 2.54 (*a*). Now, displace $\vec{Q}$ parallel to itself in such a way that the tail of $\vec{Q}$ a touches the tip of $\vec{P}$. Complete the triangle to get a new force $(\vec{P} + \vec{Q})$ running straight from the tail of $\vec{P}$ to the tip of $\vec{Q}$. According to triangle law of forces, this new force is the resultant $\vec{R}$ of the given forces $\vec{P}$ and $\vec{Q}$ such that

$$\vec{R} = \vec{P} + \vec{Q}.$$

Triangle law of forces is applicable to triangle of any shape.

Corr. It follows from triangle law of forces that if three forces are represented by the three sides of a triangle taken in order, then their resultant is zero. Thus, if three forces $\vec{A}$, $\vec{B}$ and $\vec{C}$ can be represented completely by the three sides of a

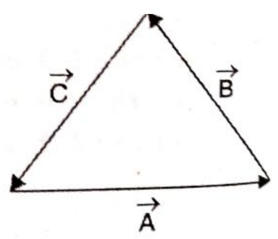


Fig. 2.55

triangle taken in order, then their force sum is zero.

$$\therefore \vec{A} + \vec{B} + \vec{C} = \vec{0}$$

[Note that the resultant (- $\vec{C}$) of $\vec{A}$ and $\vec{B}$ cancels the third force $\vec{C}$.]

If resultant of three forces is zero, then these can be represented completely by the three sides of a triangle taken in order.

2.37 PARALLELOGRAM LAW OF FORCES

Parallelogram law of forces is another useful law for the addition of two forces.

Consider two forces $\vec{P}$ and $\vec{Q}$ as shown in Fig. 2.56 (*a*). Displace $\vec{Q}$ parallel to itself till the tail of $\vec{Q}$ touches the tail of $\vec{P}$.

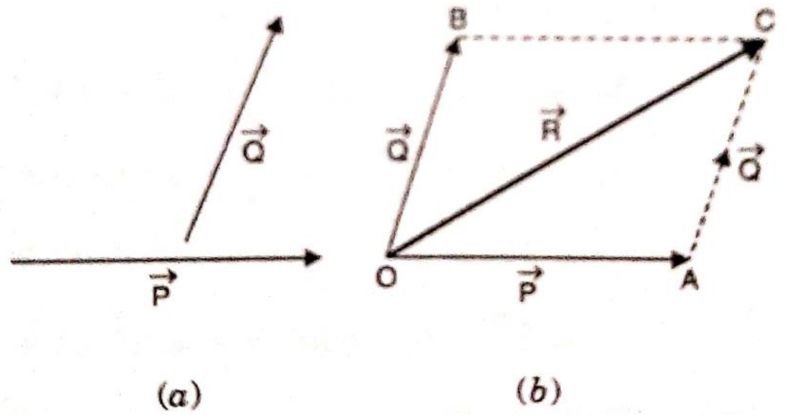


Fig. 2.56

Complete the parallelogram as shown in Fig. 2.56 (b). Applying triangle law of forces to the force triangle OAC, we get

$$\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC} \quad \text{or} \quad \vec{P} + \vec{Q} = \vec{R}$$

So, we conclude that if two forces are represented completely by the two adjacent sides, of a parallelogram, drawn from a point, then the diagonal of the parallelogram drawn through that point gives the resultant force. This is parallelogram law of forces. It is stated as follows:

"If two forces, acting simultaneously at a point, can be represented both in magnitude and direction by the two adjacent sides of a parallelogram drawn from a point, then the resultant is represented completely both in magnitude and direction by the diagonal of the parallelogram passing through that point."

In Fig. 2.57, two forces $\vec{P}$ and $\vec{Q}$ are completely represented by the two sides OA and OB respectively of a parallelogram. Then, according to parallelogram law of forces, the diagonal OC of the parallelogram will give the resultant $\vec{R}$ such that $\vec{R} = \vec{P} + \vec{Q}$.

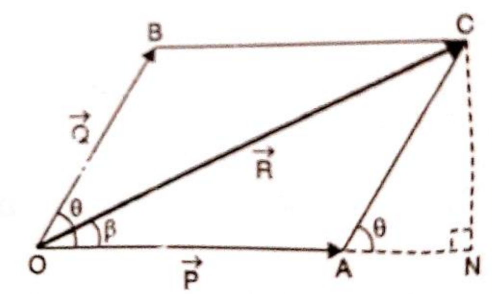


Fig. 2.57

Let us analytically calculate the magnitude and direction of the resultant force $\vec{R}$.

Let θ be the between two given forces $\vec{P}$ and $\vec{Q}$. From C, drop a perpendicular CN on OA (produced). In the right-angled ΔANC,

$$\sin\theta = \frac{CN}{AC} \text{ or } CN = AC \sin\theta$$

or $$CN = Q \sin\theta \qquad [\because AC = OB = Q] \quad \ldots(i)$$

Also, $$\cos\theta = \frac{AN}{AC} \text{ or } AN = AC\cos\theta = Q\cos\theta$$

Now, $$ON = OA + AN = P + Q\cos\theta \quad \ldots(ii)$$

Considering the right-angled ΔONC,

$$OC^2 = ON^2 + CN^2$$

or $$R^2 = (P + Q\cos\theta)^2 + (Q\cos\theta)^2$$

[From (2) and (1)]

or $$R^2 = P^2 + Q^2\cos^2\theta + 2PQ\cos\theta + Q^2\sin^2\theta$$
$$= P^2 + Q^2\cos^2\theta + Q^2\sin^2\theta + 2PQ\cos\theta$$
$$= P^2 + Q^2(\cos^2\theta + \sin^2\theta) + 2PQ\cos\theta$$

$$\therefore R^2 = P^2 + Q^2 + 2PQ\cos\theta \qquad [\because \sin^2\theta + \cos^2\theta = 1]$$

or $$\boxed{R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}}$$

which is the required expression for the magnitude of the resultant of two forces $\vec{P}$ and $\vec{Q}$ inclined to each other at an angle θ. The equation (3) is known as the **law of cosines.**

Let β be the angle which the resultant $\vec{R}$ makes with $\vec{P}$.

Then, $$\tan\beta = \frac{CN}{ON} \qquad \text{(in rt. } \angle\text{d } \Delta ONC)$$

or $\tan \beta = \frac{Q \sin \theta}{P + Q \cos \theta}$ ...(4) From 2 and 1

or $$\boxed{\beta = \tan^{-1}\left(\frac{Q \sin \theta}{P + Q \cos \theta}\right)}$$

which gives the direction of the resultant force.

Case I. When the given forces $\vec{P}$ and $\vec{Q}$ act in the same direction.

In this case, $\theta = 0°$

$$R = \sqrt{P^2 + Q^2 + 2PQ \cos 0°} \quad \text{(from equation (3))}$$

$$= \sqrt{P^2 + Q^2 + 2PQ} \quad [\because \cos 0° = 1]$$

$$= \sqrt{(P + Q)^2} = P + Q$$

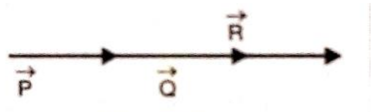


Fig. 2.58

or $$|\vec{R}| = |\vec{P}| + |\vec{Q}|$$

So, the magnitude of the resultant force is equal to the sum of the magnitude of the given forces.

Also, $\tan \beta = \frac{Q \sin 0°}{P + Q \cos 0°}$ [From equation (4)]

or $\tan \beta = 0°$ $[\because \sin 0° = 0]$

$\therefore$ $\beta = 0°$

So the resultant force points in the direction of the

Case II. When the given forces $\vec{P}$ and $\vec{Q}$ act at right angles to each other

In this case, $\theta = 90°$

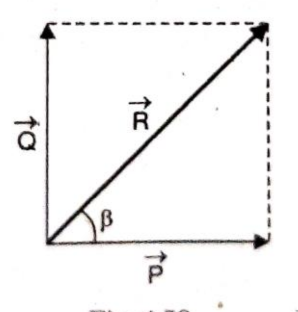


Fig. 2.59

$\therefore$ $$R = \sqrt{P^2 + Q^2 + 2PQ \cos 90°}$$

or $$R = \sqrt{P^2 + Q^2} \quad [\because \cos 90° = 0]$$

or $$|\vec{R}| = \sqrt{|\vec{P}|^2 + |\vec{Q}|^2}$$

Also, $\tan \beta = \frac{Q \sin 90°}{P + Q \cos 90°}$

or $\tan \beta = \frac{Q}{P}$ $[\because \sin 90° = 1]$

or $$\beta = \tan^{-1}\left(\frac{Q}{P}\right)$$

If P = Q, the $R = \sqrt{P^2 + P^2}$ or $R = \sqrt{2P^2} = \sqrt{2}\,P$

Also, in this case, $\tan \beta = \frac{P}{P} = 1$ or $\beta = 45°$

Case III. When the given forces $\vec{P}$ and $\vec{Q}$ act in opposite directions.

In this case, $\theta = 180°$

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos 180°}$$

$$= \sqrt{P^2 + Q^2 + 2PQ} \qquad [\because \cos 180° = -1]$$

$$= \sqrt{(P + Q)^2}$$

$\therefore$ $R = \pm (P - Q) = P - Q$ or $Q - P$

or $|\vec{R}| = |\vec{P}| \sim |\vec{Q}|$

[$|\vec{P}| \sim |\vec{Q}|$ implies positive difference between $|\vec{P}|$ and $|\vec{Q}|$.]

So, the magnitude of the resultant force of equal to the positive difference of the magnitude of the given forces.

Also, $\tan\beta = \frac{Q\sin 180°}{P + Q\cos 180°}$

or $\tan\beta = 0$

$\therefore$ $\beta = 0°$ or $180°$

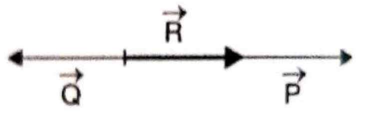


Fig. 2.60

When $|\vec{P}| > |\vec{Q}|$, then $\beta = 0°$. [Fig. 2.60]

Notes

1. While adding two forces with the help of parallelogram law of forces, we have to ensure that the two forces act either towards a point or away from a point.
2. The magnitude of the resultant of two forces is maximum when the forces act in the same direction. However, the magnitude of the resultant of two forces is maximum when the forces act in the opposite directions.
3. As θ increase from 0° to 180°, the magnitude of the resultant force decreases from $(P + Q)$ to $(P - Q)$.

When $|\vec{P}| < |\vec{Q}|$,

then $\beta = 180°$ [Fig. 2.61]

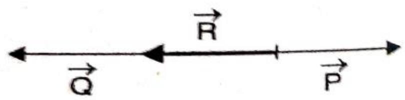


Fig. 2.61

Clearly, the resultant force acts in the direction of the bigger of the two forces.

Simple Problem 2.7. Calculate the angle between a two dyne and a three dyne force so that their sum is four dyne.

Solution. P = 2 dyne, Q = 3 dyne, R = 4 dyne, θ = ?

$$R^2 = P^2 + Q^2 + 2PQ\cos\theta$$

$$4^2 = 2^2 + 3^2 + 2 \times 2 \times 3 \times \cos\theta$$

or $$16 = 4 + 9 + 12 \cos\theta$$

or $\cos\theta = \frac{1}{4} = 0.2500$ or $\theta = \mathbf{75.52}°$

EXERCISES

1. Resultant of two forces which have equal magnitude and which act at right angles to each other is 1414 dyne. Calculate the magnitude of each force.

   [**Ans.** 1000 dyne]
2. Two forces of 5 kgf and 10 kgf are acting at an angle of 120°. Calculate the magnitude and direction of the resultant force.

   [**Ans.** 8.66 kgf, 90° with the direction of 5 kgf]
3. At what angle do the forces (A + B) and (A – B) act so that the magnitude of resultant is $\sqrt{3A^2 + B^2}$.

   [**Ans.** 60°]
4. Two forces whose magnitudes are in the ratio of 3 : 5 give a resultant of 35 N. If the angle of inclination be 60°, calculate the magnitude of each force.

   [**Ans.** 15 N, 25 N]
5. The greatest and least resultant of two forces acting at a point is 10 N and 6 N respectively. If each force is increased by 3 N, find the resultant of new forces when acting at a point at an angle of 90° with each other.

   [**Ans.** 12.1 N, 24°26']
6. The resultant of two forces P and Q is of magnitude P. Prove that if P is doubled, the resultant force will be perpendicular to Q.

2.38 POLYGON LAW OF FORCES

Polygon law of forces is used for the addition of more than two forces.

Consider four forces $\vec{P}$, $\vec{Q}$, $\vec{S}$ and $\vec{T}$ as shown in Fig. 2.62. Displace $\vec{Q}$ parallel to itself till the tail of $\vec{Q}$ touches the tip of $\vec{P}$. Similarly, displace $\vec{S}$ parallel to itself till the tail of $\vec{S}$ touches the tip of $\vec{Q}$. Again, displace $\vec{T}$ parallel to itself so that its tail touches the tip of $\vec{S}$. Now a force $\vec{R}$ running straight from the tail of $\vec{P}$ to the tip of $\vec{T}$ will be the resultant of $\vec{P}$, $\vec{Q}$, $\vec{S}$ and $\vec{T}$. This is polygon law of forces stated as follows :

"If a number of forces can be represented both in magnitude and direction by the sides of an open convex polygon taken in the same order, then the resultant is represented completely in magnitude and direction by the closing side of the polygon, taken in the opposite order."

Suppose four forces $\vec{P}$, $\vec{Q}$, $\vec{S}$ and $\vec{T}$ are represented completely by the four sides AB, BC, CD and DE respectively of a polygon, all taken in the same order as shown in Fig. 2.62. Then, according to polygon law of forces, the closing side AE of polygon taken in the opposite order will completely represent thei resultant $\vec{R}$ such that

$$\vec{R} = \vec{P} + \vec{Q} + \vec{S} + \vec{T}$$

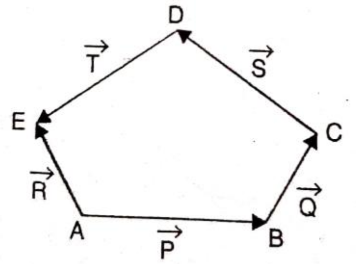


Fig. 2.62

2.39 RESOLUTION OF FORCES

The process of splitting a force is called resolution of a force. The parts obtained after resolution are known un components of the given force.

If the components of a given vector are perpendicular to each other, then they are called rectangular components. These are the most important components of a force.

Consider a force $\vec{F}$ represented by $\overrightarrow{OP}$. Through the point O, draw two mutually perpendicular axes-X-axis and Y-axis. Let the force $\vec{F}$ make an angle θ with the X-axis. From the point P, drop a perpendicular PN on X-axis.

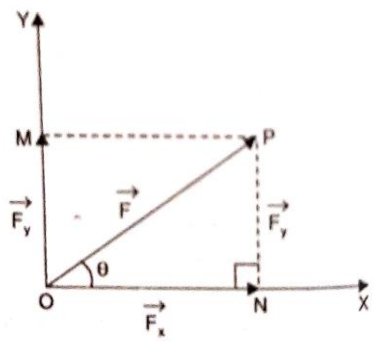


Fig. 2.63

Now $\overrightarrow{ON}$ (= $\vec{F}_x$) is the resolved part of $\vec{F}$ along X-axis. It is also known as the x-component of $\vec{F}$ or the horizontal component of $\vec{F}$. $\vec{F}x$ may be regarded as the projection of $\vec{F}$ on X-axis.

$\overrightarrow{OM}$ (=$\vec{F}_x$) is the resolved part of $\vec{F}$ along Y-axis. It is also known as the y-component of $\vec{F}$ or the vertical component of $\vec{F}$. The vertical component of $\vec{F}$ may be regarded as the projection of $\vec{F}$ on Y-axis.

So, $\vec{F}x$ and $\vec{F}y$ are the rectangular components of $\vec{F}$.

Applying triangle law of forces to the force triangle ONP, we get

$$\vec{F}_x + \vec{F}_y = \vec{F}$$

This equation confirms that $\vec{F}_x$ and $\vec{F}_y$ are the components of $\vec{F}$.

In right-angled triangle ONP,

$$\cos\theta = \frac{\vec{F}_x}{\vec{F}_y} \quad \text{or} \quad \vec{F}_x = F\cos\theta \qquad \ldots(1)$$

$$\sin\theta = \frac{\vec{F}_y}{F} \quad \text{or} \quad \vec{F}_y = F\sin\theta \qquad \ldots(2)$$

Squaring and adding (1) and (2), we get $F_x{}^2 + F_y{}^2 = F^2\cos^2\theta + F^2\sin^2\theta$.

or $$F_x{}^2 + F_y{}^2 = F^2(\cos^2\theta + \sin^2\theta)$$

$$F_x{}^2 + F_y{}^2 = F^2 \qquad [\therefore\ \cos^2\theta + \sin^2\theta = 1]$$

or $$\boxed{F = \sqrt{F_x{}^2 + F_y{}^2}}$$

This equation gives the magnitude of the given force in terms of the magnitude of the components of the given force.

EXAMPLE OF RESOLUTION OF FORCES

An example of resolution of a force' is 'walk of a man'. When a man walks, he presses the ground slantingly in the backward direction. The ground offers an equal reaction in the opposite direction. The vertical component of this reaction balances the weight of the man. The horizontal component helps the man to walk.

2.40 RESOLUTION OF A FORCE INTO THREE RECTANGULAR COMPONENTS

Let a force $\vec{F}$ be represented by $\overrightarrow{OP}$ as shown in Fig. 2.64. With O as origin, construct a rectangular parallelopiped with three edges along the three rectangular axes which meet at O. $\vec{F}$ becomes the diagonal of the parallelopiped. $\vec{F}_x$, $\vec{F}_y$ and $\vec{F}_z$ are three force intercepts along x, y and z axes respectively. These are the three rectangular components of $\vec{F}$.

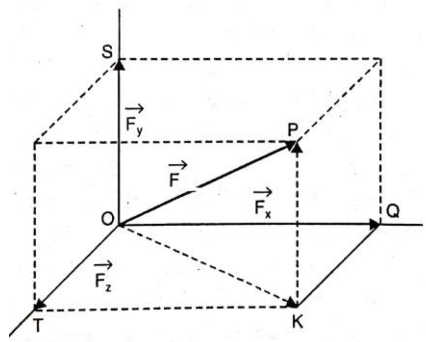


Fig. 2.64

Applying triangle law of forces, $\overrightarrow{OP} = \overrightarrow{OK} + \overrightarrow{KP}$

Applying parallelogram law of forces, $\overrightarrow{OK} = \overrightarrow{OT} + \overrightarrow{OQ}$

$\therefore$ $$\overrightarrow{OP} = \overrightarrow{OT} + \overrightarrow{OQ} + \overrightarrow{KP}$$

But $$\overrightarrow{KP} = \overrightarrow{OS}$$

$\therefore$ $$\overrightarrow{OP} = \overrightarrow{OT} + \overrightarrow{OQ} + \overrightarrow{OS}$$

$$\vec{F} = \vec{F}_z + \vec{F}_x + \vec{F}_y \quad \text{or} \quad \vec{F} = \vec{F}_x + \vec{F}_y + \vec{F}_z$$

or $$\vec{F} = \vec{F}_x i + \vec{F}_y \hat{j} + \vec{F}_z \hat{k}$$

Again $OP^2 = OK^2 + KP^2$

$OP^2 = OQ^2 + OK^2 + KP^2$ or $OP^2 = OQ^2 + OT^2 + KP^2$

$[\because OK = OT]$

or $F^2 = F_x{}^2 + F_y{}^2 + F_z{}^2$ $[\because KP = OS = F_y]$

or $F^2 = F_x{}^2 + F_y{}^2 + F_z{}^2$

or $$\boxed{F = \sqrt{F_x{}^2 + F_y{}^2 + F_z{}^2}}$$

This gives the magnitude $\vec{F}$ in terms of the magnitude of components $\vec{F}_x$, $\vec{F}_y$ and $\vec{F}_z$.

2.41 EQUILIBRIUM OF CONCURRENT FORCES

Concurrent forces are those forces whose lines of action intersect at a common point.

Fig. 2.65 shows three concurrent forces $\vec{F}_1$. $\vec{F}_2$ and $\vec{F}_3$ whose lines of action interest at O.

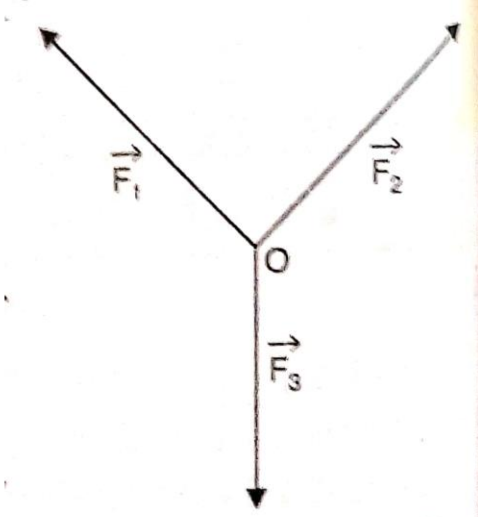


Fig. 2.65

When there is no change in the state of rest or of uniform motion of a body on which the forces act, the body is said to be in equilibrium.

A body in equilibrium may be at rest. But the body does not necessarily have to be at rest. It may be moving with uniform speed in a straight line. An example of this situation occurs when a rain drop or parachute falls with constant velocity called terminal velocity.

The condition necessary for equilibrium of a body under the action of concurrent forces is that the vector sum of all the forces acting shall be equal to zero. This condition is necessary even for the equilibrium of non-concurrent forces. This condition may also be expressed by the statement that the sum of the components in any three perpendicular directions shall be zero.

So, $\sum \vec{F} = 0$

And, $\sum \vec{F}_x = 0, \sum \vec{F}_y = 0, \text{ and } \sum \vec{F}_z = 0,$

As an illustration, consider three concurrent forces $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$ acting on a rigid body as shown in Fig. 2.66. If the rigid body is in equilibrium, then the resultant of $\vec{F}_1$ and $\vec{F}_2$ should be equal in magnitude to the magnitude of $\vec{F}_3$ and opposite in direction.

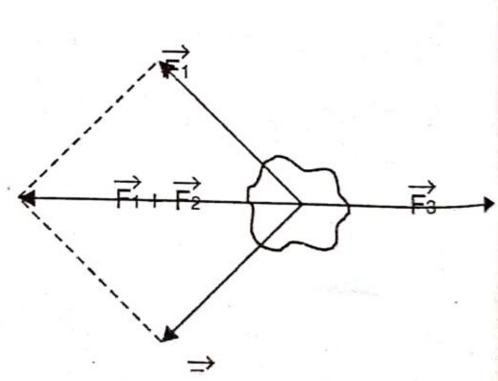


Fig. 2.66

So, $\vec{F}_3 = -(\vec{F}_1 + \vec{F}_2)$

The resultant of $\vec{F}_1$ and $\vec{F}_2$ can be found by applying parallelogram law of vectors.

In this illustration, $\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0$

If instead of throe forces, a number of forces $\vec{F}_1$, $\vec{F}_2$, $\vec{F}_3$, $\vec{F}_4$,..... act on the rigid body and the body is in equilibrium, then

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 + \ldots\ldots = 0$$

2.42 LAMI'S THEOREM

If three concurrent forces acting on a body keep it in equilibrium, then each force is proportional to the sine of the angle between the other two forces.

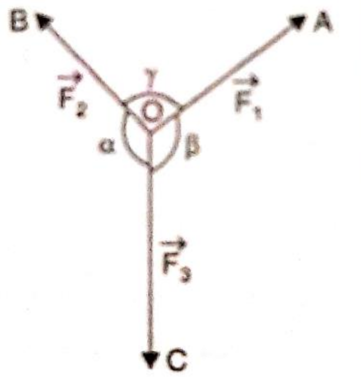


Fig. 2.67

Let $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$ be three concurrent forces in equilibrium.

Then, according to Lami's theorem,

$$\frac{\vec{F}_1}{\sin\alpha} = \frac{\vec{F}_2}{\sin\beta} = \frac{\vec{F}_3}{\sin\gamma}$$

Here α is the angle between $\vec{F}_1$ and $\vec{F}_2$. β is the angle between $\vec{F}_1$ and $\vec{F}_3$. γ is the angle between $\vec{F}_1$ and $\vec{F}_2$.

If three concurrent forces acting on a body are in equilibrium, they must be coplanar.

2.43 LAW OF SINES

In ΔONC, CN = R sin α

In ΔANC, CN = Q sin θ

$\therefore$ R sin α = Q sin θ

or $$\frac{R}{\sin\theta} = \frac{Q}{\sin\alpha} \quad \ldots(1)$$

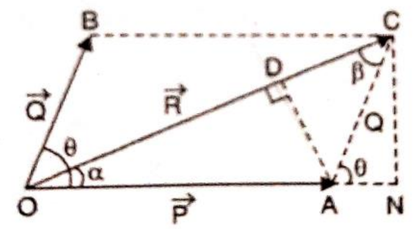


Fig. 2.68

again, in ΔODA, $\sin\alpha = \frac{AD}{P}$ or AD = P sin α

In ΔADC, $\sin\beta = \frac{AD}{Q}$ or AD = Q sin β

$\therefore$ R sin α = Q sin θ or $\frac{Q}{\sin\alpha} = \frac{P}{\sin\beta}$

Combining (1) and (2),

$$\frac{R}{\sin\theta} = \frac{P}{\sin\beta} = \frac{Q}{\sin\alpha}$$

This equation is known as the **law of sines.**

Note. $\sin\alpha = \frac{Q}{R}\sin\theta$

Sample Problem 2.8. A car whose weight is W is on a ramp which makes an angle a to the horizontal. How large a perpendicular force must the ramp with. stand if it is not to break under the car's weight ?

Solution. The weight W of the car acts vertically downwards. It can be resolved into two rectangular components : W cos θ acts normally to the ramp while the component W sin θ acts parallel to the ramp. The ramp must balance the normal component W cos θ if the car is not to crash through the ramp.

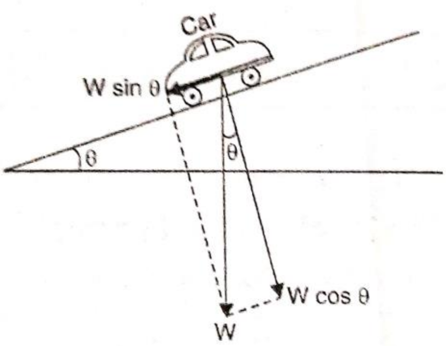


Fig. 2.69

EXERCISE

1. A force is inclined at 50° to the horizontal. If its rectangular component in the horizontal direction be 50 N, find the magnitude of the force and its vertical component.

[**Ans.** 77.8 N, 59.6 N]

PARABOLIC MOTION

2.44 WHAT IS A PROJECTILE

A body which is in flight through the atmosphere but is not being propelled by any fuel is called a projectile.

Examples. (i) A bomb released from an aeroplane in level flight.
(ii) A bullet fired from a gun. (iii) A javelin thrown by an athlete.
(iv) An arrow released from bow.

The path followed by a projectile is called trajectory.

The motion of a projectile is a two-dimensional motion. In projectile motion, we study the process of compounding the vectors $\vec{r}(0)$, $\vec{u}(0)$ and $\vec{a}$.

When we consider the motion of a projectile, the following assumptions are made :

(i) There is no resistance due to air. (ii) The effect due to curvature of Earth is negligible. (iii) The effect due to rotation of Earth is negligible. (iv) For all points of the trajectory, the acceleration due to gravity 'g' is constant in magnitude and direction.

2.45 TWO TYPES OF PROJECTILES

(i) Horizontal Projectile. If a body is projected horizontally from certain height with a certain velocity, then the body is called a horizontal projectile.

(ii) Oblique Projectile. If a body is projected at a certain angle with the horizontal, then the body is called an oblique projectile.

2.46 PRINCIPLE OF PHYSICAL INDEPENDENCE OF MOTIONS

The motion of a projectile is a two-dimensional motion. So, it can be discussed in two parts.

(i) horizontal motion (ii) vertical motion.

These two motions take place independent of each other. This is called the principle of physical independence of motions.

At any instant, the velocity of a projectile has two components (i) horizontal component (ii) vertical component.

The horizontal component remains unchanged throughout the flight. The vertical component is continuously affected by the force of gravity.

Thus, while the horizontal motion is a uniform motion, the vertical motion is a uniformly accelerated motion.

2.47 HORIZONTAL PROJECTION

(i) Nature of trajectory. Consider a projectile thrown horizontally from a point O, with horizontal velocity $\vec{v}$, at a certain height above the ground.

Through the point O, take two axes—X-axis and Y-axis. Let x and y be the horizontal and vertical distances respectively covered by the projectile in time t. At time t, the projectile is at P (Fig. 2.70).

The horizontal motion of the projectile is uniform motion. This is because the only force acting on the projectile is force of gravity. This force acts in the vertically downward direction and its horizontal component is zero.

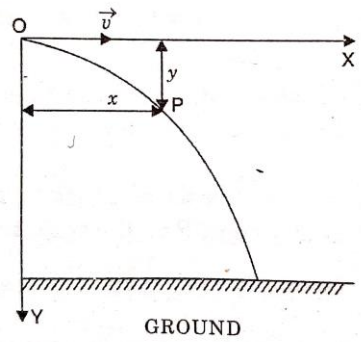


Fig. 2.70

Using S = $ut + \frac{1}{2}at^2$ we get

$$x = vt + 0 \quad \text{or} \quad x = vt$$

or $$t = \frac{x}{v}$$

The vertical motion of the projectile is controlled by force of gravity and is an accelerated motion. The initial velocity in the vertically downward direction is zero. Since Y-axis is taken downwards, therefore, the downward direction will be regarded as positive direction. So, the acceleration of the projectile is + g.

Using $$S = ut + \frac{1}{2}at^2 \text{ we get}$$

$$y = 0 + \frac{1}{2}gt^2 \quad \text{or} \quad y = \frac{1}{2}gt^2$$

Combining (1) and (2), we get

$$y = \frac{1}{2}g\left(\frac{x}{v}\right) \quad \text{or} \quad y = \frac{g}{2v^2}x^2 \quad \text{or} \quad y = kx^2 \qquad \ldots(3)$$

where $k = \left(\frac{g}{2v^2}\right)$ is a constant.

The equation (3) is a second degree equation in x and a first degree equation in y. This is the equation of a parabola.

Conclusion. A body thrown horizontally from a certain height above the ground follows a parabolic trajectory till it hits the ground.

(ii) Time of flight (T). It is the time of descent of the projectile from the point of projection to the ground.

Let h be the vertical height of the point of projection above the ground. Considering vertically downward motion,

$$S = ut + \frac{1}{2}at^2$$

Putting values, $h = 0 + \frac{1}{2}aT^2$ or $T = \sqrt{\frac{2h}{g}}$

(iii) Horizontal Range (R). It is the horizontal distance travelled by the projectile during the time of flight.

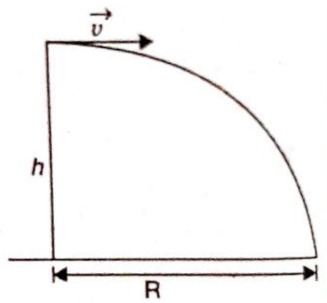


Fig. 2.71

Using $S = ut + \frac{1}{2}at^2$,

$$R = vt + 0 = vt = v\sqrt{\frac{2h}{g}}$$

(iv) Instantaneous Velocity. It is the resultant velocity at any instant of time.

Let $\vec{V}$ be the resultant velocity of the projectile at any point P on the trajectory. The projectile reaches P (Fig. 2.72) in time t, Let $\vec{V_x}$ and $\vec{V_y}$ be the horizontal and vertical components of $\vec{V}$.

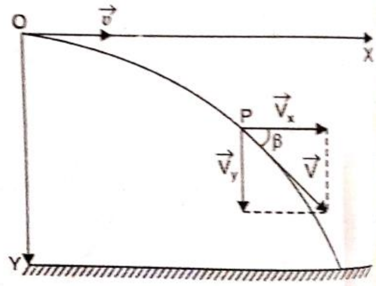


Fig. 2.72

Since the horizontal motion of the projectile is uniform motion,

$\therefore \qquad V_x = v$

The vertical motion is a uniformly accelerated motion.

We know that $v = u + at \qquad V_y = gt$

The magnitude of resultant velocity $\vec{V}$ is given by

$$V = \sqrt{V_x{}^2 + V_y{}^2} \quad \text{or} \quad V = \sqrt{v^2 + g^2t^2}$$

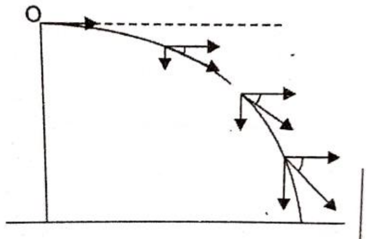


Fig. 2.73

If β is the angle which the resultant velocity $\vec{V}$ makes with the horizontal then

$$\tan\beta = \frac{V_y}{V_x} \quad \text{or} \quad \tan\beta = \frac{gt}{v}$$

or $$\beta = \tan^{-1}\left(\frac{gt}{v}\right)$$

This gives the direction of the resultant velocity.

Fig. 2.73 shows the velocity of the projectile at different points on the trajectory. While the horizontal velocity remains constant, the vertical velocity goes on increasing.

2.48 PROJECTION AT AN ANGLE [TRAJECTORY OF OBLIQUE PROJECTILE]

Consider a projectile thrown with velocity v at an angle θ with the horizontal (Fig. 2.74). The velocity v can be resolved into two rectangular components m(i) $v \cos\theta$ along X-axis and (ii) $v \sin\theta$ along Y-axis. The motion of the projectile is a two-dimensional motion. It can be supposed to be made up of two motions- horizontal motion (along X-axis) and vertical motion (along Y-axis). The horizontal motion of the projectile is uniform motion. This is because the only force acting on the projectile is the force of gravity. This force acts in the vertically downward direction and its horizontal component is zero. Thus, the equations of motion of the projectile for the horizontal direction are simply the equations of uniform motion in a straight line. The horizontal motion takes place with constant velocity $v \cos\theta$. If x be the horizontal distance covered in time t, then

$$x = (v \cos\theta)\ t \qquad \text{or} \qquad t = \frac{x}{v\cos\theta}$$

The vertical motion of the projectile is controlled by the force of gravity. The projectile increases its height up to a maximum where its vertical velocity $v_y(t)$ becomes zero. After this, the projectile reverses its vertical direction and returns to earth striking the ground with a speed v which is the same as the initial speed of the projectile.

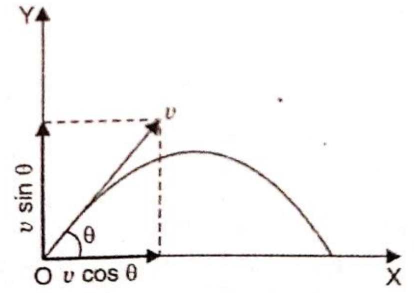


Fig. 2.74

Let y be the vertical distance covered by the projectile in time t. Let us now consider the vertical motion of the projectile.

$$u = v \sin\theta,\ a = -g,\ 't' = t,\ S = y$$

we know that $\qquad S = ut + \frac{1}{2}at^2$

Substituting values, $\qquad y = v \sin\theta\ t - \frac{1}{2}at^2$

Using equation (1), $y = v \sin\theta \left(\frac{x}{v\cos\theta}\right) - \frac{1}{2}at^2\left(\frac{x}{v\cos\theta}\right)^2$

or $y = v\sin\theta - \frac{gx^2}{2v^2\cos^2\theta}$

This is a first degree equation in y and a second degree equation in x. This is the equation of a parabola. So, the path followed by the projectile, i.e., **the trajectory of the projectile is parabolic.**

2.49 RESULTANT VELOCITY OF OBLIQUE PROJECTILE

Suppose the projectile is at P at time t. Let V be the resultant velocity of the projectile at time t, This velocity is along the tangent to the trajectory at point P.

Since the horizontal motion of the projectile is uniform motion therefore the horizontal component of velocity will remain unchanged.

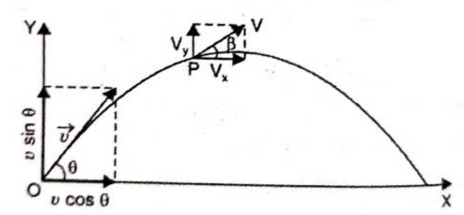


Fig. 2.75

$\therefore \quad V_x = v\cos\theta \qquad \ldots(1)$

Let us now calculate the vertical component V_y of the velocity at P. For this, we shall consider vertical motion of the projectile.

$u = v\sin\theta$, $a = -g$, 't' $= t$, $v = V_y = ?$

We know that $v = u + at$

$V_y = v\sin\theta - gt \qquad \ldots(2)$

Applying parallelogram law of vectors, we find that

$$V^2 = {V_x}^2 + {V_y}^2$$

$$V^2 = (v\cos\theta)^2 + (v\sin\theta - gt)^2$$

[From equations (1) and (2)]

or $V^2 = v^2\cos^2\theta + v^2\sin^2\theta + g^2t^2 - 2vg\,t\sin\theta$

or $V^2 = v^2(\cos^2\theta + \sin^2\theta) + g^2t^2 - 2vg\,t\sin\theta$

But $\cos^2\theta + \sin^2\theta = 1$

$$V^2 = v^2 + g^2 t^2 - 2v\, g\, t \sin\theta$$

or $$V = \sqrt{v^2 + g^2t^2 - 2vgt\sin\theta}$$

This equation gives the magnitude of the resultant velocity of the projectile at time t.

If β is the anglo which the resultant velocity $\vec{V}$ makes with the horizontal,

then $$\tan\beta = \frac{V_y}{V_x} \quad \text{or} \quad \tan\beta = \frac{v\sin\theta - gt}{v\cos\theta}$$

or $$\beta = \tan^{-1}\left(\tan\beta = \frac{gt}{v\cos\theta}\right)$$

This equation gives the direction of the resultant velocity.

2.50 MAXIMUM HEIGHT

It is the maximum uheight to which a projectile rises above the horizontal plane of projection. It is denoted by h_{max}. or H. It is also known as vertical range.

In order to calculate the maximum height H, we make use of the fact that the velocity of the projectile at the maximum height is zero. If t_1 be the time taken by the projectile to reach maximum height, then from equation (2),

$$0 = v\sin\theta - gt_1 \text{ or } gt_1 = v\sin\theta$$

or $$t_1 = \frac{v\sin\theta}{g}$$

So, when '= 't' = t_1, S = H.

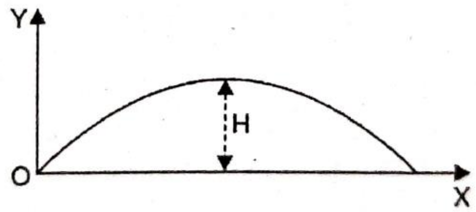


Fig. 2.76

∴ $$H = (\,gt)\, t_1 - \frac{1}{2}gt_1^{\,2}$$

or $$H = v\sin\theta \times \frac{v\sin\theta}{g} - \frac{1}{2}\, g\left(\frac{v\sin\theta}{g}\right)^2$$

or $$H = \frac{v^2\sin^2\theta}{g} - \frac{v^2\sin^2\theta}{2g} \quad \text{or} \quad \boxed{H = \frac{v^2\sin^2\theta}{2g}}$$

2.51 TIME OF FLIGHT

It is the total time taken by the projectile to return to the same level from where is was thrown.

Time of flight is equal to twice the time taken by the projectile to reach the maximum height. This is because the time of ascent is equal to the time of decent. This fact is also clear from the symmetry of the curve.

Time of flight, T = $2t$

where *t is the time taken by the projectile to reach maximum height.

Now, $v = 0,\ a = -g,\ u = v \sin\theta$

We know that $v = u + at$

Substituting values, $0 = v \sin\theta - gt$ or $gt = v \sin\theta$

or $t = \dfrac{v \sin\theta}{g}$

∴ $T = \dfrac{2v \sin\theta}{g}$

2.52 HORIZONTAL RANGE

It is the total horizontal distance from the point of projection to the point where the projectile comes back to the plane of projection. It is denoted by R.

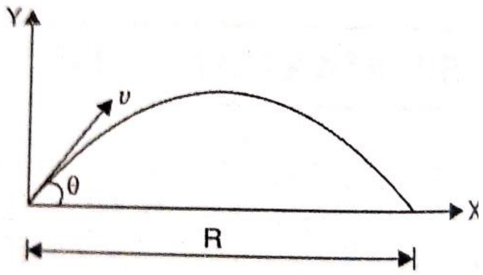


Fig. 2.77

In order to calculate horizontal range R, we shall consider horizontal motion of the projectile. The horizontal motion is uniform motion. It takes place with constant velocity $v \sin\theta$.

If θ is with vertical, even then $R = \dfrac{v^2 \sin 2\theta}{g}$

$\therefore \quad R = v \sin\theta \times \text{time of flight}$

$$= v \sin\theta \times \frac{2v\sin\theta}{g}$$

or $$R = \frac{v^2\,(2\sin\theta\cos\theta)}{g}$$

or $$\boxed{R = \frac{v^2 \sin 2\theta}{g}} \qquad (\because 2\sin\theta\cos\theta = \sin 2\theta)$$

2.53 MAXIMUM HORIZONTAL RANGE

For a given velocity of projection and at a given place, the value of R will be maximum when the value of sin 2θ is maximum i.e., 1.

For R to be maximum, sin 2θ = 1 (maximum value)

or $$\sin 2\theta = \sin 90° \quad \text{or} \quad \theta = 45°$$

So, for a given velocity, the angle of projection for maximum range is 45° i.e., $\frac{\pi}{4}$.

Maximum horizontal range, $R_{max} = \frac{v^2}{g}$

2.54 TWO ANGLES OF PRODUCTION FOR THE SAME RANGE

Again, $$R = \frac{v^2\sin 2\theta}{g} = \frac{v^2\sin(180°-2\theta)}{g} \qquad [\because \sin(180° - 2\theta = \sin 2\theta]$$

or $$R = \frac{v^2\sin 2\theta}{g} = \frac{v^2\sin 2(90°-\theta)}{g}$$

This shows that there are two angles of projection for the same horizontal range i.e., 0 and (90° - θ) with the horizontal. The projectile will cover the same horizontal range whether it is thrown at an angle θ or (90° - θ) with the horizontal. This means that the horizontal range is the same whether the projectile is thrown at an angle θ with the horizontal or at an angle θ with the vertical.

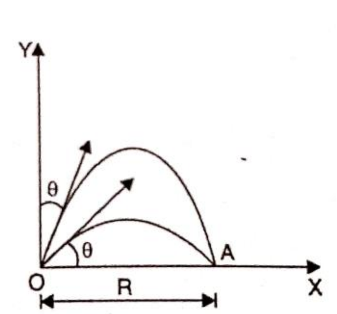


Fig. 2.78

Sample Problem 2.9. A projectile has a range of 50 m and reaches a maximum height of 10 m. What is the elevation of the projectile ?

Solution. We know that horizontal range,

$$R = \frac{v^2 \sin 2\theta}{g} = \frac{2v^2 \sin\theta \cos\theta}{g}$$

Maximum height $$H = \frac{v^2 \sin^2\theta}{2g}$$

Now, $$\frac{H}{R} = \frac{v^2 \sin^2\theta}{2g} \times \frac{g}{2v^2 \sin\theta \cos\theta}$$

or $$\frac{H}{R} = \frac{1}{4} \tan\theta \text{ or } \tan\theta = \frac{4H}{R}$$

or $$\tan\theta = \frac{4 \times 5}{50} = \frac{4}{5} = 0.8 \text{ or } \mathbf{38.66}°$$

Sample Problem 2.10. Prove that the velocity at the end of flight of an oblique projectile is the same in magnitude as at the beginning but the angle that it makes with the horizontal is negative of the angle of projection.

Solution. Let $\vec{V}$ be the velocity of the projectile at the end of flight. Let V_x and V_y be the horizontal and vertical components respectively. Since horizontal motion is uniform motion,

$$\therefore \quad V_x = v\cos\theta$$

Again, $$V_y = v\cos\theta - gT$$

where T is the time of flight

$$V_y = v\cos\theta - g \times \frac{2v\sin\theta}{g}$$

or $$V_y = -\,v\cos\theta$$

now $$V = \sqrt{{V_x}^2 + {V_y}^2}$$

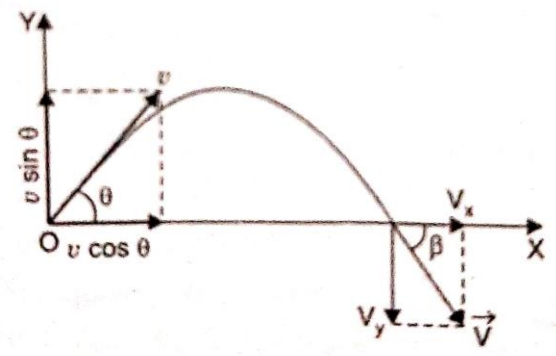


Fig. 2.79

or $$V = \sqrt{(v\cos\theta)^2 + (-\,v\cos\theta)^2} = \sqrt{v^2(\cos^2\theta + \sin^2\theta)} = v$$

So, the magnitude of the velocity at the end of the flight is equal to the magnitude of the velocity of projection.

If β is the angle which velocity $\vec{V}$ makes with the horizontal. Then

$$\tan\beta = \frac{V_y}{V_x} = \frac{-\,v\sin\theta}{v\cos\theta}$$

or $\tan\beta = -\tan\theta$ or $\tan\beta = \tan(-\theta)$

or $\beta = -\theta$

So, the velocity vector at the end of the flight makes an angle '- θ' with the horizontal. This angle is negative of the angle of projection.

EXERCISES

1. Two paper screens A and B are separated by a distance of 100 m. A bullet pierces A and then B. The hole in B is 10 cm below the hole in A. If the bullet is travelling horizontally at the time of hitting the screen A, calculate the velocity of the bullet when it hits the screen A. Neglect the resistance of paper and air

   [**Ans.** 700 m s^{-1}]
2. A projectile thrown from a horizontal plane reaches back the plane after covering a horizontal distance of 40 m. If the horizontal velocity of the projectile is 10 m s^{-1} then what is its initial vertical velocity ? Given : g = 10 m s^{-2}

   [**Ans.** 20 m s^{-1}]
3. Prove that the maximum horizontal range is four times the maximum height attained by the projectile when fired at an inclination so as to have the maximum horizontal range.
4. Prove that a gun will shoot three times as high when its angle of elevation is 60° as when it is 30°, but will carry the same horizontal distance.
5. A bullet is fired with a velocity of 10 m s^{-1} in a direction making an angle of 30° with the vertical. Calculate its time of flight and the maximum height reached by it.

   [**Ans.** 1.8 s, 3.8 m]
6. The maximum vertical height of a projectile is 10 m. If the magnitude of the initial velocity is 28 m s^{-1}, what is the direction of the initial velocity ? Given : g = 9.8 m s^{-2}.

   [**Ans.** 30°]

CONCEPT OF FRICTION AND ITS APPLICATIONS

2.55 FRICTION

Friction is the retarding force which is called into play when a body actually moves or tends to move over the surface of another body.

Consider a block of mass m which is projected with initial velocity v along a long horizontal table. The block will finally come to rest. This means that while it is moving, it experiences an opposing force that points in a direction opposite to its motion. This opposing force is called force of friction.

Whenever the surface of one body slides over that of another, each body exerts a frictional force on the other. The frictional force on each body is in a direction opposite to its motion relative to the other body. Frictional forces always oppose relative motion and never help it. Even when no relative motion is actually present but there is onlyl a tendency for relative motion, frictional forces exist between surfaces.

2.56 CAUSE OF FRICTION

(a) Old view. The surfaces of bodies are never perfectly smooth but have irregularities. Even if the surface of a body appears smooth, it is not actually so. When such a surface is minutely observed under a powerful microscope, it shows irregularities.

When a body is placed on another body, there is an interlocking of the irregular projections of the two surfaces AB and CD as shown in Fig. 2.80. When some force is applied to make one body move over the surface of another body the motion is resister due to the interlocking of the large number of irregularities. The opposing force is called friction. The direction of the force of friction is always opposite to the direction of motion.

(b) Modern theory of friction or surface adhesion theory of friction. On the atomic scale, even the most finely polished surface is far from plane. When two bodies are placed in contact, the actual microscopic area of contact is much less than the apparent macroscopic area of contact. These areas can be easily in the ratio of 1 : 10^4. Since the actual area of contact is very small therefore the pressures at the points of contact will be very large. So, the contact points deform plastically and many contact points actually become "cold-welded" together. This phenomenon called surface adhesion occurs because at the contact points, the molecules on opposite sides of the surface are so close together that they exert strong intermolecular forces (forces of adhesion) on each other, When one body is pulled across another, the frictional opposition is associated with the rupturing of these thousands of tiny welds. These tiny welds continually reform as new chance contacts are made.

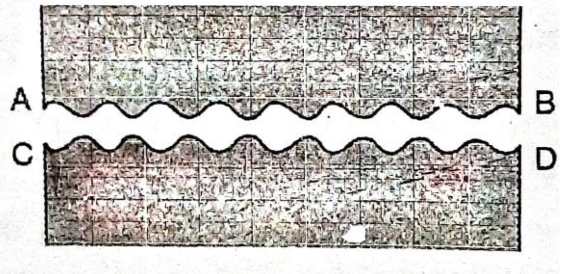


Fig. 2.80

2.57 STATIC FRICTION

It is the force of friction which exactly balances the applied force during the stationary state of the body. This frictional force exists when the bodies in contact are at rest with respect to each other. The force of static friction is a self-adjusting force i.e., it adjusts its magnitude and direction so as to become exactly equal and opposite to the applied pull. The direction of the force of friction remains always opposite to the direction of the applied force.

Consider a block resting on a horizontal surface [Fig. 2.81]. Let a small pull P be applied on the as shown. Let f_s be the resulting force of static friction. In the equilibrium position, the weight W of the body will be balanced by the normal reaction R. And the applied pull P will be balanced by the frictional force f_s.

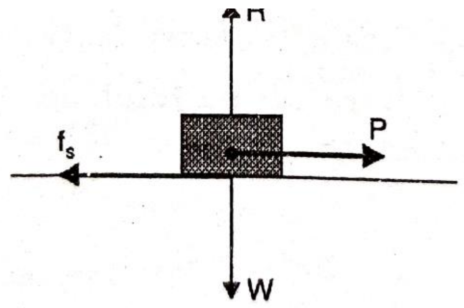


Fig. 2.81

In vector notation, $\vec{W} = -\vec{R}$

and $\vec{P} = -\vec{f_s}$

2.58 LIMITING FRICTION

Limiting friction in the maximum value of static friction which is called into play when a body in just going to start sliding over the surface of another body.

When the applied pull P is increased, the static frictional force f_s also increase, However, there is a particular limit upto which the static frictional force f_s can increase. Boyond this limit, the applied pull P will be able to produce motion in the body.

2.59 LAWS OF LIMITING FRICTION

These laws are based upon experimental observations only.

1. The direction of the force of limiting friction is always opposite to that in which the motion tends to take place.
2. The limiting friction acts tangentially to the two surfaces in contact.
3. The magnitude of the limiting friction is directly proportional to the normal reaction between the two surfaces.
4. The limiting friction depends upon the material and the nature of the surfaces in contact and their state of polish.
5. For any two given surfaces, the magnitude of the limiting friction is independent of the shape or the area of the surfaces in contact so long as the normal reaction remains the same.

2.60 DYNAMIC OR KINETIC FRICTION

Dynamic or kinetic friction comes into play if the two bodies in contact are in relative motion. It acts in a direction opposite to the direction of the instantaneous velocity.

The dynamic or kinetic friction is of the following two types :

(1) Sliding friction. It comes into play when a solid body slides over surface of another body.

(2) Rolling Friction. It comes into play when a body rolls over the surface of another body.

2.61 LAWS OF SLIDING FRICTION

(i) The sliding friction opposes the applied force and has a constant value, depending upon the nature of the two surfaces in relative motion.

(ii) The force of sliding friction is directly proportional to the normal reaction R.

(iii) The sliding frictional force is independent of the area of the contact between the two surfaces so long as the normal reaction remains the same.

(iv) The sliding friction does not depend upon the velocity, provided the velocity is neither too large nor too small.

2.62 variation of frictional force with the applied force

It is illustrated graphically in Fig. 2.82. When there is no relative motion between the two bodies in contact, the frictional force increases at the same rate as the applied force.

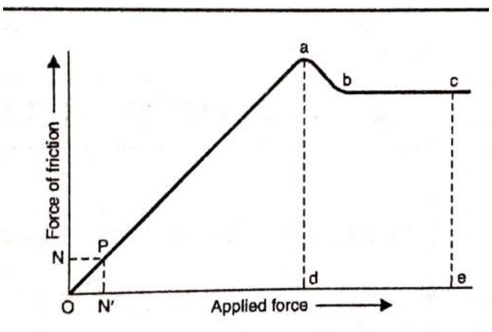


Fig. 2.82

If ON' is the applied force, then ON is the frictional force such that

$$ON' = ON$$

The slope of the curve Oa is constant and is equal to unity.

When the applied force is equal to Od, the static frictional force becomes maximum. So, *ad* represents the limiting friction. When the applied pull exceeds the value *Od*, the body begins to move. At this stage, the frictional force suddenly decreases by a small amount and acquires a constant value *ce*. This value represents the dynamic or kinetic or sliding frictional force.

2.63 COEFFICIENT OF STATIC FRICTION

For any two surfaces in contact, it is the ratio of the limiting friction f_{ms} and the normal reaction R between them. It is denoted by μ_s.

$$\mu_s = \frac{f_{ms}}{R}$$

Since μ_s is a pure ratio therefore it has no units. The value of μ_s depends upon the state of polish of the two surfaces in contact. If the surfaces are smooth, the value of μ_s is small.

The force of static friction f_s is equal to the applied force. So, f_s can have any value from 0 to f_{ms}.

$$f_s \leq f_{ms}$$

[The equality sign holds only when f_s has its maximum value.]

$$f_s \leq \mu_s R$$

2.64 COEFFICIENT OF KINETIC FRICTION

It is defined as the ratio of friction and normal reaction. It is denoted by μ_k.

$\therefore$ $$\mu_s = \frac{f_s}{R}$$

Now, $$\frac{\mu_s}{\mu_k} = \frac{f_{ms}}{R} \times \frac{R}{f_k} = \frac{f_{ms}}{f_k}$$

But $$f_{ms} > f_k$$

$\therefore$ $$\mu_s > \mu_k$$

It is the angle which the resultant of the force of limiting friction $\overrightarrow{f_{ms}}$ and the normal reaction $\vec{R}$ makes with the normal reaction $\vec{R}$.

Consider a block of weight $\vec{W}$ resting on a horizontal surface. The weight $\vec{W}$ will be balanced by the normal reaction $\vec{R}$ (Fig. 2.83).

In vector notation, $\vec{W} = -\vec{R}$ (Newton's 3rd law of motion)

Now, apply a horizontal force $\vec{P}$ of such a magnitude that the block is about to move. Then, CB will represent the maximum force of static friction i.e., limiting friction. The resultant of the limiting friction and the normal reaction is represented by the diagonal CL of the parallelogram CBLA. The angle θ which the result and makes with the normal reaction is called the angle of friction.

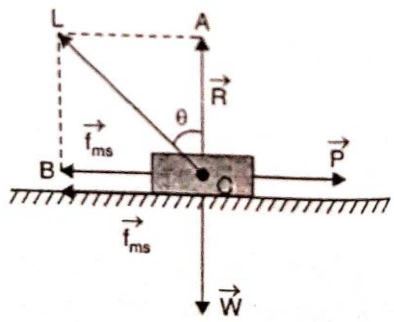


Fig. 2.83

In ΔCAL, $\tan\theta = \frac{AL}{CA} = \frac{CB}{CA} = \frac{f_{ms}}{R}$

But $\frac{f_{ms}}{R} = \mu_s$ (definition of coefficient of friction)

∴ $\tan\theta = \mu_s$

So, the tangent of the angle of friction is equal to the coefficient of static friction.

2.66 ANGLE OF SLIDING OR ANGLE OF REPOSE

It is the angle that an inclined plane makes with the horizontal when a body placed on it is about to start sliding down.

Consider an inclined plane OB whose angle of inclination with the horizontal surface OA can be changed (Fig. 2.84). Suppose a block of weight W is placed on the inclined plane. The weight W will act vertically downwards through the centre of gravity of the block.

Increase the angle of inclination very slowly till the block just begins to slide down the plane. This particular value of the angle of inclination is called the angle of sliding or angle of repose. It is denoted by ϕ. The value of ϕ depends upon the material and the nature of the two surfaces in contact.

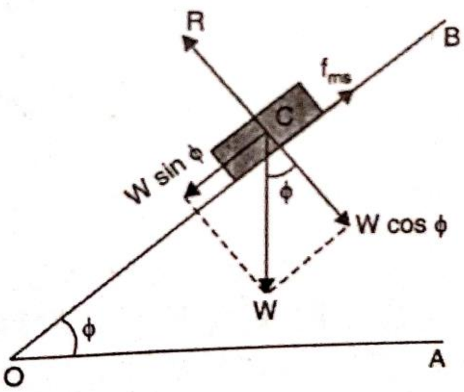


Fig. 2.84

The weight W can be resolved into two rectangular components: W cos ϕ and W sin ϕ. The component W cos ϕ balances the normal reaction R while the component W sin ϕ is equal to the limiting friction f_{ms}.

$\therefore \quad \frac{W \sin \phi}{W \sin \phi} = \frac{f_{ms}}{R}$ or $\tan \phi = \frac{f_{ms}}{R}$. But $\frac{f_{ms}}{R} = \mu_s$

$$\boxed{\tan \phi = \mu_s}$$

The tangent of the angle of repose is equal to the coefficient of static friction. This fact is used for finding the coefficient of static friction in the laboratory.

But $\quad \mu_s = \tan \theta \quad$ (θ is angle of friction)

$\therefore \quad \tan \phi = \tan \theta \quad$ or $\quad \boxed{\phi = \theta}$

The angle of repose is equal to the angle of friction.

2.67 BODY ACCELERATING DOWN AN INCLINED PLANE

Consider a block of mass m and weight W resting on an inclined plane OB [Fig. 2.85). Increase the angle of inclination to a such that a is greater than angle of sliding. Let 'a' be the acceleration with which the block begins to slide down the inclined plane.

W cos α and W cos α are the two rectangular components of W. The components W cos α balances the normal reaction R.

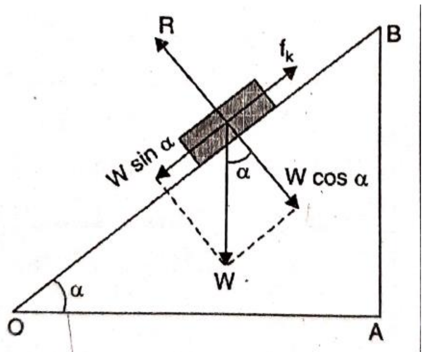


Fig. 2.85

Resulting force (parallel to the inclined plane and downwards)

$$= W \sin a - f_k$$

Where f_k is the force of kinetic friction.

Applying Newton's second law of motion,

$$ma = W \sin a - f_k$$

But $\quad W = mg$

and $\quad f_k = \mu_k R = \mu_k W \cos \alpha = \mu_k mg \cos \alpha$

$\therefore \quad ma = mg \sin \alpha - \mu_k mg \cos \alpha$

or $\quad a = g (\sin \alpha - \mu_k \cos \alpha)$

which gives the acceleration of the body sliding down a rough inclined plane.

2.68 ROLLING FRICTION

When a body rolls or tends to roll over the surface of another body, then both the rolling body and the surface on which it rolls are compressed by a small amount. As a result, the rolling body has to continuously climb a hill as shown [Fig. 2.86]. Apart from this, the rolling body has to continuously detach itself from the surface on which it rolls. This is opposed by the adhesive force between the two surfaces in contact. On account of both these factors, a force originates which retards the rolling motion. This retarding force is called the rolling friction. It is denoted by f_r. It is given by

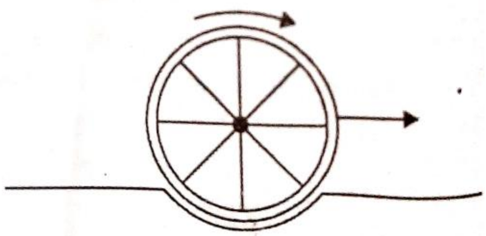


Fig. 2.86

Note1. equation (i) is applicable only of the motion is a purely rolling motion. If there is slipping. Then equation (i) cannot be applied.

Note2. μ_r is different from μ applying the principle of homogeneity of dimensions, we find that μ_r has the units of length.

$$f_r = \mu_r . \times \frac{R}{r}$$

where, μ_r is the coefficient of rolling friction, R is the normal reaction and the radius of the rolling body.

Comparison. For the same magnitude of normal reaction, the sliding friction is much greater than the rolling friction. That is why we prefer to convert sliding friction into rolling friction. The ball and roller bearings make use of this principle.

Illustration. The sliding friction of steel on steel is 100 times more than the rolling friction of steel on steel.

2.69 APPLICATIONS

Friction is a Necessity

(i) Without friction between our feet and the ground, it will not be possible to walk. When the ground becomes slippery after rain, it is made rough by spreading sand, etc.

(ii) The tyres of the vehicles are made rough to increase friction.

(iii) Various parts of a machine are able to rotate due to friction between belt and pulley.

(iv) It would be impossible to climb, to fix a nail, to drive if there were no friction.

Friction is an Evil

(i) Wear and tear of the machinery is due to friction.

(ii) Friction between different parts of the rotating machines produces heat and causes damage to them.

(iii) We have to apply extra power to machines in order to overcome friction. Thus, the efficiency of the machines decreases.

2.70 METHODS OF REDUCING FRICTION

(i) Polishing. The interlocking and the projections between the two surfaces are minimised and therefore the friction is reduced. This makes their life long.

(ii) Lubrication. A lubricant is a substance (a solid or a liquid) which forms thin layer between the two surfaces in contact. It fills the depressions present in the surfaces of contact and hence friction is reduced.

(iii) Streamlining. When a body moves past a fluid (liquid or air), the particles of the fluid move past jt in regular lines of flow called streamlines. It is found that the resistance offered by the fluid to the body is minimum when its shape resembles that of streamlines. Thus the shape of automobiles is so designed that it resembles the streamline pattern and the resistance offered by the fluid is minimum.

(iv) Avoiding moisture. When the moisture is present, the friction is more. So, we must avoid moisture between the two surfaces.

(v) Use of alloys. Friction is reduced by lining the moving parts with alloys because alloys have low coefficients of friction.

(vi) Use of ball bearings or roller-bearings. The rolling friction is much less than the sliding friction. So, we convert sliding friction into rolling friction. Even the axle is not allowed to move directly in the hub. The friction is further minimized by the use of roller bearings or ball bearings [Fig. 2.87].

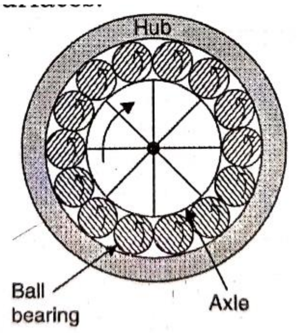


Fig. 2.87

Sample Problem 2.11. A horizontal force of 1200 gf is applied to a 1500 g block, which rests on a horizontal surface. If the coefficient of friction is 0.3, find the acceleration produced in the block.

Solution. Mass, $m = 1500\ g = 1.5$ kg

Force, F = 1200 gf = 1200 × 981 dyne

$$= \frac{1200 \times 981}{10^5} \text{N} = 11.77 \text{ N}$$

Frictional force = μ_k R = μ_k mg = 0.3 × 1.5 × 9.8 N = 4.41 N

Net force i.e., Acceleration force = (11.77 – 4.41) N

Acceleration, $a = \frac{\text{force}}{\text{mass}} = \frac{7.36 \text{ N}}{1.5 \text{ kg}}$ = **4.9 m s^{-2}**

Sample Problem 2.12. A block slides down a rough inclined plane of inclination θ with constant velocity. If this block is projected up the plane with a velocity v_v then at what distance along the inclined plane, the block will come to rest ?

Solution. Since the block moves down the plane therefore the force of friction equals the component of weight parallel to the inclined plane.

$$f = mg \sin \theta$$

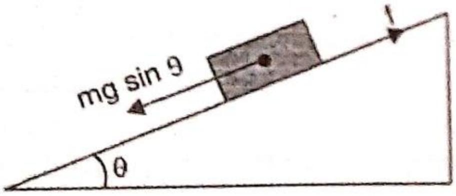


Fig. 2.88

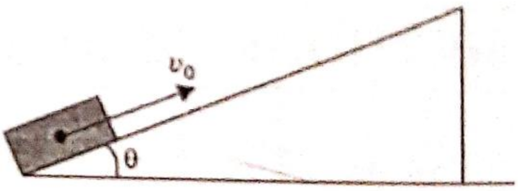


Fig. 2.89

When the block is moved up the inclined plane, then retarding force

$$= mg \sin \theta + f = mg \sin \theta + mg \sin \theta = 2mg \sin \theta$$

$$\therefore \quad a = \frac{-2\, mg \sin \theta}{m} = -2\, g \sin \theta$$

Now, 'u' = v_0, 'v' = 0, 'a' = - $2g$ sin θ, S = ?

We know that $v^2 - u^2 = 2as$

Substituting values, $0^2 - {v_0}^2 = 2(-2g \sin \theta)$ S

or $$S = \frac{{v_0}^2}{4g \sin \theta}$$

EXERCISES

1. A cube placed on a rough horizontal surface is imparted a velocity v_0 The cube stops after travelling a distance of 10 m, Calculate v_0. Given : coefficient of friction = 0.5 and g = 10 m s^{-2},

   [**Ans.** 10 m s^{-1}]
2. A piece of ice slides down a 45° inclined plane in twice the time it takes to slide down a frictionless 45° incline. What is the coefficient of friction between the ice and the incline ?

   [**Ans.** 0.75]
3. Consider an automobile moving along a straight horizontal road with a speed v_0. If the coefficient of static friction between the tyres and the ground is μ_s, what is the shortest distance in which the automobile can be stopped ?

   [**Ans.** ${v_0}^2/2\, g\, \mu_s$]

4. A car with a mass of 1200 kg is travelling at the rate of 700 km h^{-1}. Suddenly the brakes are applied causing all the tyres to skid. How far will the car travel before coming to rest? Given : $\mu = 0.2$.

[**Ans.** 96.45 m]

2.71 IMPORTANCE TERMS AND CONCEPTS OF CIRCULAR MOTION

(i) Measuring an angle

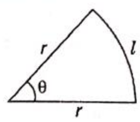


Fig. 2.90

The angle o is defined by the equation

$$\theta = \frac{\text{arc length}}{\text{radius of arc}} = \frac{l}{r}$$

Here I is the curved distance along the arc of radius r.

The SI unit of angle is 'radian'. One *radian is the angle when the arc length is the same as the radius of the circle.

(ii) Angular displacement

The angular displacement of a particle in a given time interval is the angle which the position vector of the particle sweeps out in that time interval.

(iii) Angular velocity

(a) Average angular velocity. It is the ratio of the angular displacement to the time taken by the particle to undergo this displacement. It is denoted by $\vec{\omega}$ or ω_{av}.

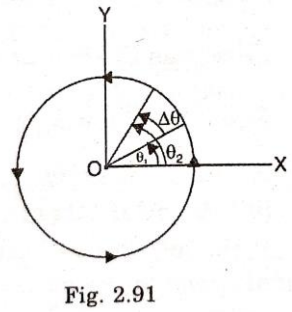


Fig. 2.91

Suppose the particle is at angular position θ_1 at time t_1 and at angular position θ_2 at time t_2. The average angular velocity of the particle in the time interval Δt from t_1 to t_2 is given by :

$$\omega_{av} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

Here 40 is the angular displacement that occurs in time Δt.

(b) Instantaneous angular velocity. It is the limiting value of the average angular velocity of the particle in a small time interval as the time interval approaches zero. It is denoted by ω.

$$\omega_{av} = \underset{\Delta t \to 0}{\text{LT}} \frac{\Delta\theta}{\Delta t} = \frac{\Delta\theta}{\Delta t}$$

The SI unit of ω is rad s^{-1}. Its dimensional formula is $[M^0L^0T^{-1}]$.

(c) Uniform angular velocity. If a particle describes equal angler in equal intervals of time, then the angular velocity of the particle is said to be uniform.

(iv) ω in terms of T and v

The time period of a particle in circular motion is the time taken by the particle to complete one revolution. It is denoted by T.

The frequency of revolution of a particle in circular motion is the number of revolutions completed by the particle in 1 second. It is denoted by v. $v = \frac{1}{T}$.

When the particle completes one revolution, $\theta = 2\pi$ and $t = T$.

$\therefore \quad \omega = \frac{\theta}{t} = \frac{2\pi}{T}$

Again, $\omega = 2\pi \left(\frac{1}{T}\right) = 2\pi\upsilon$.

(v) Relation between linear and angular velocities

Let the particle travel from A to B along a circular path in time t with linear speed υ.

Thus, $\quad AB = \upsilon t \quad \ldots(1)$

Also, $\quad AB = r\,\theta \quad \ldots(2)\ [\because \theta = \frac{l}{r}]$

Here, r is the radius of the circle along which the particle is moving.

Equating (1) and (2), $\upsilon t = r\,\theta$ or $\upsilon = r = \frac{l}{\theta}\, r\,\omega$

$\therefore$ linear velocity = radius of the circular path × angular velocity

(vi) Angular Acceleration

If the angular velocity of a particle is not constant, then the particle has angular acceleration.

(a) Average angular acceleration. It is the ratio of change in angular velocity of a particle to the time taken to undergo this change in angular velocity.

Let ω_1 and ω_2 be the angular velocities at times t_1 and t_2 respectively. The average angular acceleration in the interval from t_1 to t_2 is given by :

$$\alpha_{av} = \frac{\omega_1 - \omega_2}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

Here $\Delta\omega$ is the change in angular velocity that occurs during the time interval Δt.

Again, $$\alpha_{av} = \frac{\omega_1 - \omega_2}{t_2 - t_1} = \frac{2\pi n_2 - 2\pi n_1}{t_2 - t_1} = \frac{2\pi(n_2 - n_1)}{t_2 - t_1}$$

Here n_1 and n_2 are the number of revolutions made by the particle in one second.

(b) Instantaneous angular acceleration. It is the limiting value of the angular acceleration of the particle in a small time interval as the time interval approaches zero.

(vii) Relation between linear acceleration and angular acceleration Let v_1 be the linear velocity of the particle at A where its angular velocity is ω_1. Let v_2 be the linear velocity of the particle at B where its angular velocity is ω_2.

Then, $v_1 = r\omega_1$ and $v_2 = r\omega_2$

Now, $$\alpha = \frac{\omega_1 - \omega_2}{t} = \frac{\frac{v_2}{r} - \frac{v_1}{r}}{t} = \frac{v_2 - v_1}{t}$$

or $$\alpha = \frac{l}{r} \times \frac{v_2 - v_1}{t} = \frac{a}{r} \qquad [\therefore \text{ acceleration } a = \frac{v_2 - v_1}{t}]$$

or $$a = ra$$

2.72 CENTRIPETAL FORCE AND ACCELERATION

If a particle moves in a circle with uniform speed, it is said to be moving with uniform circular motion. In this case, the velocity changes only in direction and not in magnitude. Applying Newton's first law of motion, we find that some external force must be acting on the particle. Without the presence of external force, the velocity cannot change. Since the velocity is changing continuously, therefore, the external force must also be acting continuously.

Let us now investigate the nature of this external force. The direction of the external force is such that only the direction of the velocity vector changes. The magnitude of the velocity remains unchanged. This is possible only if the force always acts at right angles to the velocity vector. If the force acts in any other direction, it will have a component along the direction of motion which would change the speed of the particle.

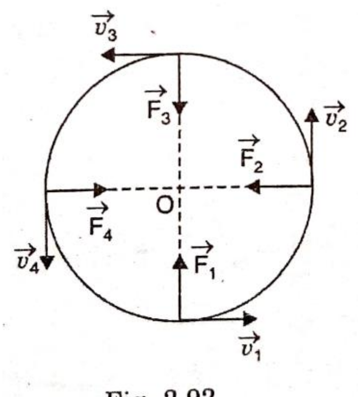


Fig. 2.92

The direction of the velocity of the particle is along the tangent to the circle at every point. So, the external force must act at right angles to the tangent to circle at every point. In other words, the force is always directed towards the centre of the circle at every point. This force is called *centripetal force. The corresponding acceleration is called centripetal acceleration.

Centripetal force is that external force which deflects a particle from its linear path to make it move along a circle and is always directed radially inwards.

Expressions for Centripetal Force. Consider a particle moving along a circle of radius r with a constant speed v. Suppose it moves from A to B in a small time interval Δt (Fig. 2.93(a)]

Let $\vec{v}_1$ and $\vec{v}_2$ be the velocities of the particle at A and B respectively such that

$$|\vec{v}_1| = |\vec{v}_2| = v$$

However, the two velocity vectors have different directions. Let θ be the angle between the directions of $\vec{v}_1$ and $\vec{v}_2$. $\vec{v}_1$ is represented by a vector drawn along the tangent at A. $\vec{v}_1$ is represented by a vector drawn along the tangent at B.

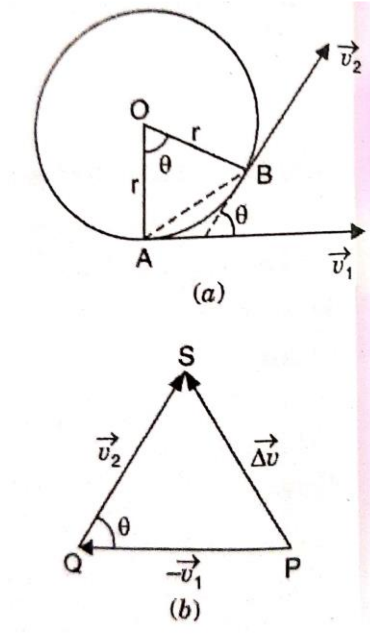


Fig. 2.93

The angle between two tangents is equal to the angle between the corresponding radii. $\therefore \angle AOB = \theta$.

The change in velocity $\overrightarrow{\Delta v}$ due to change in direction as particle moves from A to B is depicted separately in Fig. 2.93(b).

Applying triangle law of vectors to the vector triangle PQS, we get

$$\overrightarrow{PQ} + \overrightarrow{QS} = \overrightarrow{PS}$$

$$-\vec{v}_1 + \vec{v}_2 = \overrightarrow{\Delta v} \quad \text{or} \quad \overrightarrow{\Delta v} = \vec{v}_2 - \vec{v}_1$$

Triangles AOB and PQS are isosceles triangles having the same vertex angle θ. So, these triangles are similar,

$$\therefore \quad \frac{PS}{AB} = \frac{QS}{OB} \quad \text{or} \quad \frac{\Delta v}{AB} = \frac{v}{r}$$

Here, v (constant speed) is the magnitude of $\vec{v}$ and Δv is the magnitude of $\overrightarrow{\Delta v}$.

Since Δt is very small, $\therefore$ AB (chord AB) = AB (are AB) = $v\,\Delta t$

$$\therefore \quad \frac{\Delta v}{v\,\Delta t} = \frac{v}{r} \quad \text{or} \quad \frac{\Delta v}{\Delta t} = \frac{v^2}{r}$$

Acceleration, $a = \underset{\Delta t \to 0}{\text{LT}} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} \quad \therefore \quad \boxed{a = \frac{v^2}{r}}$

The equation gives the magnitude of the centripetal acceleration. This acceleration is along the radius of the circle and is directed towards the centre of the circle. It is also known as radial acceleration.

If m be the mass of the particle, then the centripetal force, F = ma

or $$F = \frac{mv^2}{r} \qquad ...(1)$$

Also $$v = r\omega$$

where ω is the angular velocity (also known as angular frequency).

Then, $$F = \frac{m(r\omega)^2}{r}$$

or $$F = mr\omega^2 \qquad ...(2)$$

But $$\omega = \frac{2\pi}{T}$$

where T is the period of revolution, *i.e.,* the time taken by the particle to complete one revolution.

∴ $$F = mr\left(\frac{2\pi}{T}\right)^2$$

or $$F = \frac{4\pi^2 mr}{T^2} \qquad ...(3)$$

Also $$\frac{1}{T} = n$$

where n is the frequency of revolution, *i.e.,* the number of revolutions completed in unit time.

$$F = 4\pi^2 mrn^2 \qquad ...(4)$$

Equations (1), (2), (3) and (4) give the different expressions for the centripetal force.

Direction of Centripetal Force. The direction of centripetal force is the same as that of $\overrightarrow{\Delta v}$ when Δt approaches zero.

When $\Delta t \to 0$, $\theta \to 0$ ∴ $B \to A$

In the limiting case, $\overrightarrow{\Delta v}$ points along the radius of the circle and towards the centre of the circle. So, the centripetal force will also act along the radius of the circle and towards the centre of the circle.

Examples of Centripetal Force

1. When a stone tied to a string is revolved in a circle, the tension in the string supplies the necessary centripetal force.
2. In the case of the motion of Earth around the Sun, the gravitational force of attraction between the Sun and the Earth provides the necessary centripetal force.

3. For an electron revolving around the nucleus, the centripetal force is provided by the electric force of attraction between the nucleus and the electron.

When the centripetal force ceases to act, the particle would move in a straight line along tangent to the circular path at the point where the force has ceased to act.

In the absence of the centripetal force, the particle has a natural tendency to move in a straight line in accordance with Newton's first law of motion.

2.73 CENTRIFUGAL FORCE

Suppose a body is rotating in circular path. Let the centripetal force suddenly vanish. Now, the body would leave the circular path. For an observer standing outside the circular path, the body appears to fly off tangentially at the point of release.

For an observer rotating with the body with the same velocity, the body appears to be stationary before it is released. It appears to the observer as if it has been thrown off along the radius away from the centre by some force. This force is called centrifugal force. Its magnitude is the same as that of the centripetal force, i.e., $\frac{mv^2}{r}$.

Note. Centrifugal force is not a force of reaction. It is a fictitious force which has a concept only in a rotating frame of reference.

2.74 MOTION OF CAR ON CIRCULAR LEVEL ROAD

Consider a car of weight mg moving on a circular level road of radius 'r' with constant velocity 'v' [Fig. 2.94]. While taking the round, the tyres of the car tend to leave the road and move away from the centre of curve. So the forces of friction f_1 and f_2 act inward to the *two tyres. If R_1 and R_2 are the normal reactions of ground on the tyres, then

$$f_1 = \mu R_1 \quad \text{and} \quad f_2 = \mu R_2$$

where μ is the coefficient of friction.

Total frictional force,

$$f = \mu R_1 + \mu R_2$$

or $$f = \mu (R_1 + R_2) \quad \text{or} \quad f = \mu R$$

where 'R' is the reaction of the ground on the car.

Again, R = mg (weight of car)

$\therefore \quad f = \mu mg$

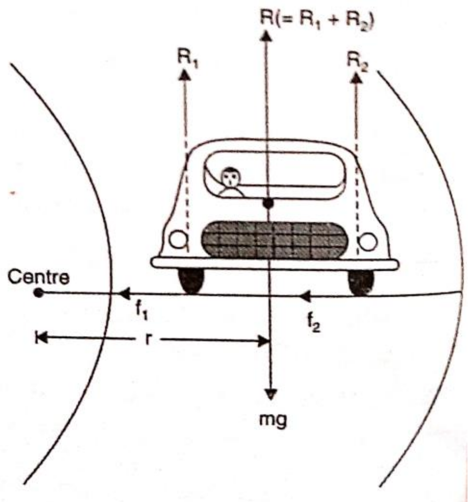


Fig. 2.94

Required centripetal force = $\frac{mv^2}{r}$

Since this force is to be provided only by the force of friction.

∴ $\frac{mv^2}{r} \le f$ or $\frac{mv^2}{r} \le \mu mg$

or $\frac{v^2}{r} \le \mu g$ or $v^2 \le \mu gr$ or $v \le \sqrt{\le \mu gr}$

∴ Maximum speed, $v_{max.} = \sqrt{\le \mu gr}$

if the car is driven at a speed greater than , then the car will skid and go off the road in a circle of radius greater than *r*. This is because even the maximum available friction will be inadequate to provide the necessary centripetal force.

2.75 APPLICATION TO BANKING OF ROADS

If a cyclist is to take a turn, he can bend from his vertical position. However, this is not possible in the case of a vehicle like car or truck or train. The tilting the vehicle is achieved by raising the outer edge of the circular track, slightly above the inner edge. This is known as banking of curved track. The angle through which the outer edge is raised above the inner edge is called angle of banking.

In Fig. 2.95, a car is coming out of the paper and is turning to the left after negotiating a curve of radius *r*. The normal reaction R acts along normal to the banked track. It has two components, vertical component R. cos θ and horizontal component R sin θ (Fig. 2.96).

The vertical component of normal reaction balances the weight of the car. The horizontal component of normal reaction provides the necessary centripetal force.

∴ $R \cos\theta = mg$

and $R \cos\theta = \frac{mv^2}{r}$

∴ $\frac{\sin\theta}{\cos\theta} = \frac{mv^2}{r} \times \frac{1}{mg}$

or $\tan\theta = \frac{v^2}{rg}$

∴ $v^2 = gr \tan\theta$

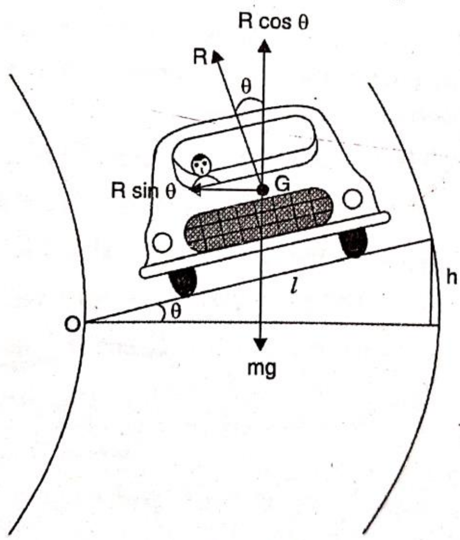


Fig. 2.95

or $\qquad v = \sqrt{gr \tan \theta}$

This gives us the **maximum safe speed** of the vehicle. In actual practice some frictional forces are always present. So, the maximum safe velocity is always much greater than that given by the above equation. While constructing the curved track, the value of is calculated for fined values of v_{max} and r.

This explains as to why along the curved roads, the speed limit at which the curve is to be negotiated is clearly indicated on sign boards.

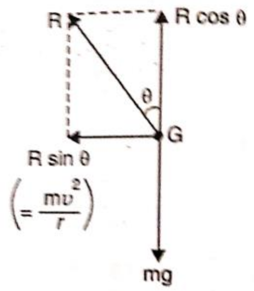


Fig. 2.96

The outer side of the road is raised by

$$\boldsymbol{h \times l \times \theta}.$$

When is small, then $\tan \theta = \sin \theta = \frac{h}{l}$;

Also $\tan \theta = \frac{v^2}{rg}$

$\therefore \qquad \frac{v^2}{rg} = \frac{h}{l} \qquad$ or $\qquad h = \frac{v^2}{rg} \times l$

This gives us the height through which outer edge is raised above the inner edge.

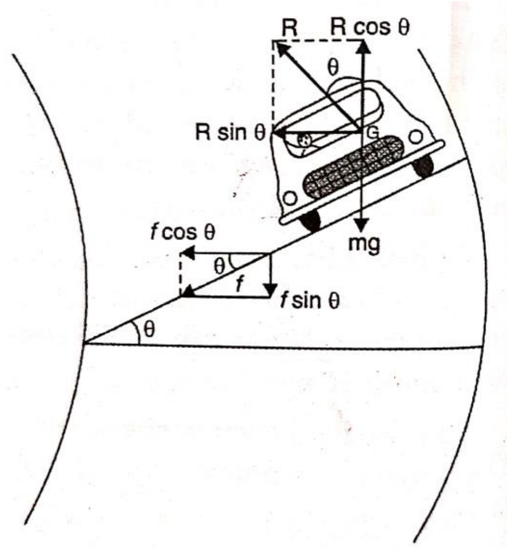


Fig. 2.97

Let us now take into account the actual frictional force which is always present during motion between the tyres and the road.

In this case, we shall consider the following three forces acting on the car [Fig. 2.97].

(i) Weight mg of the car acting vertically downwards through the centre of gravity G of the car.

(ii) The reaction R of the track on the car. This reaction acts normally to the banked track.

(iii) The frictional force f between the tyres and the road.

R cos θ and R sin 0 are the two rectangular components of R.

$f \cos \theta$ and $f \sin \theta$ the two rectangular components of f.

The car does not have any vertical motion.

$\therefore \qquad mg + f \sin \theta = R \cos \theta$ or $mg = R \cos \theta - f \sin \theta$

But $\qquad f = \mu R$, where $\mu \leq \mu_s$

∴ mg = R cos θ – μ R sin θ ...(1)

The forces R sin θ and f cos θ together provide the necessary centripetal force.

∴ $$\frac{mv^2}{r} = \text{R} \sin \theta + f \cos \theta$$

or $$\frac{mv^2}{r} = \text{R} \sin \theta + \mu \text{ R} \cos \theta \qquad ...(2)$$

dividing equation (2) by equation (1), we get

$$\frac{mv^2}{r} = \frac{\text{R} \sin \theta + \mu \text{R} \cos \theta}{\text{R} \cos \theta - \mu \text{R} \sin \theta}$$

or $$\frac{v^2}{rg} = \frac{\sin \theta + \mu \cos \theta}{\cos \theta - \mu \sin \theta}$$

or $$\frac{v^2}{rg} = \frac{\sin \theta\,(\tan \theta + \mu)}{\cos \theta(1 - \mu \tan \theta)} \quad \text{or} \quad v^2 = \frac{\tan \theta + \mu}{1 - \mu \tan \theta}\, rg$$

or $$v = \sqrt{\frac{\tan \theta + \mu}{1 - \mu \tan \theta}}\, rg$$

The best speed to negotiate a curve would be obtained by putting μ = 0.

∴ $$v = \sqrt{rg \tan \theta}$$

With this speed, there will be minimum wear and tear of the tyres.

Sample Problem 2.13. An aircraft executes a horizontal loop of radius 1 km with a steady speed of 900 km h^{-1}. Compare its centripetal acceleration with the acceleration due to gravity.

Solution. R = 1 km = 10^3 m ; v = 900 km h^{-1} = 900 × $\frac{5}{18}$ m s^{-1} = 250 m s^{-1}

Centripetal acceleration, $a_c = \frac{v^2}{r} = \frac{250 \times 250}{10^3}$ m s^{-2} = 62.5 m s^{-2}

Now, $$\frac{a_c}{g} = \frac{62.5}{9.8} = \mathbf{6.38}$$

Sample Problem 2.14. The radius of curvature of a railway line at a place when the train is moving with a speed of 36 km h^{-1} is 1000m, the distance between the two rails being 1.5 metre. Calculate the elevation of the outer will rail above the inner rail so that there may be no side pressure on the rails.

Solution. Velocity, v = 36 km h^{-1} = $\frac{36 \times 100}{10^3}$ m s^{-1} = 10 m s^{-1}

Radius, r = 1000 m ; $\tan\theta = \frac{v^2}{rg} = \frac{10 \times 10}{100 \times 9.8} = \frac{1}{98}$

Let h be the height through which outer rail is raised. Let l be the distance between the two rails.

Then, $\tan\theta = \frac{h}{l}$ or $h = l \tan\theta$ [∴ θ is very small]

$$h = 1.5 \times \frac{1}{98} \textbf{ m = 0.0153 m}$$

EXERCISES

a. A gramophone disc rotates at 60 rpm. A coint of mass 0.013 kg is placed at a distance of 0.08 m from its centre. Calculate the centripetal force acting on it. Take π^2 = 9.87.

[**Ans.** 0.041 N]

b. The driver of a car travelling at a speed v suddenly sees a wall at a distance r directly in front of him. To avoid collision, should he apply the brakes or turn the car sharply away from the wall.

[**Ans.** He should apply the brakes]

c. A train is running at 20 m s^{-1} on railway line whose radius of curvature in 4 × 109 m. The distance between the two rails is 1.5 m. What is the elevation of the outer rail above the inner rail so that the train may be able to run safely? Given: g = 9.8 10 m s^{-2}.

[**Ans.** 1.53 × 10^3 m]

d. At what angle must a track with a bend of 200 m radius be banked for safe running of trains at a speed of 72 km s^{-2}? Given: g = 9.8 m s^{-2}.

[**Ans.** 11° 32']

e. A 2000 kg car has to go over a turn whose radius is 750 metre and the angle of slope is 5°. The coefficient of friction between the car wheels and the road is 0.5. What should be the maximum speed of the car so that it may go over the turn without slipping?

[**Ans.** 67.2 m s^{-1}]

f. For traffic moving at 60 km h^{-1} if the radius of the curve is 0.1 km, what is the correct angle of banking of the road ? Given : g = 10 m s^{-2}.

SUMMARY

- Vector quantities are those quantities which possess both magnitude and direction and obey the laws of vector addition.
- While dot product of two vectors gives a scalar, the cross product of two vectors in a vector.
- The time rate of change of displacement gives velocity.
- The time rate of change of velocity gives acceleration.
- The equations of motion ($v = u + at$, $S = ut + \frac{1}{2} at^2$ and $v^2 - u^2 = 2aS$) hold good for uniformly accelerated motion.
- Newton's first law of motion gives definition of force.

- Newton's second law of motion gives the measurement of force.
- Newton's third law of motion gives the properties of force.
- For composition of two forces, we may use triangle law of forces or parallelogram law of forces.
- For composition of more than two forces, we use polygon law of forces.
- The motion of projectile is parabolic.
- There are two angles of projection for the same horizontal range.
- The maximum value of static friction is called limiting friction.
- If the two bodies in contact are in relative motion, then the force of friction is called dynamic or kinetic friction.
- The coefficient of static friction is greater than the coefficient of kinetic friction.
- The acceleration of a body sliding down an inclined plane is $g(\sin \alpha - \mu_k \cos \alpha)$, where α is the angle of inclination and μ_k is the coefficient of kinetic friction.
- Friction is a necessary evil.
- If m be the mass of a particle moving with speed v in a circle of radius r, then the centripetal force is $\frac{mv^2}{r}$
- Centripetal force acts along the radius of the circle and towards the centre of this circle.
- When a circular track is banked, a component of the normal reaction provides the necessary centripetal force.

TEST YOURSELF

i. What are sculars and vectors ? Give examples.

ii. What do you mean by the following ?
 (a) Equal vectors
 (b) Co-initial vectors
 (c) Co-terminus vectors
 (d) Collinear vectors
 (e) Unit vector

iii. What do you understand by dot and cross product of vectors ?

iv. Define the following terms:
 (a) Average velocity
 (b) Instantaneous velocity
 (c) Average acceleration
 (d) Instantaneous acceleration

v. Derive the following equations of motion :
 (a) $v = u + at$
 (b) $S = ut + \frac{1}{2} at^2$
 (c) $v^2 - u^2 = 2aS$

6. State Newton's second law of motion and derive formula for force.

7. What are the absolute and gravitational units of force ? How are they related to each other ?

8. What are the three laws of composition of forces ?

9. Using parallelogram law of forces, determine the magnitude and direction of the resultant of two forces.

10. A projectile is thrown horizontally from a certain height above the ground. Show that the trajectory is parabolic.

11. A projectile is thrown at a certain angle with the horizontal. Show that the trajectory is parabolic. Find formulae for
(a) time of ascent
(b) time of flight
(c) maximum height
(d) horizontal range

12. What do you understand by the following terms ?
(a) force of static friction
(b) force of limiting friction
(c) angle of friction
(d) angle of repose
(e) coefficient of static friction
(f) coefficient of kinetic friction

13. What is centripetal force ? Derive formula for centripetal force.

14. Applying the concept of centripetal force to banking of roads.

SECTION - B

3 WORK, POWER AND ENERGY

LEARNING OBJECTIVES

- Work.
- Units of work
- Work done in moving a body over a rough horizontal surface.
- Work done in moving a body up a rough inclined plane.
- Concept of power.
- Units of power.
- Power of an agent pulling a body on a rough horizontal surface.
- Power of an agent pulling a body up a rough inclined plane with a constant velocity.
- Energy
- Concept of kinetic energy.
- Expression for kinetic energy.
- Work-energy theorem.
- Relation between momentum and kinetic enrgy.
- Concept of potential energy.
- Conservative forces.
- Non-conservative forces.
- Principles of conservation of energy.
- Conservation of energy in the case of freely flailing bodies.
- Transformation of energy in vibrating simple pendulum.
- Potential energy of a spring.

3.1 WORK

Work is said to be done by a force when the force produces a displacement in the body on which it acts in any direction except perpendicular to the direction of the force.

Suppose a constant force $\vec{\mathbf{F}}$ applied on a body produces a displacement $\vec{\mathbf{S}}$ in the body in such a way that $\vec{S}$ is inclined to $\vec{F}$ at an angle θ. Now the work done will be given by the dot product of force and displacement.

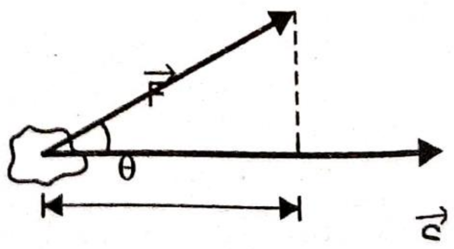


Fig. 3.1

∴ $\boxed{W = \vec{F} \cdot \vec{S}}$ or $\boxed{W = FS \cos \theta}$

Since work is the dot product of two vectors therefore, it is a scalar quantity.

Special cases

(a) Zero work. **(i) When θ - 90°,** then W = FS cos 90° = 0

So, work done by a force is zero if the body is displaced in a direction perpendicular to the direction of the force.

1. Consider a body moving in a circle with constant speed. At every point of the circular path, the *centripetal force and the displacement are mutually perpendicular (Fig. 3.2). So, the work done by the centripetal force is zero. The same argument can be applied to a satellite moving in a circular orbit. In this case, the gravitational force is always perpendicular to displacement. So, work done by gravitational force is zero.

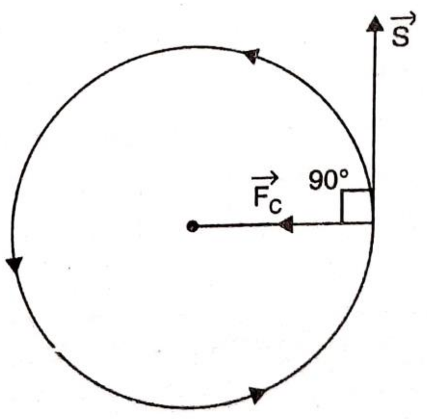


Fig. 3.2

2. The tension in the string of a simple pendulum is always perpendicular to displacement (Fig. 3.3). So, work done by the tension is zero.

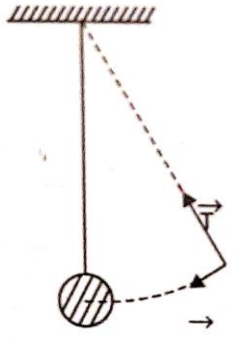


Fig. 3.3

(ii) When S = 0, then W = 0.

So, work done by a force is zero if the body suffers no displacement on the application of a force.

Example. A person carrying a load on his head and standing at a given place does no work.

(b) Positive work. When 0° ≤ θ < 90° (Fig. 3.4), then cos θ is positive. Therefore, W (= FS cos θ) is positive.

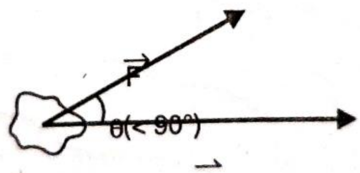


Fig. 3.4

Work done by a force is said to be positive if the applied force has a component in the direction of the displacement.

Examples :

1. When a horse pulls a cart, the applied force and the displacement are in the same direction. So, work done by the horse is positive.

2. When a load is lifted, the lifting force and the displacement act in same direction. So, work done by the lifting force is positive.

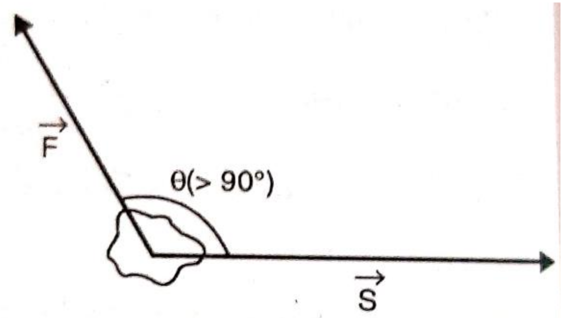


Fig. 3.5

(c) Negative work. When 90° < θ < 180° [Fig. 3.5), then cos θ is negative. Therefore, W (= FS cos θ) is negative.

Work done by a force is said to be negative. If the applied force has a component in a direction opposite to that of the displacement.

Examples:

1. When brakes are applied to a moving vehicle, the work done by the braking force is negative. This is because the braking force and the displacement act in opposite directions.

2. When a body is dragged along a rough surface, the work done by frictional force is negative. This is because the frictional force acts in a direction opposite to that of the displacement.

3.2 UNITS OF WORK

Unit work is the amount of work done when a unit force displaces a body through a unit distance in the direction of the force. So, the units of work will depend upon the units of force and distance.

(1) Absolute units of work. Work done is said to be one absolute unit if an absolute unit of force moves a body through unit distance in the direction of the force.

(i) In cgs system, the absolute unit of work is **erg.**

One erg of work is said to be done when a force of one dyne displaces a body through one centimetre in its own direction.

∴ $1 \text{ erg} = 1 \text{ dyne} \times 1 \text{ cm} = 1 \text{ g} \times 1 \text{ cm s}^{-2} \times 1 \text{ cm} = 1 \text{ g cm}^2 \text{ s}^{-2}$

Note. Erg is also called dyne centimetre.

(ii) In SI i.e., International System of units, the absolute unit of work is joule (abbreviated as J). It is named after the famous British physicist James Prescott Joule (1818--1869).

One joule of work is said to be done when a force of one newton displaces a body through one metre in its own direction.

1 joule = 1 newton × 1 metre = 1 kg ×1 m s^{-2} × 1 m = 1 kg cm^{-2} s^{-2}

Note. Another name for joule is newton metre.

Relation between joule and erg

1 joule = 1 newton × 1 metre

or **1 joule =** 10^5 dyne × 10^2 cm = 10^7 dyne cm

or **1 joule = 107 erg**

or 1 erg = 10^{-7} joule

(2) Gravitational Units of Work. Work stone said to be one gravitational unit if a gravitational unit of force moves a body through unit distance in the direction of the force.

(i) In egs system, the gravitational unit of work is **g cm.**

One gram centimetre of work is said to be done when a force of one gram weight displaces a body through one centimetre in its own direction.

1 g cm = 1 g wt × 1 cm or **1 g cm** = 981 dyne × 1 cm

or $\boxed{\mathbf{1\ g\ cm = 981\ erg}}$

In general, 1 g cm = g erg

(ii) ln **SI,** the gravitational unit of work is **kilogram metre.**

One kilogram metre of work is said to be done when a force of one kilogram weight displaces a body through one metre in its own direction.

1 kg m = 1 kg wt × 1 cm or **1 kg m** = 9.81 N × 1 cm

or $\boxed{\mathbf{1\ kg\ m = 9.81\ J}}$

In general, 1 kg m = g J

3.3 WORK DONE IN MOVING A BODY OVER A ROUGH HORIZONTAL SURFACE

Consider a block of weight W resting on a rough horizontal surface (Fig. 3.6). Let a force P be applied horizontally so that block just begin to move. Let f_k be the force of kinetic friction. Let S be the distance through which the body is moved.

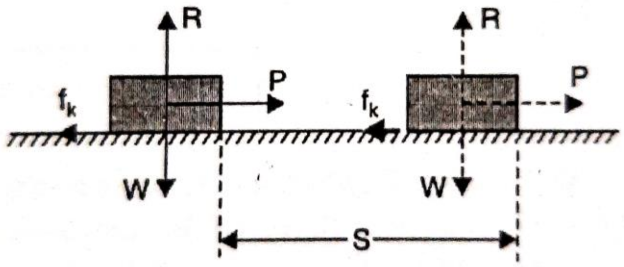


Fig. 3.6

Work done against friction = $f_k \times$ S

But $f_k = \mu_k$ R = μ_k W

Work = μ_k WS

or Work = μ_k mg S [$\therefore$ W = mg]

3.4 WORK DONE IN MOVING A BODY UP A ROUGH INCLINED PLANE

Let o be the angle of inclination. Let a force P be applied on a block placed on the inclined plane as shown (Fig. 3.3. Suppose the block just begins to slide up the inclined plane.

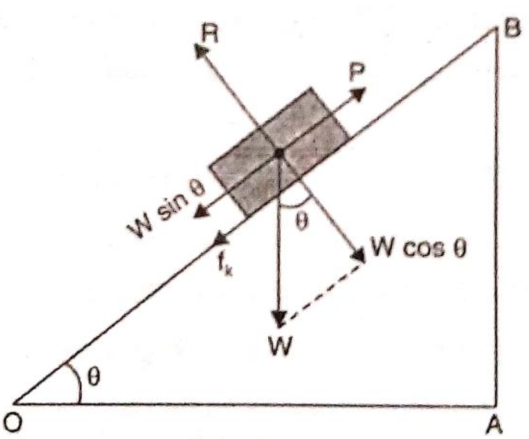


Fig. 3.7

Resolve the weight W into two rectangular components: W cos θ and W sin θ. The component W cos θ will balance the normal reaction R. If f_k be the force of friction, then the applied force is given by

$$P = W \sin\theta + f_k$$

But $\quad W = mg$ and $f_k = \mu_k R = \mu_k W \cos\theta = \mu_k mg \cos\theta$

$\therefore \quad P = mg \sin\theta + \mu_k mg \cos\theta$

Let the block be pulled through a distance S.

Work done = P × S = ($mg \sin\theta + \mu_k mg \cos\theta$) S

or $\quad$ Work done = $mg (\sin\theta + \mu_k \cos\theta)$ S

If the body is moved down the inclined plane with constant speed, then the work shall have to be done only against the force of friction whereas $mg \sin\theta$ will help the motion of the body. In this case, the work done is given by

$$\text{Work done} = mg(\mu_k \cos\theta - \sin\theta)\, S$$

Sample Problem 3.1. A gardener moves a lawn roller through a distance of 100 metre with a force of 50 newton. Calculate his wages if he is to be paid 10 paise for doing 25 joule of work. It is given that the applied force is inclined at 60° to the direction of motion.

Solution. Force, F = 50 N : Distance, S = 100 m;

Angle, θ = 60°

Work, $\quad W = FS \cos\theta = 50 \times 100 \times \cos 60°$ joule

or $\quad W = 50 \times 100 \times \frac{1}{2} J = 2500$ J $\qquad [\therefore \cos 60° = \frac{1}{2}]$

Wages = $\frac{2500}{25} \times 10$ paise = **10 rupees**

EXERCISES

1. Calculate the amount of work done by a boy when

(a) He holds a bundle of books of mass 5 kg for 5 minute.

(b) He walks with the same bundle of books along a lovel road at a speed of 5m s^{-1}

(c) He lifts up the same bundle of books by 1 m in order to keep it on a book shelf.

(Ans. (i) 0 (ii) 0 (iii) 49J)

2. A gardener pulls a lawn roller along the round through a distance of 20 m. If he applies a force of 20 kg wt in a direction inclined at 60° to the ground, find the work done by him.

[**Ans.** 1960 J]

3. Calculate the work done by a coolie in carrying a loud of mass 10 kg on his lead when he walks a distance of 5 in in the (i) horizontal direction is vertical direction.

[**Ans.** (i) Zero (ii) 490 J]

4. A body starts sliding down an inclined plane, the top half of which is perfectly
smooth and the lower half is rough Calculate the ratio of the force of friction nd the weight of the body, if the body is brought to rest just when it reaches the bottom. Given that inclination of the plane with the horizontal is 30°

3.5 CONCEPT OF POWER

Power is defined as the time rate of doing work. When the time taken to complete a given amount of work is important, we measure the power of the agent doing work.

The average power ($\overline{P}$ or P_{av}) delivered by an agent is given by

$$\overline{P} \text{ or } P_{av} = \frac{W}{t}$$

where W is the amount of work done in time *t*.

Power is the ratio of two scalars-work and time. So, power is a scalar quantity. If time taken to complete a given amount of work is more, then power is less.

Now, $$\overline{P} = \frac{\vec{F} \cdot \vec{S}}{t} \qquad [\because W = \vec{F} \cdot \vec{S}]$$

or $$\overline{P} = \vec{F} \cdot \frac{\vec{S}}{t} = \vec{F} \cdot \vec{v} \qquad [\because \frac{\vec{S}}{t} = \text{velocity } \vec{v}]$$

By definition of dot product, $P = Fv \cos \theta$

where θ is the smaller angle between $\vec{F}$ and $\vec{v}$.

SPECIAL CASE. If both $\vec{F}$ and $\vec{v}$ point in the same direction, then

$$\theta = 0°$$

$$\overline{P} = Fv \cos 0° = Fv \qquad [\because \cos 0° = 1]$$

Instantaneous Power is the power at any given instant. It is denoted by P. Suppose an agent does an infinitesimally small amount of work dW in an infinitesimally small time dt.

Then, $$\overline{P} = \frac{dW}{dt}; \qquad \text{But } dW = \vec{F} \cdot \overrightarrow{dS}$$

where $\overrightarrow{dS}$ is the infinitesimally small displacement in infinitesimally small time *dt*.

$$\therefore \qquad \overline{P} = \frac{\vec{F}\,.\,\vec{S}}{t} = \vec{F}\,.\,\frac{\overrightarrow{dS}}{dt} = \vec{F}\,.\,\vec{v}$$

So, power of an agent at any instant in equal to the dot product of the applied force and the velocity at that instant.

When an agent delivers power at a uniform rate, the average power is equal to the instantaneous power.

3.6 UNITS OF POWER

A unit power is the power of an agent which does unit work in unit time.

Absolute Units. The absolute unit of power in cgs system is erg per second.

The power of an agent is said to be one erg per second if it does one erg of work in one second.

The absolute unit of power in SI is joule per second. It is also known as watt (W).

The power of an agent is said to be one watt if it does one joule of work in one second.

1 watt = 1 joule/second = 10^7 erg/second [$\therefore$ 1 joule = 10^7 erg]

Also, 1 watt = $\frac{1 \times \text{newton} \times 1 \text{ metre}}{1 \text{ second}}$ = 1 N m s^{-1}

Gravitational Units. The gravitational unit of power in cgs system is gram centimetre per second (g cm s^{-1}).

The power of an agent is said to be one gram centimetre per second if it does one gram centimetre of work in one second.

The gravitational unit of power in SI is kilogram metre per second (kg m s^{-1}).

The power of an agent is said to be one kilogram metre per second if it does one kilogram metre of work in one second.

Practical Units. The power of an electric appliance is generally measured in watt or kilowatt.

1 kilowatt = 1000 watt (1 kW = 1000 W)

The power of an automobile is expressed in horse power (hp).

1 horse power = 746 W = 75 kg m s^{-1}

Power has 1 dimension in mass, 2 dimensions in length and – 3 dimensions in time.

CALCULATION OF POWER IN SIMPLE CASES

3.7 POWER OF AN AGENT PULLING A BODY ON A ROUGH HORIZONTAL SURFACE

Let a block of weight w be moved up a rough horizontal surface with uniform velocity v. Let F be the force applied by the external agent. Since the acceleration of the block is zero.

$\therefore F = f$

or $F = \mu_k R = \mu_k W$ $(\because R = W)$

Power = Fv

$= \mu_k W v$

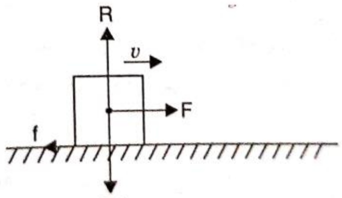


Fig. 3.8

If m be the mass of the block, then the power of the agent pulling the block is $\mu_k W v$.

3.8 POWER OF AN AGENT PULLING A BODY UP A ROUGH INCLINED PLANE WITH A CONSTANT VELOCITY

Let a block of weight W being moved up a rough inclined plane with uniform velocity v. Let F be the force applied by the external agent.

Clearly, $F = W \sin\theta + f$

$= W \sin\theta + \mu_k R$

$= W \sin\theta + \mu_k W \cos\theta$

Power = Fv

$= W(\sin\theta + \mu_k \cos\theta)\, v$

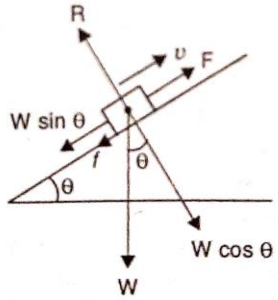


Fig. 3.9

3.9 POWER OF AN AGENT IN THE CASE OF A BODY MOVING DOWN A ROUGH INCLINED PLANE WITH CONSTANT VELOCITY

Consider a block of weight W being moved down a rough inclined plane with constant velocity v. Let F be the force applied by the external agent.

Now, $F + f = W \sin\theta$

or $F + \mu_k R = W \sin\theta$

or $F + \mu_k W \cos\theta = W \sin\theta$

or $F = W(\sin\theta - \mu\cos\theta)$

$$\text{Power} = Fv = W(\sin\theta - \mu\cos\theta)\, v$$

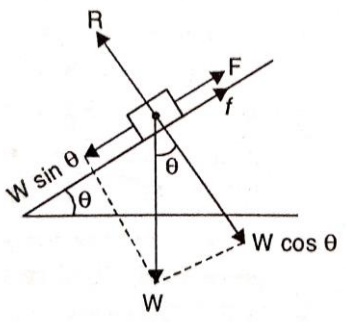


Fig. 3.10

Sample Problem 1.1. One coolie take one minute to rule box through a height of 2 metre. Another one takes 30 second for the same job and does the same amount of work. Which one of the two has greater power and which one uses greater energy?

Solution. Power of first coolie $= \frac{\text{Work}}{\text{Time}} = \frac{M \times g \times S}{t} = \frac{M \times 9.8 \times 2}{60}$ J s^{-1}

Power of second coolie $= \frac{M \times 9.8 \times 2}{30}$ J s^{-1} $= 2\left(\frac{M \times 9.8 \times 2}{60}\right)$ J s^{-1}

$= 2 \times$ Power of first coolie

So, the power of the second cootie in double that of the first.

Both the coolies spend the same amount of energy.

In 'power' time is important.
In 'work' time is not relevant.

Alter. We know that $W = Pt$

For the same work, $W = P_1 t_1 = P_2 t_2$

or $\frac{P_2}{P_1} = \frac{t_1}{t_2} = \frac{1 \text{ minute}}{30 \text{ s}} = 2$ or $P_2 = 2P_1$

Sample Problem 3.3. A car of mass 2000 kg is lifted up a distance of 30 m by a crane in 1 minute. A second crane does the same job in 2 minute. Do the cranes consume the same or different amounts of fuel? What is the power supplied by each crane? Neglect power dissipation against friction.

Solution. Work done W by the first crane is the same as the work done by the second crane. This is because the work done is independent of time. Both the cranes consume the same amount of fuel (energy).

$$W = 2000 \text{ kg wt} \times 30 \text{ m} = 2000\ 9.8 \text{ N} \times 30 \text{ m}$$

$$= 2000\ 9.8 \times 30 \text{ joule} = \mathbf{5.88 \times 10^5 \text{ joule}}$$

$$\text{Power of first crane} = \frac{\text{Work}}{1 \text{ minutes}} = \frac{\text{Work}}{60 \text{ s}} = \frac{5.88 \times 10^5}{60} \text{ J s}^{-1} \text{ or watt}$$

$$= 0.098 \times 10^5 \text{ watt} = \mathbf{9800 \text{ watt}}$$

$$\text{Power of second crane} = \frac{\text{Work}}{2 \text{ minutes}} = \frac{\text{Work}}{120 \text{ s}} = \frac{5.88 \times 10^5}{120} \text{ watt} = \mathbf{4900 \text{ watt}}$$

EXERCISES

1. The power of a pump motor is 2 kilowatt. How much water per minute can it raise to a height of 10 metre? Given : $g = 10 \text{ m s}^{-2}$.

[**Ans.** 1200 kg]

2. An engine develops 10 kW of power. How much time will it take to lift a mass of 200 kg through a height of 40 m ? Given : $g = 10 \text{ m s}^{-2}$.

[**Ans.** 8 second]

3. A lift pump of power 2000 W pumps water to an average height of 15 m to fill a tank 3 m × 2 m × 1 m. The efficiency of the pump is 75%. What is the time required to fill the tank? Given : $g = 10 \text{ m s}^{-2}$.

[**Ans.** 10 minute]

4. A man cycles up a hill, whose slope is 1 in 25, at 6 km h^{-1}.. The mass of the man and cycle is 150 kg. Find the power.

5. An engine of 150 kW power is drawing a train of total mass 150,000 kg up an inclination of 1 in 5. The frictional resistance is 4 kg wt/1000 kg. Find its maximum speed. Given : $g = 10 \text{ m s}^{-2}$.

[**Ans.** 0.5 m s^{-1}]

3.10 ENERGY

Energy of a body is defined as the capacity or ability of the body to do work.

It is a scalar quantity.

The dimensional formula of energy is $[ML^2T^{-2})$. It is the same as that of work.

The SI units of energy is the same as that of work i.e., joule.

It should be noted here that energy and power are different from each other Energy of a body shall give us an idea of the total amount of work that the body can do. It has nothing to do with the time taken to do the work. On the other hand, power depends upon the time in which the work is done.

Energy is of many 'types-mechanical energy, sound energy, heat energy, light energy, chemical energy, atomic energy, nuclear energy etc. The mechanical energy is of two types-kinetic energy and potential energy.

Alternative Units of Work/Energy in J

erg	10^{-7} *J*
electron volt (eV)	1.6×10^{-19} *J*
calorie (cal)	*4.186 J*
kilowatt hour (kWh)	3.6×10^{-6} *J*

3.11 concept of kinetic energy

Kinetic energy is the energy possessed by a body by virtue of its motion. It is logical that a body moving with higher speed should possess more kinetic energy than a body moving with lower speed.

A few examples of bodies possessing kinetic energy are given below.

(i) A bullet in motion (ii) A moving hammer

(iii) Running water (iv) Air in motion

(v) Moving piston of a locomotive engine.

3.12 EXPRESSION FOR KINETIC ENERGY

(i) Non-calculus method. Consider body of mass m lying on a smooth horizontal surface as shown as Fig. 3.11. Let a constant force $\vec{F}$ displace this body in its own direction through $\vec{S}$. Let the velocity of the body, after travelling distance S, be v. If W be the work done by the force, then

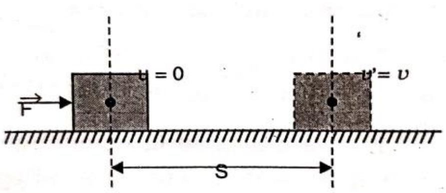


Fig. 3.11

$$W = \vec{F} \cdot \vec{S} = FS \cos \theta$$

Both the force and the displacement are in the same direction.

$\therefore$ $\theta = 0°$

$\therefore$ $W = FS \cos 0° = FS$

If a be the acceleration produced in the body, then $a = \frac{F}{m}$.

Also, $v^2 - 0^2 = 2as$ or $v^2 = 2\left(\frac{F}{m}\right)S$ or $FS = \frac{1}{2} mv^2$

From equation (1), $W = \frac{1}{2} mv^2$

This work done becomes the kinetic energy of the body.

So, the kinetic energy of a body of mass m moving with velocity v is $\frac{1}{2} mv^2$

(ii) Calculus method. If dW be the small amount of work done in giving an infinitesimally small displacement $d\vec{S}$ to the body, then $dW = \vec{F} \cdot \overrightarrow{dS}$.

Here, $\vec{F}$ is the applied force.

Suppose force and displacement have same direction.

Then, $dW = F\, dS \cos 0° = F\, dS$ $[\because \cos 0° = 1]$

Let m be the mass of the body and a be the acceleration. Then $F = ma$.

$$\therefore \quad dW = (ma)\, dS = m\frac{dv}{dt}dS \quad \text{or} \quad dW = m\,\frac{dv}{dt}dv = mv\, dv$$

Let W be the total work done in increasing the velocity from zero to v.

$$\text{Then, } W = \int_0^v mv\, dv = m\int_0^v v\, dv = m\left[\frac{v^2}{2}\right]_0^v = m\left(\frac{v^2}{2} - 0\right) = \frac{1}{2} mv^2$$

This work done becomes the kinetic energy.

$\therefore$ Kinetic energy $= \frac{1}{2} mv^2$

In words, the kinetic energy of a body is one half the product of the mass of the body and the square of its speed.

The kinetic energy of a body is directly proportional to (i) mass m of the body (ii) square of the velocity v of the moving body.

If m is measured in gram and v in cm s^{-1}, then the kinetic energy is measured in **erg.** If m is measured in kilogram and v in m s^{-1}, then the kinetic energy is measured in **joule.** It may be noted that the units of kinetic energy are the same as those of work. Infact, this is true of all forms of energy since they are interconvertible.

3.13 WORK-ENERGY THEOREM

We know that $dW = mv\ dv$

If W be the work done in increasing the velocity from v (0) to v, then

$$W = \int_{v(0)}^{v} mv\ dv = m\int_{v(0)}^{v} v\ dv = m\left[\frac{v^2}{2}\right]_{v(0)}^{v} = \frac{1}{2}mv^2 - \frac{1}{2}mv^2$$

So, the work done on a body by a resultant force is equal to the increase the kinetic energy of the body. This is called the Work-Energy Theorem or Work-Energy Principle.

A more general relation in which force varies both in magnitude and direction can also be derived on the same lines.

Discussion of Work-Energy Theorem

(i) If there is no change in the speed of a particle, there is no change in kinetic energy. So, work done by the resultant force is zero.

Example. When a particle moves with constant speed in a circle, there is no change in the kinetic energy of the particle. So, according to work-energy principle, the work done by centripetal force is zero.

(ii) If the kinetic energy of the body decreases, then the work done on the body is negative. In this case, force and displacement are oppositely directed i.e., the force opposes the motion of the body.

(iii) In the above discussion, it has been assumed that the work done by the force is effective only in changing the kinetic energy of the body. However, it should be clearly noted that the work done on a body may also be stored in the body as potential energy.

3.14 RELATION BETWEEN MOMENTUM AND KINETIC ENERGY

Consider a body of mass m moving with velocity v. Linear momentum of the body, $p = mv$.

Kinetic energy of the body, $E_k = \frac{1}{2}mv^2$

or $$E_k = \frac{1}{2}(m^2v^2)$$

or $$E_k = \frac{p^2}{2m}$$

or $$p = \sqrt{2mE_k}$$

Sample Problem 3.4. A bullet of mass 20 g is found to pass two points 30 m apart in time interval of 4 second. Calculate the kinetic energy of the bullet if it moves with constant speed.

Solution. Constant speed,

$$v = \frac{\text{distance covered}}{\text{time taken}} = \frac{30 \text{ m}}{4 \text{ s}} \; 7.5 \text{ m J s}^{-1}$$

mass, $m = 20 \text{ g} = 0.02 \text{ kg}$

kinetic energy $= \frac{1}{2} \times 0.02 \times (7.5)^2 \text{ J} = \mathbf{0.5625 \text{ J}}$

Sample Problem 3.5. In a ballistics demonstration, a police officer fires a bullet of mass 50.0 g with speed 200 m s^{-1} on soft plywood of thickness 2.00 cm. The bullet emerges with only 10% of its initial kinetic energy. What is the emergent speed of the bullet?

Solution. Initial kinetic energy, $K_i = \frac{1}{2} \times \frac{500}{1000} \times 200 \times 200 \text{ J} = 1000 \text{ J}$

Final kinetic energy, $K_i = \frac{10}{100} \times 1000 \text{ J} = 100 \text{ J}$

If v_f is the emergent speed of the bullet, then

$$\frac{1}{2} \times \frac{50}{100} \times {v_f}^2 = 100 \quad \text{or} \quad {v_f}^2 = 4000 \text{ or } v_f = \mathbf{63.2 \text{ s}^{-1}}$$

Note that the speed is reduced by approximately 68% and not 90%.

EXERCISES

1. Establish relation between momentum and kinetic energy. What is the percentage change in the kinetic energy of a body if its momentum is increased by 2% ?

[**Ans.** $p = \sqrt{2mE_k}$; + 4%]

2. A bullet of mass 20 g strikes a target with a velocity of 150 m s^{-1} and is brought to rent after piercing 10 cm into it. What is the average resistance offered by the target?

[**Ans.** 2250 N]

3. A bullet of mass 20 g moving with a velocity of 500 m s^{-1} strikes a tree and goes out from the other side with a velocity of 400 m s^{-1}. Calculate the work done in passing through the tree.

[**Ans.** 900 joule]

4. The linear momentum of a body is increased by 10%. What is the percentage change in its kinetic energy?

[**Ans.** Percentage increase in kinetic energy = 21%]

3.15 CONCEPT OF POTENTIAL ENERGY

It is the energy possessed by a body by virtue of its position in a field of force or by its configuration.

Potential energy is also known as mutual energy or energy of configuration. The energy of a system can be altered by the application of a force.

Broadly speaking, the potential energy is of the following three types:

(i) gravitational potential energy (ii) elastic potential energy (iii) electrostatic potential energy.

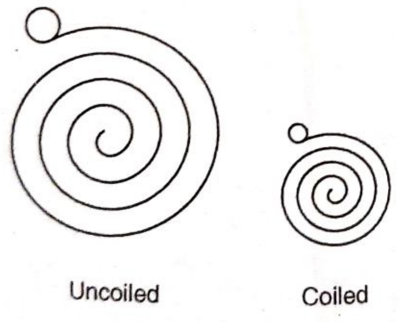


Fig. 3.12

Examples of bodies possessing PE due to position

(i) water stored in a reservoir (ii) lifted weight.

Examples of bodies possessing PE due to configuration

(i) The coiled spring of a watch or gramophone [Fig. 3.12].

(ii) A stretched how

(iii) Compressed gas

(iv) Compressed or elongated spring.

3.16 EXPRESSION FOR POTENTIAL ENERGY

Consider a body of mass m lying on the surface of Earth at a place where the value of acceleration due to gravity is g [Fig. 3.13].

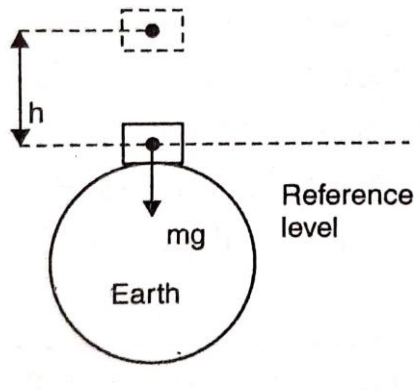


Fig. 3.13

Weight mg of the body acts vertically downwards. In order to lift the body, a force mg is required to be applied in the upward direction. Let h be the height through which the body is lifted.

$$\text{Work, } W = mg \times h$$

This work done is stored in the body as gravitational potential energy.

$\therefore$ gravitational potential energy, $U = mgh$

Strictly speaking, mgh is the gravitational potential energy of Earth-body system.

3.17 CONSERVATIVE FORCES

A force is said to be conservative if work done by or against the force in moving a body depends only on the initial and final positions of the body and not on the nature of path followed between the initial and final positions.

Examples of Conservative Forces

(i) Gravitational force, not only due to the Earth but in its general form as given by the universal law of gravitation, is a conservative force. (Note that the gravitational field is a conservative field.]

(ii) Elastic force in a stretched or compressed spring is a conservative force.

(iii) Electrostatic force between two stationary electric charges is a conservative force. (Note that the electrostatic field is a conservative field.]

(iv) Magnetic force between two magnetic poles is a conservative force.

Properties of Conservative Forces

(i) Work done by or against a conservative force depends only on the initial and final positions of the body.

(ii) Work done by or against a conservative force does not depend upon the nature of the path between initial and final positions of the body.

(iii) Work done by or against a conservative force in a round trip is zero.

(iv) The work done by a conservative force is completely recoverable.

3.18 NON-CONSERVATIVE FORCES

A force is said to be non-conservative if work done by or against the force in moving a body depends upon the path between the initial and final position.

The frictional forces are non-conservative forces. This is because the work done against friction depends on the length of the path along which a body is moved. It does not depend only on the initial and final positions. Note that the work done by frictional force in a round trip is not zero.

The velocity-dependent forces such as air resistance, viscous force, magnetic force etc. are non-conservative forces.

3.19 PRINCIPLE OF CONSERVATION OF ENERGY

The total mechanical energy of a system is conserved if the forces doing work on it are conservative. If some of the forces involved are non-conservative, part of the mechanical energy may get transformed into other forms such as heat, light and sound. However, the total energy of an isolated system does not change, as long as one accounts for all forms of energy.

The total energy E of a system is the sum of the system's mechanical energy, thermal energy and any other form of internal energy in addition to thermal energy.

Energy may be transformed from one form to another but the total energy of an isolated system remains constant. Energy can neither be created, nor destroyed.

The universe as a whole may be viewed as an isolated system. So, the total energy of the universe is constant. If one part of the universe loses energy, another part must gain an equal amount of energy.

It follows from the principle of conservation of energy that work W done on a system is given by

$$W = \Delta E = \Delta E_{mech} + \Delta E_{th} + \Delta E_{int}$$

where ΔE_{mech} is any change in the mechanical energy of the system, ΔE_{th} is any change in the thermal energy of the system and ΔE_{th} is any change in any other form of internal energy of the system. Included in ΔE_{mech} are changes ΔK in kinetic energy and changes ΔU in potential energy (elastic, gravitational or any other form we might find).

3.20 CONSERVATION OF ENERGY IN THE CASE OF FREELY FAILING BODIES

When a body falls from a certain height, its kinetic energy begins to increase. The gain in kinetic energy is at the expense of the gravitational potential energy which is reduced. But the total mechanical energy remains conserved.

Let us first discuss the case of a **freely falling body.** Consider a body of mass m to be at a position A. Let h be the height of the body above the reference level which is of course close to the ground [Fig. 3.14].

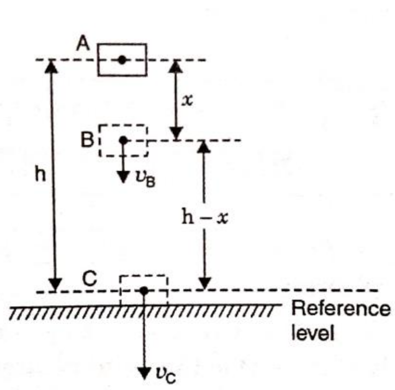


Fig. 3.14

At the position A

Kinetic energy of the body = 0 [∴ The body is at rest.]

Potential energy of the body = mgh

Total energy of the body = mgh ...(1)

At the position B

Let the body be at the position B at any instant after having fallen through distance x.

Potential energy of the body = $mg(h - x)$

If v_B be the velocity of the body at B, then of $v_B^2 - 0^2 = 2gx$ or $v_B^2 = 2gx$

Kinetic energy of the body = $\frac{1}{2} \times mv_B{}^2 = \frac{1}{2}m \times 2gx = mgx$ ...(2)

At the position C

When the body is at position C, its height above the ground may be regarded is zero.

∴ Potential energy of the body = 0

If v_C be the velocity with which the body just touches the ground, then

$$v_C^2 - 0 = 2gh \qquad \text{or} \qquad v_C^2 = 2gh$$

$\therefore$ Kinetic energy of the body $= \frac{1}{2} \times mv_C^2 = \frac{1}{2}m \times 2gh = mgh$

Total energy of the body $= mgh$ ...(3)

From equations (1), (2) and (3), it is clear that the sum of the kinetic and potential energies of freely failing body is constant at all stages of motion. As the body falls, its potential energy decrease while its kinetic energy increases. However, the total mechanical energy is conserved.

The graph shown in Fig. 3.15 depicts the values of kinetic energy, potential energy at different heights above the reference level. As is clear from the graph, the potential energy decrease linear with decrease in height. On the other hand, the kinetic energy increases with decrease in height. The total mechanical energy is represented by a thick straight line parallel to the height-axis. This shows that the total mechanical energy is conserved.

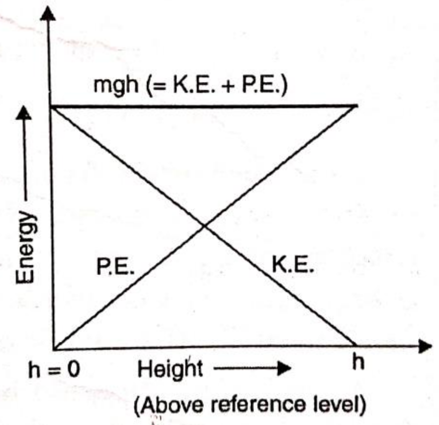


Fig. 3.15

3.21 TRANSFORMATION OF ENERGY IN VIBRATING SIMPLE PENDULUM

The conversion of energy from one form to another is called **transformation of energy.**

In Fig. 3.16, a vibrating simple pendulum is shown. O is the point of suspension. When the centre of gravity of the bob is vertically below the point of suspension, the position of the pendulum is called the rest position or mean position or equilibrium position.

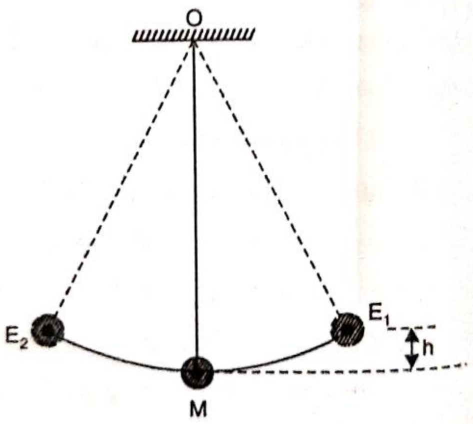


Fig. 3.16

Suppose the bob is displaced to the position E_1. When the bob is released from this position, the pendulum will start vibrating between the two extreme positions E_1 and E_2.

When the bob moves from mean position to the extreme position E_1, its centre of gravity is raised vertically through a height h.

∴ Potential energy of the bob at the extreme position = mgh.

where m is the mass of the bob and g is the value acceleration due to gravity at the place where pendulum is suspended.

As the bob moves from the extreme position to the mean position, its potential energy goes on decreasing while the kinetic energy goes on increasing. However, the sum of the two energies is constant. At the mean position, the whole of the potential energy is converted into kinetic energy. If v be the velocity of the bob at the mean position, then

kinetic energy at mean position = $\frac{1}{2} \times mv^2$

applying the law of conservation of energy.

$$mgh = \frac{1}{2} \times mv^2 \qquad \text{or} \qquad v = \sqrt{2gh}$$

As the bob moves from mean position to the other extreme position E_2, its kinetic energy decreases and potential energy increases. Energy becomes wholly potential at E_2. As the bob moves from E, to the mean position, the potential energy again goes on decreasing and kinetic energy goes on increasing. At the mean position, the energy is once again wholly kinetic.

Practical application of conversion of gravitational potential energy into kinetic energy.

The conversion of gravitational potential energy into kinetic energy has many practical application. The most important application is 'hydroelectric power generation'. A large amount of water is stored in lakes by dame which are built at high levels. The water stored in the lake possesses a very large amount of gravitational potential energy. When this water is made to run through the pipes, the gravitational potential energy in converted into kinetic energy. Now, the fast-flowing water is employed to rotate the turbines of electric generators. This produces electric energy.

3.22 POTENTIAL ENERGY OF A SPRING

(i) Block-spring system. One end of a massless, elastic spring is fixed to a support which is rigid and of infinite mass. The other end of the spring is connected to a small block of mass *m*. The block is placed on a smooth horizontal surface (Fig. 3.17]. The gravitational pull on the block is balanced by the reaction force of the surface. So, the only force that will come into play in this problem is the elastic force of the spring.

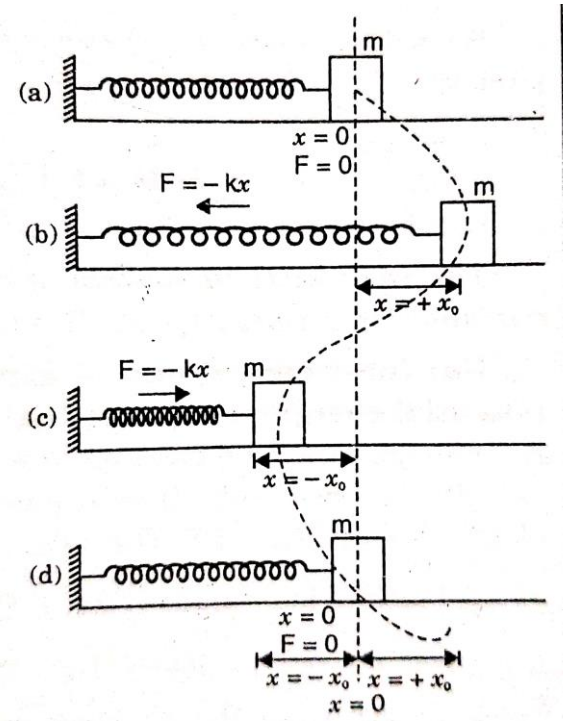


Fig. 3.17

Fig. 3.17(a) shows the upstretched and uncompressed position of the spring. The position of the body when the spring is of normal length is taken as $x = 0$.

(ii) Oscillating Block. If the block is pulled to the right through a small distance x [Fig. 3.17(b)], a restoring force F will be set up in the spring which, according to Hooke's law is given by $F = -kx$.

where k is known as spring constant or force constant of spring factor or stiffness constant or only, stiffness of the sring.

The restoring force shall try to bring the block back to the equilibrium position. When the block is released, it will move towards the mean position. On account of inertia of motion, it will overshoot its mean position ($x = 0$) and the spring will be compressed (Fig. 3.17(c)] through a distance x. Now the restoring force shall be directed to the right. This shall again try to bring the block back to the mean position and so on. In this way, the block will begin to oscillate about its mean position.

(iii) Potential energy of spring. The force required to stretch the spring through a distance x is '+ kx'. The work done in stretching the spring is stored as potential energy in the spring.

If dW is the work done in stretching the spring though a small distance dx, then dW = F dx.

Total work done in stretching the spring from 'x' = 0, to 'x' = x_0 is given by

$$\int d\text{W} = \int_0^{x_0} \text{F}\, dx \qquad \text{or} \qquad \text{W} = \int_0^{x_0} kx\, dx = k \int_0^{x_0} x\, dx.$$

$$= k = \left[\frac{x^2}{2}\right]_0^{x_0} = \frac{1}{2}\, k{x_0}^2$$

$$\int x^n dx = \frac{x^{n+1}}{n+1}, \text{ provided } n \neq -1$$

This work done is stored in the spring as potential energy.

∴ Potential energy of the spring = $\frac{1}{2} k{x_0}^2$

Note. The work done in changing the elongation of the spring from 'x' = x_1 to 'x' = x_2 is given by

$$\text{W} = \int_{x_1}^{x_2} kx\, dx = k \int_{x_1}^{x_2} x\, dx = k = \left[\frac{x^4}{2}\right]_{x_1}^{x_2} = \frac{1}{2}\, k\, ({x_2}^2 - {x_1}^2)$$

The work done in stretching a spring is a special case of work done by a variable force which varies linearly with displacement.

(iv) Inter-conversion of kinetic and potential energies of the oscillating block. At the right extreme position (x = + x_0), the velocity and hence the kinetic energy of the block are zero [Fig. 3.18]. The potential energy of the block is maximum ($\frac{1}{2}\, k{x_0}^2$). Obviously, the total energy is also $\frac{1}{2}\, k{x_0}^2$. When the block moves from the extreme position to the mean position, its velocity and hence the kinetic energy begin to increase and the potential energy begins to decrease, At any position x between the mean and the extreme positions, the kinetic energy will be $\frac{1}{2}\, k\, ({x_0}^2 - x^2)$. The corresponding potential energy will be $\frac{1}{2} kx^2$.

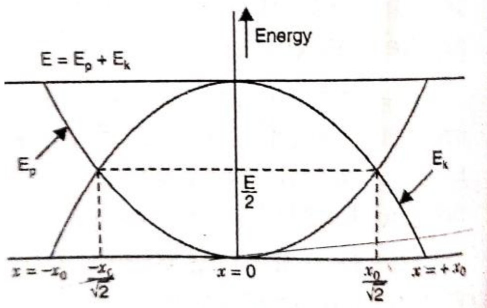


Fig. 3.18

The total mechanical energy will be again $\frac{1}{2}\,k{x_0}^2$.

At the mean position, the velocity and hence the kinetic energy of the oscillator will be maximum ($\frac{1}{2}\,k{x_0}^2$.). The potential energy will be zero.

As the block moves from the mean position to the left extreme position, the velocity and the kinetic energy begin to increase.

At x = - x_0, K.E. = 0 and P.E. = $\frac{1}{2}\,k{x_0}^2$ (maximum value)

The total energy is still $\frac{1}{2}\,k{x_0}^2$.

Sample Problem 3.6. Calculate the velocity of the bob of a simple pendulum at its mean position if it is able to rise to a vertical height of 10 cm. Given: g = 980 cm s^{-2}.

Solution. If v be the velocity at the mean position, then

kinetic energy at mean position = $\frac{1}{2}\,mv^2$.

Potential energy at a height of 10 cm = $mg \times 10$ *erg*

Equation, $\frac{1}{2}\,mv^2 = mg \times 10$

or $\quad v^2 = 2 \times g \times 10 \quad$ or $\quad v^2 = 2 \times 980 \times 10 = 19600$

or $\quad v = \sqrt{19600}$ cm s^{-1} = 140 cm s^{-1} = **1.40 m s^{-1}**

Sample Problem 3.7. The leaning tower of Pisa Is 45 m high. A ball of mass 4 kg is raised to its top and dropped from the top.

(a) Calculate the work done in raising the body to the top.

(a) Calculate the value of potential energy at the top.

(c) What is the value of the kinetic energy Just before hitting the ground?

(d) What is the velocity just before hitting the ground ?

Take g = 10 m s^{-1}

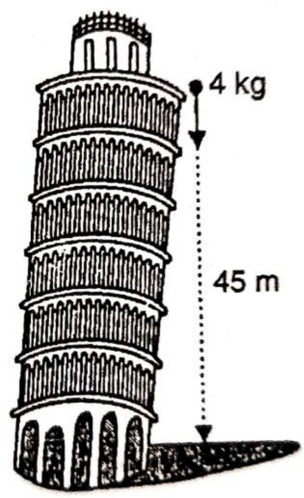


Fig. 3.19

Solution. (a) Work done, $\quad$ W = FS = mgs

$= 4 \times 10 \times 45$ J = **1800 J**

(b) P.E. = Work done = **1800 J**

(c) K.E. = **1800 J**

(d) $\frac{1}{2}\, mv^2 = 1800$

$$v^2 = 1800 \times \frac{2}{m} = 1800 \times \frac{2}{4} = 900$$

or $v = \sqrt{900}$ cm s^{-1} = **30 m s^{-1}**

EXERCISES

1. A trapeze artist with a mass of 65 kg swings on a trapeze (cord attached at O) from platform A to platform B. (Fig. 3.20)
   (i) Calculate the change in gravitational potential energy of the artist between platforms A and B.
   (ii) Calculate the kinetic energy of the artist on arriving at platform B.
   (iii) Calculate the speed of the artist on arrival at platform B. Take g = 10 m s^{-2}

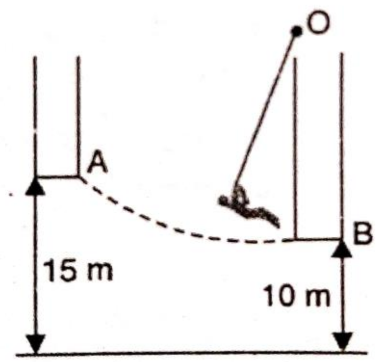


Fig. 3.20

[**Ans.** (i) 3250 J (ii) 3250 J (iii) 10 m s^{-1}]

2. A steel ball of mass 0.5 kg drops from a height of 2.8 m above the floor. The steel ball strikes and breaks a glass table-top 1 m above the floor. The ball finally reaches the floor at a velocity of 6 m s^{-1}. Calculate the work the ball does in breaking the table top.
   Take g = 10 m s^{-2}.

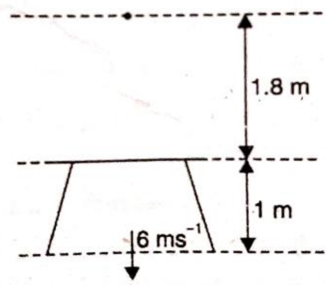


Fig. 3.21

[**Ans.** 5 J)

3. Consider the stationary situation shown in the sketch. The effect of friction is negligible. The 3 kg mass is joined to the 4 kg trolley by a string passing over his pulley.
   When the sky mass piece released it falls and accelerates the trolley 2 m to the top of the runway. In the process, the trolley rises 1 m. Calculate:
   (i) the rain in potential energy of the trolley on reaching the top of the runway; and
   (ii) the speed of the pulley at the top of the runway. Take g = 10 m s^{-2}.

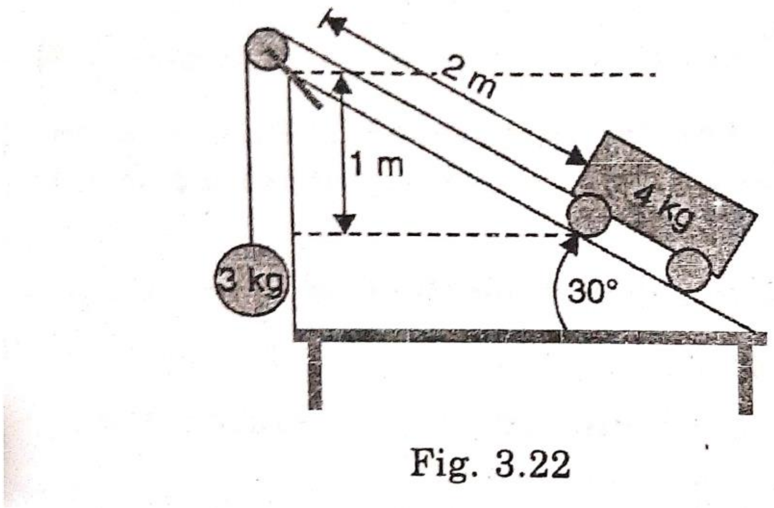


Fig. 3.22

[**Ans.** (i) 40 J (ii) 2.4 m s^{-2}]

SUMMARY

- Work is said to be done by a force if the applied force produces a displacement in any direction except perpendicular to the direction of force.
- If 0° < θ < 90°, then the work done is positive.
- If 90° < θ < 180°, then the work done is negative.
- One joule of work is said to be done if a force of one newton displaces a body through one metre in its own direction.
- The time rate of doing work is called power.
- The energy possessed by a body by virtue of motion is called kinetic energy.
- The energy possessed by a body by virtue of its position or configuration is called potential energy.
- According to principle of conservation of energy, the sum total of energy remains conserved.

TEST YOURSELF

1. Define work. Give examples of (i) zero work (ii) positive work and (iii) negative work.
2. What are the different units of work and how are they related to each other?
3. Derive an expression for the work done in moving a body on a rough horizontal surface.
4. Derive an expression for the work done in moving a body up a rough inclined plane.
5. What is the power? What are its units?
6. Define kinetic energy and potential energy. Give examples.

7. Derive expressions for kinetic energy and potential energy.
8. What is principle of conservation of energy? Show that total mechanical energy conserved in the case of a freely falling body.

4 ROTATIONAL AND SIMPLE HARMONIC MOTIONS

LEARNING OBJECTIVES

- Moments of inertia.
- Physical significance of moment of inertia.
- Radius of gyration.
- Theorem of perpendicular axis.
- Theorem of parallel axes.
- Moment of inertia of a ring.
- Moment of inertia of a disc.
- Moment of inertia of a solid sphere.
- Torque.
- Dependence of torque on lever arm.
- Work and power in rotational motion.
- Angular momentum.
- Geometrical meaning of angular momentum.
- Kepler's second law.
- Relation between angular momentum and torque.
- Rotational kinetic energy.
- Angular momentum in terms of moment of inertia.
- Angular momentum conservation and the case of rigid bodies.
- Applications/Illustrations of the law of conservation of angular momentum.
- Concept of rolling motion.
- Kinetic energy of rolling body.
- Important terms.
- Simple harmonic motion.
- Geometric definition of simple harmonic motion.
- Displacement in SHM.
- Amplitude in SHM.
- Velocity in SHM.
- Acceleration in SHM.
- Restoring force and force constant.
- Time period or periodic time of SHM.
- Frequency.
- Phase.
- Epoch.
- Dynamics of harmonic oscillation.

- Graphical representation of particle displacement, particle velocity and particle acceleration.
- Expression for time period and frequency in SHM.
- Differential equation of angular simple harmonic motion.
- Motion of cantilever.
- Free vibrations.
- Forced vibrations.
- Resonant vibrations and resonance.

4.1 MOMENT OF INERTIA

(i) Moment of inertia of a rigid body about a fixed axis is defined as the sum of the products of the masses of all the particles constituting the body and the squares of their respective distances from the axis of rotation. It is a scalar quantity.

Let YY' ne the axis about which the rigid body is rotating [Fig. 4.1]. Let the body he composed of n particles of masses m_1, m_2,........, m_n. Let r_1, r_2, ..., r_n be their respective distances from the axis of rotation. The moment of inertia of the rigid about the given axis YY' is given by

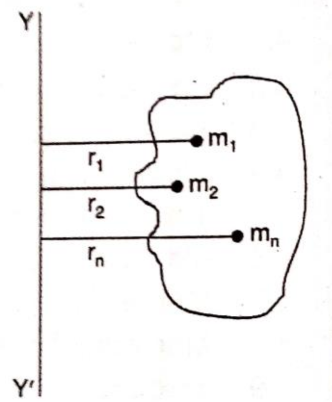


Fig. 4.1

$$I = m_1 r_1^2 + m_2 r_2^2 + m_n r_n^2 = \sum_{i=1}^{n} m_i r_i^2$$

(ii) In cgs system, the unit of moment of inertia is g cm^2. In SI, moment of inertia is measured in kg m^2.

(iii) Moment of inertia depends on the following factors:

1. Mass of the body.
2. Position of the axis of rotation.
3. Distribution of mass about the axis of rotation.

4.2 PHYSICAL SIGNIFICANCE OF MOMENT OF INERTIA

In rotational motion, a torque is required to produce angular acceleration in a body. In other words, a body cannot change by itself its state of rotational motion. This inability or property of the body is called rotational inertia or moment of inertia. It plays the same role in rotational motion as is played by mass in linear motion.

Larger the value of moment of inertia of a body, more is the rotation inertia of the body. This property has been put to a **great practical use**. The machines such as steam engine and the automobile engine, etc. that produce rotational motion have a disc with a large moment of inertia, called a flywheel.

4.3 RADIUS OF GYRATION

It is defined as the distance from the axis of rotation at which if whole mass of the body were supposed to be concentrated, the moment of inertia would be the same as with the actual distribution of mass.

It is denoted by K.

The moment of inertia of a body of mass M and radius of gyration K is given by

$$I = MK^2$$

4.4 THEOREM OF PERPENDICULAR AXIS

This theorem is applicable to bodies which are planar. A planar body (or a plan lamina) is a flat body whose thickness is very small compare to other dimensions such as length, breadth or radius. The theorem is stated as follows:

The moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of the moments of inertia of the lamina about any two mutually perpendicular axes in its own plane intersecting each other at the point through which the perpendicular axis passes.

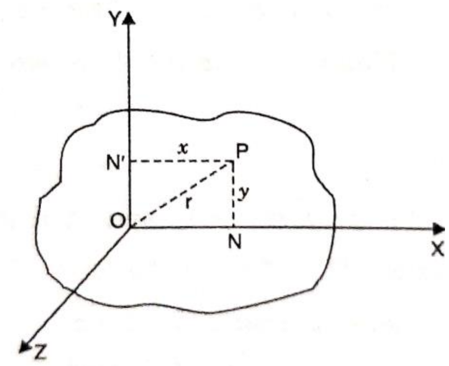


Fig. 4.2

Proof. Consider a plane lamina lying in the XOY plane [Fig. 4.2). The lamina can be supposed to be made up of a large number of particles. Consider a particle of mass m at P. From P, drop perpendiculars PN and PN' on X-axis and Y-axis respectively.

Now, PN' = x and PN = y

Moment of inertia of particle about X-axis = my^2

Moment of inertia of the whole of lamina about X-axis, $I_x = \sum my^2$

Moment of inertia of the whole of lamina about Y-axis, $I_y = \sum mx^2$

Moment of inertia of the whole of lamina about Z-axis, $I_z = \sum mr^2$

But $r^2 = x^2 + y^2$

$\therefore \quad I_z = \sum m\,(x^2 + y^2) = \sum mx^2 + \sum my^2$ or $I_z = I_y + I_x$.

4.5 THEOREM OF PARALLEL AXES

This theorem is applicable not only to a plane lamina but also to a three- dimensional body. It is stated as follows:

The moment of inertia of a body about any axis is equal its moment of inertia about a parallel axis through its centre of gravity plus the product of the mass of the body and the square of the perpendicular distance between the two parallel axes.

Proof. Let I be the moment of inertia of a Plane lamina about an axis YY [Fig. 4.3]. Let G be the centre of gravity of the lamina. Y'Y' is an axis parallel to the given axis and passing through the centre of gravity G of the lamina. Let I_g be be moment of inertia of the lamina about Y'Y'.

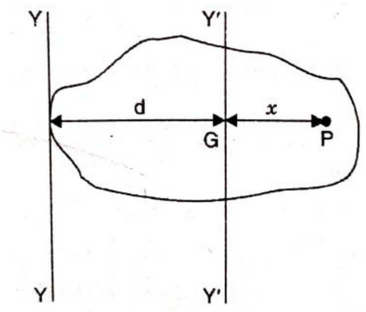


Fig. 4.3

Consider a particle of mass m at P. Let d be the perpendicular distance between parallel axes YY and Y'Y'. Let GP = x.

Moment of inertia of the particle about YY = $m(x + d)^2$

Moment of inertia of the whole of lamina about YY,

$$I = \sum m(x + d)^2 = \sum m(x^2 + d^2 + 2xd)$$

or $$I = \sum mx^2 + \sum md^2 = \sum 2mxd$$

But $\sum mx^2 = I_g$ and $\sum md^2 = (\sum m)\, d^2 = Md^2$

where M $(= \sum m)$ is the mass of the lamina.

Also, $\sum 2mxd = 2d \sum mx$

$$\therefore \quad I = I_g + Md^2 + 2d \sum mx \qquad \ldots(i)$$

The lamina will balance itself about its centre of gravity. So, the algebraic sum of the moments of the 'weights of constituent particles about the centre of gravity G should be zero.

$$\therefore \qquad \sum mx \times x = 0 \qquad \text{or} \qquad g\sum mx = 0$$

or $$\sum mx = 0 \qquad [\because g \neq 0]$$

From equation (1), $\boxed{I = I_g + Md^2}$

which is the mathematical statement of the theorem of parallel axes.

4.6 MOMENT OF INERTIA OF A RING

(i) **About an axis passing through the centre of the ring and perpendicular to the plane of the ring.** Consider a ring of mass M and radius R [Fig. 4.4]. Let XOY be an axis passing through the centre of the ring and perpendicular to the plane of the ring. Let dx be the length of an infinitesimally small element of the ring.

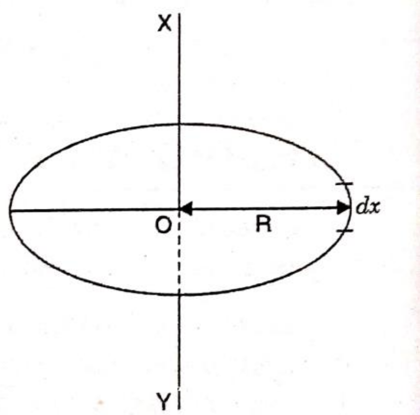


Fig. 4.4

Mass per unit length of ring = $\frac{M}{2\pi R}$,

Mass of element = $\frac{M}{2\pi R}dx$

Moment of inertia of element about the axis XOY

$$= \left(\frac{M}{2\pi R}\, dx\right)R^2 = \frac{M}{2\pi R}dx$$

Let I be the moment of inertia of the whole of ring about the axis XOY.

$$I = \int_0^{2\pi R} \frac{MR}{2\pi} dx = \frac{MR}{2\pi} \int_0^{2\pi R} dx = \frac{MR}{2\pi} [x]_0^{2\pi R}$$

$$= \frac{MR}{2\pi}(2\pi R - 0) \qquad \text{or} \qquad I = MR^2$$

So, the moment of inertia of a ring about an axis passing through the centre of the ring and perpendicular to the plane of the ring is the product of mass of the ring and the square of the radius of the ring.

(ii) About a diameter of the ring. Let I_d be the moment of inertia of the ring about diameter AB [Fig. 4.5]. Since ring has a symmetrical shape, therefore its moment of inertia about any other diameter would be the same. So, moment of inertia of the ring about CD is also I_d.

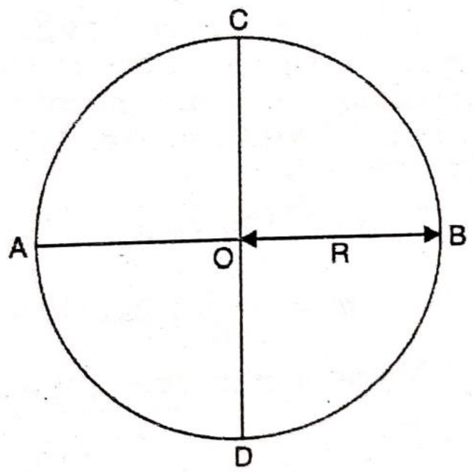


Fig. 4.5

Applying theorem of perpendicular axis we get

$$I_d + I_d = MR^2 \quad \text{or} \quad 2I_d = MR^2$$

or $$I_d = \frac{1}{2} MR^2$$

(iii) About an axis parallel one of the diameters and touching the ring. Let I_T be the moment of inertia of the ring about the tangent EBF [Fig. 4.6].

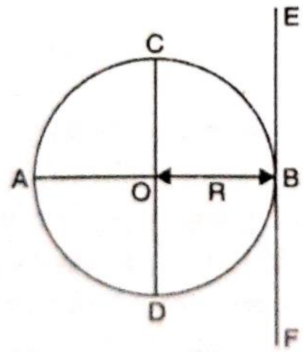


Fig. 4.6

Applying the theorem of parallel axes, we get I_T = M.I. about diameter CD + MR^2

or $$\boxed{I_T = \frac{1}{3} MR^2}$$

(lv) About an axis tangential to the and perpendicular to the plane the ring. Let I_T be the moment of inertia of the case.

What is the moment of inertia of a ring about a tangent to the circle of the ring, in the plane of the ring?

Applying the theorem of parallel axes, we get

I_T = M.I. about an axis passing through the centre of the ring and perpendicular to the plane of the ring + MR^2

Or $$I_T = MR^2 + MR^2 \quad \text{or} \quad \boxed{I_T = 2MR^2}$$

4.7 MOMENT OF INERTIAL OF A DISC

(i) About an axis passing through the centre of the disc and perpendicular to the plane of the disc. Consider a uniform disc of mass M and radius R as shown in Fig. 4.7. XoY is an axis passing through the centre O of the disc and perpendicular to the plane of the disc.

Area of the disc = πR^2, Mass per unit

area of the disc = $\frac{M}{\pi R^2}$

The disc can be imagined to be made up of a larger number of concentric rings. Consider one such concentric ring of radius x. Let its width be dx. Area of the ring is $2\pi x\, dx$.

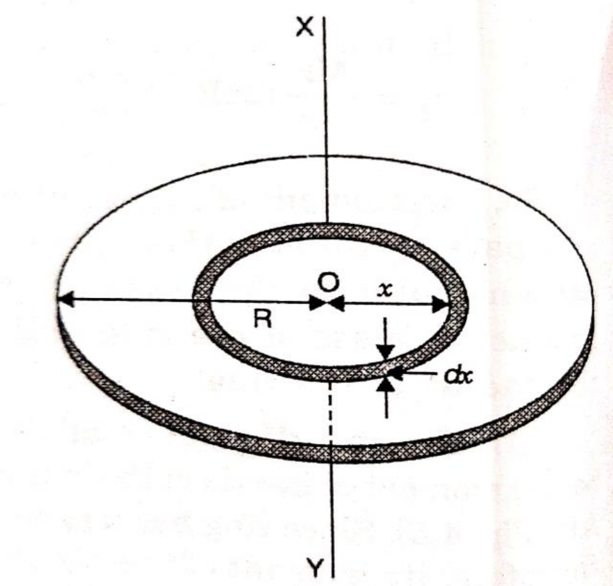


Fig. 4.7

Mass of the ring,

$$m = \left(\frac{M}{\pi R^2}\right) 2\pi x \, dx = \frac{2Mx \, dx}{R^2}$$

Moment of inertia of the ring about XOY

$$= mx^2 = \frac{2Mx \, dx}{R^2} \times x^2 \; \frac{2Mx^3 \, dx}{R^2}$$

If I be the moment of inertia of the disc about XOY, then

$$I = \int_0^R \frac{2Mx^3 \, dx}{R^2} = \frac{2M}{R^2} \int_0^R x^3 \, dx = \frac{2M}{R^2} \left[\frac{x^4}{4}\right]_0^R = \frac{2M}{4R^2} [R^4 - 0]$$

$$= \frac{M}{2R^2} \times R^4 \text{ or } \boxed{I = \frac{1}{2} MR^2}$$

So, the moment of inertia of a disc about an axis passing through the centre of the disc and perpendicular to the plane of the disc is half the product of mass of the disc and the square of the radius of the disc.

What is the moment of inertia of the disc about one of its diameters?

(ii) About a diameter of the disc. Let I_d be the moment of inertia of the disc about a diameter AB [Fig. 4.8]. Since disc has a symmetrical shape, therefore, its moment of inertia about any other diameter would be the same. So, moment of inertia of the disc about CD is also I_d.

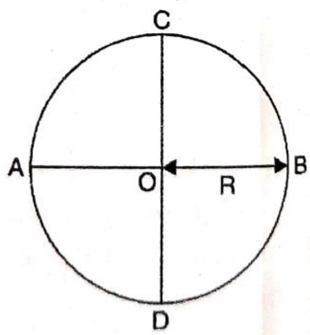


Fig. 4.8

Applying the theorem of perpendicular axis, moment of inertia of the disc about an axis passing through the centre of the disc and perpendicular to its plane = $I_d + I_d$.

or $\frac{1}{2} MR^2 = 2I_d$ or $\boxed{I_d = \frac{1}{4} MR^2}$

(iii) About an axis parallel to one of the diameters and touching the disc. Let us now find the moment of inertia I_T of the disc about a tangent EBP in the plane of the disc (Fig. 4.9). This tangent is parallel to the diameter CD of the disc. Applying the theorem of parallel axes, we may write

I_T = moment of inertia of disc about CS + MR^2

or $I_T = \frac{1}{4} MR^2 + MR^2$ or $\boxed{I_T = \frac{5}{4} MR^2}$

(iv) About an axis tangential to the disc and perpendicular to the plane of the disc. Let I_T be the moment of inertia in this case.

Applying the theorem of parallel axes, we get I_T – M.I. about an axis passing through the centre of the disc and perpendicular to the place of the disc + MR^2.

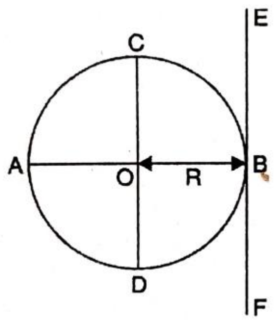


Fig. 4.9

$\therefore I_T = \frac{MR^2}{2} + MR^2$ or $\boxed{I_T = \frac{3}{2} MR^2}$

4.8 MOMENT OF INERTIA OF A SOLID SPHERE

Case I. About a diameter

Fig. 4.10. shows a section through the centre O, of a solid sphere. Let M and R be the mass and radius respectively of the solid sphere.

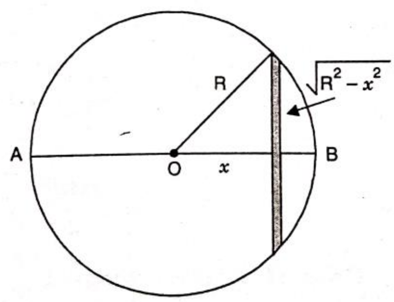


Fig. 4.10

Let, M = mass of sphere

R = radius of sphere

Mass per unit volume (or density) = $\frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}$

Consider a thin circular slice (shown shaded in Fig. 4.10) of the sphere. Let its thickness be dx. Let x e its distance from the centre O of the sphere. The slice is clearly a disc of radius $\sqrt{R^2 - x^2}$.

Area of slice = $\pi(R^2 - x^2)$; Volume of slice = $\pi(R^2 - x^2)\, dx$

Mass of slice = Volume × Density

$$= \pi(R^2 - x^2)\ dx \times \frac{3M}{4\pi R^3} = \frac{3M\ (R^2 - x^2)dx}{4\pi R^3}$$

M.I. of slice (disc) about AB = mass × $\frac{(\text{radius})^2}{2}$

$$= \frac{3M\ (R^2 - x^2)dx}{4R^3} \times \frac{R^2 - x^2}{2} = \frac{3M\ (R^2 - x^2)dx}{8R^3}$$

M.I. of whole of sphere about AB, I = $2\int_0^R \frac{3M\ (R^2 - x^2)dx}{8R^3}$

$$= \frac{2 \times 3M}{8R^3}\int_0^R (R^2 - x^2)^2 dx$$

$$= \frac{3M}{4R^3}\int_0^R (R^4 + x^4 - 2R^2x^2)^2 dx$$

$$= \frac{3M}{4R^3}\left[\int_0^R R^4\ dx + \int_0^R x^4\ dx - \int_0^R 2R^4\ x^4\ dx\right]$$

$$= \frac{3M}{4R^3}\left[R^4\ \left|x\right|_0^R + \left|\frac{x^5}{5}\right|_0^R - 2R^2\left|\frac{x^3}{3}\right|_0^R\right]$$

$$= \frac{3M}{4R^3}\left[R^5 + \frac{R^5}{5} - \frac{2R^5}{3}\right]$$

$$= \frac{3M}{4R^3} \times R^5\left[1 + \frac{1}{5} - \frac{2}{3}\right] = \frac{3MR^2}{4} \times \frac{15 + 3 - 10}{15}$$

$$= \frac{3MR^2}{4} \times \frac{8}{15} = \mathbf{\frac{2}{5}MR^2}$$

Case II. About a tangent

Applying theorem of a parallel axes, moment of inertia of sphere about a tangent

$$= \frac{2}{5}MR^2 + MR^2 = \mathbf{\frac{7}{5}MR^2.}$$

Sample Problem 4.1. A thin uniform wire of length *l* and mass *m* is bent in the form of a semi-circle. Calculate its moment of inertia about an axis (Y'OY') passing though the free ends.

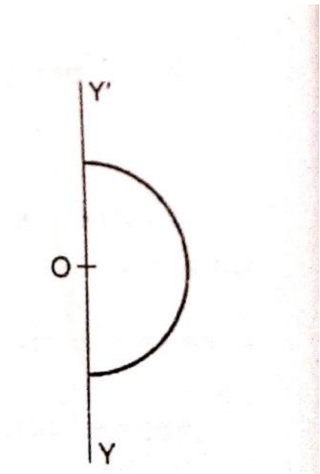


Fig. 4.11

Solution. $\pi r = l$ or $r = \frac{l}{\pi}$

$$I = \frac{1}{2}mr^2 = \frac{1}{2}m\frac{l^2}{\pi^2} = \mathbf{\frac{ml^2}{2\pi^2}}$$

Simple Problem 4.2. The moment of Inertia of a uniform circular disc about a diameter is I. What is the moment of inertia about an axis perpendicular to its plane and passing through a point on its rim?

Solution. Applying theorem of parallel axes,

$$\text{M.I. about the given axis} = \frac{1}{2}MR^2 + MR^2$$

$$= \frac{3}{2}MR^2 = 6 \times \frac{1}{2}MR^2 = \mathbf{6.1}$$

EXERCISES

1. What is the moment of inertia of a ring of mass 2 kg and radius 0.50 m about an axis passing through its centre and perpendicular to its plan? Also find the moment of inertia about a parallel axis though its edge.

[**Ans.** 0.5 kg m^2, 1.0 kg m^2]

2. Use the theorem of parallel axes to calculate moment of inertia of a disc of mass 400 g and radius 7 cm about an axis passing through its edge and perpendicular to the plane of the disc.

[**Ans.** 2.94×10^4 g cm^2]

3. Prove that the radii of gyration of a circular ring and circular disc of the same radius about an axis passing through their centres and perpendicular to their plane are in the ratio $\sqrt{2} : 1$.

4. The moment of inertia of a uniform circular disc about its diameter is 100 g cm^2. What is its moment of inertia (i) about a tangent in the plane of the disc (ii) about its central axis perpendicular to the plane of the disc?

[**Ans.** (i) 500 g cm^2, (ii) 200 g cm^2]

4.9 TORQUE

(a) The rotational analogue of force is moment of force. It is also referred to as torque. This quantity measures the turning effect of a force.

The **torque (or moment of force)** about an axis of rotation is a vector quantity, whose magnitude is equal to the product of magnitude of force and the perpendicular distance of the line of action of force from the axis of rotation and its direction is perpendicular to the plane containing the force and perpendicular distance.

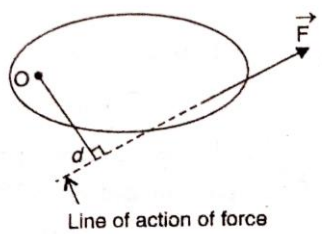


Fig. 4.12

Fig. 4.12 shows a force $\vec{F}$ applied on a rigid body. The body is free to rotate about an axis passing through a point O and perpendicular to the plane of paper. If d is the perpendicular distance of the line of action of force from the point O, then the torque t about the axis of rotation is : $\tau = Fd$.

The symbol + stands for the Greek letter **tau.**

The torque is taken as positive if it tends to rotate the body anticlockwise. If the torque tends to rotate the body clockwise, then it is taken as negative.

The SI unit of torque is Nm. Its dimensional formula is (ML^2T^2).

The dimensions of torque are the same as those of work or energy. It is, however, a very different physical quantity than work. Moment of force is a vector, while work is a scalar.

(b) Torque in Vector Notation. If a force $\vec{F}$ acts on a single particle at a point P whose position with respect to the origin O is given by the position vector $\vec{r}$, then the moment of force, acting on the particle, with respect to the origin O is given by

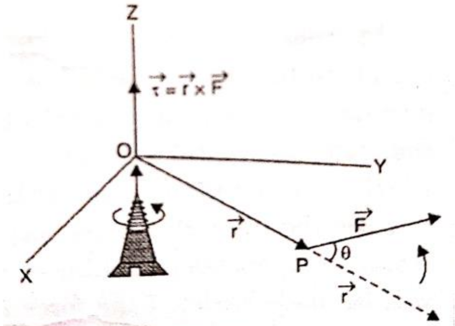


Fig. 4.13

$$\vec{\tau} = \vec{r} \times \vec{F}$$

The direction of $\vec{\tau}$ is perpendicular to the plane of $\vec{r}$ and $\vec{F}$. Its direction is given by right handed screw rule or right hand thumb rule.

The magnitude of $\vec{\tau}$ is given by

$$\tau = rF \sin \theta$$

where r is the magnitude of the position vector $\vec{r}$ i.e., the length OP, F is the magnitude of the force and is the angle between $\vec{r}$ and $\vec{F}$

Now, $\sin \theta = \frac{d}{r}$ or $d = r \sin \theta$

From = n (1), $\tau = F\,(r \sin \theta) = Fr\perp = r\,F_0$

Again $\tau = r\,(F \sin \theta) = r\,F_{\perp} = r\,F_0$

Special Cases. (i) If $r = 0$, then $\tau = 0$. Clearly, a force has no torque if it passes through the point O about which torque is to be calculated. This explains as to why we cannot open or close a door by applying force at the hinges of the door.

(ii) If θ = 0° or 180°, then sin θ = 0.

$\therefore \qquad \tau = r F \sin\theta = 0$

In this case, the line of action of the force passes through point O. Thus, if the line of action of force passes through point O, the torque is zero.

(iii) If θ = 90°, then sin θ = sin 90° = 1 (max. value). So, t is maximum.

$$\tau_{max.} = r F.$$

This explains as to why a handle is fixed perpendicular to the plane of door.

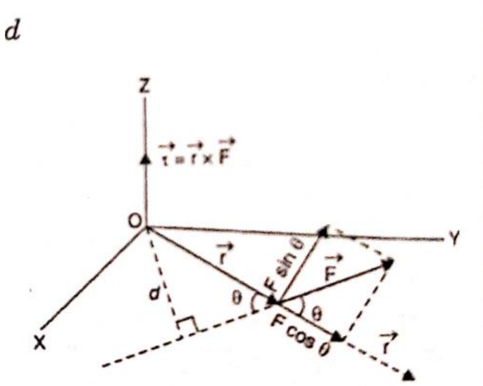


Fig. 4.14

4.10 DEPENDENCE OF TORQUE ON LEVER ARM

In order to increase the turning effect of force, it is not necessary to increase the magnitude of the force itself. We may increase the turning effect of force by changing the point of application of force and by changing the direction of the force. Let us take the case of a heavy door. If we apply a force at a point which is close to the hinges of the door, we may find it quite difficult to open or close the door. But if the same force is applied at a point which is at the maximum distance from hinges, we can easily close or open the door. The task will be made easier if the force is applied at right angles to the plane of the door.

4.11 WORK AND POWER IN ROTATIONAL MOTION

Consider a rigid body which is capable a rotation about an axis through a point O of the rigid body and perpendicular to the plane of the paper.

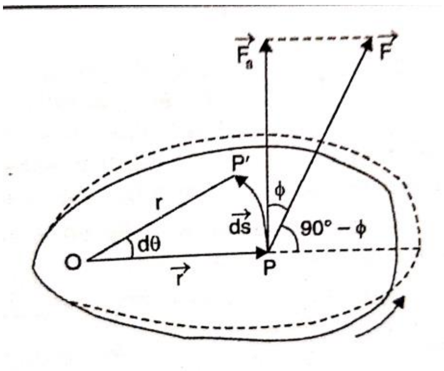


Fig. 4.15

Consider a point P such that the position vector of P with respect to 0 is $\vec{r}$ [Fig.4.15). Suppose an external force $\vec{F}$ is applied at the point P as shown. Let the body turn through an infinitesimally small angle $d\theta$ in a short time dt so that P moves to new position P' such that $\overrightarrow{PP'} = \overrightarrow{ds}$.

*In magnitude, $ds = r\,d\theta$

Work dW done in rotating the body through a small angle $d\theta$ is given by

$$dW = \vec{F} \,.\, \overrightarrow{ds} = \cos\phi$$

where ϕ is the angle between $\vec{F}$ and $\overrightarrow{ds}$.

Now, $dW = (F\cos\phi)\,ds = F_s\,r\,d\theta$

where F_s (= F cos ϕ) is the component of $\vec{F}$ in the direction of $\overrightarrow{ds}$. $\overrightarrow{F_s}$ is perpendicular to $\vec{r}$.

Again, $dW = (F\cos\phi)\,r\,d\theta = (r\,F\cos\phi)\,d\theta$

But $rF\cos\phi = rF\sin(90° - \phi) = \tau$

where $(90° - \phi)$ is the angle between $\vec{r}$ and $\vec{F}$.

$\therefore \qquad dW = \tau = d\theta$

$\int dW = \int_0^\theta \tau\,d\theta$ or $W = \tau\int_0^\theta d\theta = \tau(\theta - 0) = \tau\theta$ or $W = \tau\,\theta$

Here it is assumed that is constant. Thus, the work done in rotating the body through a given angle is equal to the product of the torque and the angular displacement of the body.

Power,

$$P = \frac{dW}{dt} = \frac{d}{dt}(\tau\theta)$$

or $$P = \tau = \frac{d}{dt}(\theta)$$

or $\qquad P = \tau\,\omega$

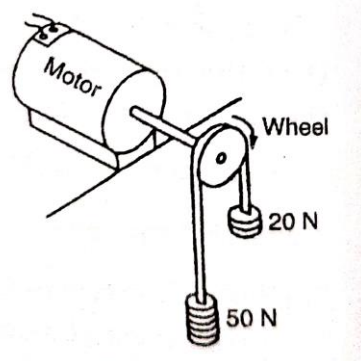


Fig. 4.16

Sample Problem 4.3. The diagram shown an arrangement used to find the output power of an electric motor. The wheel attached to the motor's axle has a circumference of 0.5 m and the belt which passes over it is stationary when the weights have the values shown.

Solution. $\qquad P\,\tau\omega = FR\omega = FR(2\pi v)$

or $\qquad P = F(2\pi R)v$

$\qquad = (50 - 20)\ 0.5 \times 20$ W

= **300 W**

4.12 ANGULAR MOMENTUM

(a) The rotational analogue of momentum is moment of momentum. It is also referred to as angular momentum. This quantity is a measure of the twisting or turning effect associated with the momentum of the particle.

The **angular momentum (or moment of momentum)** about an axis of rotation is a vector quantity, whose magnitude is equal to the product of the magnitude of momentum and the perpendicular distance of the line of action of momentum from the axis of rotation and its direction is perpendicular to the plane containing the momentum and the perpendicular distance.

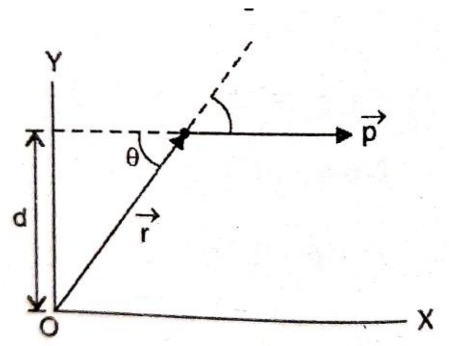


Fig. 4.17

Fig. 4.17 shows a particle having linear momentum $\vec{p}$. Its position vector with reference to point O is $\vec{r}$. The perpendicular distance of the line of action of momentum from O is d. The angular momentum of the particle about an axis passing through O and perpendicular to the plane of the paper is given by :

$$\mathrm{L} = pd$$

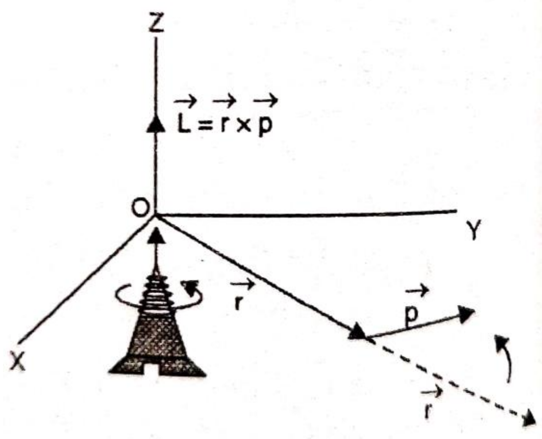


Fig. 4.18

The cgs and SI units of L are g cm^2 s^{-1} and kg m^2 s^{-1} respectively. Its dimensional formula is (ML^2T^{-1}).

(b) Angular Momentum in vector notation. Fig. 4.18 shows position vector $\vec{r}$ and momentum $\vec{p}$ of a particle P in XOY plane. The angular momentum of the particle P with respect to the origin O is given by :

$$\vec{L} = \vec{r} \times \vec{p}$$

The direction of $\vec{L}$ is obtained by applying the right hand rule for the vector product of two vectors. In this case, $\vec{L}$ acts along OZ.

The angular momentum is taken as positive for anti-clockwise rotation and negative for clockwise rotation.

The magnitude of $\vec{L}$ is given by

$$L = rp \sin \theta \qquad \text{...(i)}$$

where is the magnitude of the position vector $\vec{r}$ i.e., the length OP, p is the magnitude of momentum $\vec{p}$ and θ is the angle between $\vec{r}$ and $\vec{p}$ as shown.

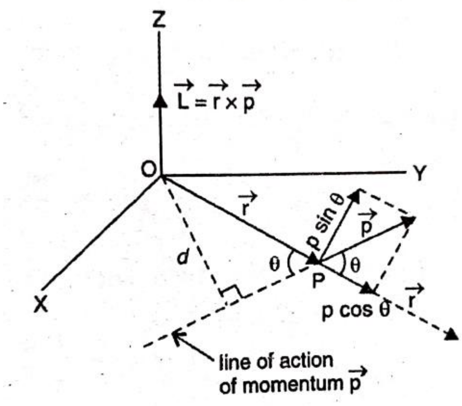


Fig. 4.19

Now, $\sin \theta = \frac{d}{r}$ or $d = r \sin \theta$

From eqn. (i). $L = p\,(r \sin \theta) = pr_{\perp} = pd$

Again, $L = r\,(p \sin \theta) = pr_{\perp} = rp_{\theta}$

Special Cases. (i) If $r = 0$, then L = 0. A particle at O has zero angular momentum about O.

(ii) If θ = 0° or 180°, then sin θ = 0.

∴ $L = rp \sin \theta = 0$

In this case, the line of action of the momentum passes through the point O. Thus, if the line of action of momentum passes through point O, the angular momentum is zero.

(iii) If θ = 90°, then sin θ = sin 90° = 1 (max. value). So, L is maximum.

$$L_{max.} = rp$$

Sample Problem 4.4. An electron of mass 9 × × 10^{-31} kg revolves in a circle of 0.53 A around a nucleus of hydrogen with a velocity of 2.2 × 10^{-31} m s^{-1}. Show that its angular momentum is equal to $\frac{h}{2\pi}$, where his Planck's constant of value 6.6 × 10^{-34} J s. Given: π 3.142.

Solution. Mass, $m = 9 \times 10^{-31}$ kg

Radius, $r = 0.53$ A = 0.53×10^{-10} m

Velocity, $v = 2.2 \times 10^{-6}$ m s^{-1}.

Planck's constant, $h = 6.6 \times 10^{-34}$ J s.

Angular momentum, L = mvr

$= 9 \times 10^{-31} \times 2.2 \times 10^{-6} \times 0.53 \times 10^{-10}$ J s

$= 1.0494 \times 10^{-34}$ J s ...(i)

Again, $\frac{h}{2\pi} = \frac{6.6 \times 10^{-34}}{2 \times 3.142}$ J s = 1.0503×10^{-34} J s ...(ii)

From (i) and (2), angular momentum = $\frac{h}{2\pi}$

4.13 GEOMETRICAL MEANING OF ANGULAR MOMENTUM

Consider a particle of mass m rotating in XY plane, about Z-axis. Let the particle be at P at any time t such that $\overrightarrow{OP} = \vec{r}$. Let the particle be at Q at time $t + \Delta t$ such that $\overrightarrow{OQ} = \vec{r} + \overrightarrow{\Delta r}$. Displament of the particle in time interval Δt is $\overrightarrow{PQ}$. Using triangle law of vectors.

$$\vec{r} + \overrightarrow{PQ} = \vec{r} + \overrightarrow{\Delta r} \quad \text{or} \quad \overrightarrow{PQ} = \overrightarrow{\Delta r}$$

Let $\overrightarrow{\Delta A}$ be the area vector swept by the position vector in time Δt.

$$\overrightarrow{\Delta A} = \frac{1}{2} \vec{r} \times \overrightarrow{\Delta r} \qquad \ldots(1)$$

The direction of $\overrightarrow{\Delta A}$ is perpendicular to the XY plane, i.e., along Z-axis. Dividing both sides of equation (1) by Δt. We get

$$\frac{\overrightarrow{\Delta A}}{\Delta t} = \frac{1}{2} \vec{r} \times \frac{\overrightarrow{\Delta r}}{\Delta t}$$

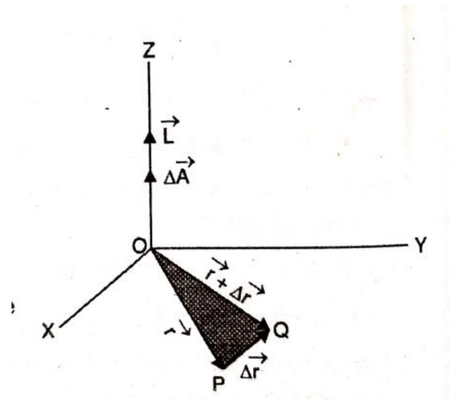


Fig. 4.20

Proceeding to limit as Δt tends to zero, we get

$$\underset{\Delta t \to 0}{\text{Lt}} \frac{\overrightarrow{\Delta A}}{\Delta t} = \underset{\Delta t \to 0}{\text{Lt}} \frac{1}{2} \vec{r} \times \frac{\overrightarrow{\Delta r}}{\Delta t}$$

or $$\frac{\overrightarrow{dA}}{dt} = \frac{1}{2} \vec{r} \times \frac{\overrightarrow{dr}}{dt}$$

but $\frac{\overrightarrow{dr}}{dt}$ = Velocity $\vec{v}$. $\quad \therefore \frac{\overrightarrow{dA}}{dt} = \frac{1}{2} \vec{r} \times \vec{v}$

Time rate of change of positions vector is velocity vector.

or $\quad 2\frac{\overrightarrow{dA}}{dt} = \vec{r} \times \vec{v}$ or $2m\frac{\overrightarrow{dA}}{dt} = \vec{r} \times m\,\vec{v}$

or $\quad 2m\frac{\overrightarrow{dA}}{dt} = \vec{r} \times \vec{p}$ or $\vec{L} = 2m \times \frac{\overrightarrow{dA}}{dt}$

$\frac{\overrightarrow{dA}}{dt}$ is the area swept by the position voter per unit time. It us called areal velocity of the position vector of particle.

The angular momentum of a particle about a given axis in twice the product of mass of the particle and areal velocity of position vector of the particle. This is the geometrical meaning of' angular momentum.

4.14 KEPLER'S SECOND LAW

In the case of a planet moving around the Sun in an elliptical orbit with Sun at one of the foci of the ellipse, the gravitational force always acts along the line joining the planet with the Sun. Thus, the force is always radial or the angular component of the force is always zero.

Since the torque, $\tau = rF_\theta$ therefore, torque acting on the planet is always zero.

Now, $\vec{\tau} = \frac{\overrightarrow{dL}}{dt}$

$\therefore \quad \frac{\overrightarrow{dL}}{dt} = 0$, i.e., the angular momentum ($\vec{L}$) of the planet during its motion is constant.

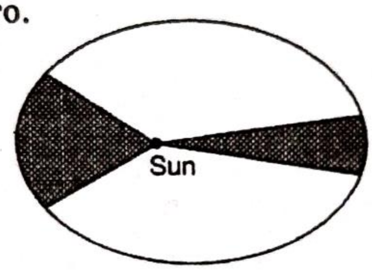


Fig. 4.21

Areal velocity of the planet = $\frac{\text{Angular momentum}}{2m}$. Since mags of the planet and its angular momentum are constant, areal velocity is also constant. This is Kepler's second law of planetary motion which states that the area swept by radius vector of a planet in equal intervals of time is same or areal velocity of a planet is constant. The areas swept in equal time intervals are shown shaded a Fig. 4.21.

4.15 RELATION BETWEEN ANGULAR MOMENTUM AND TORQUE

We know that, $\vec{L} = \vec{r} \times \vec{p}$

Differentiating both sides w.r.t. t, we get

$$\frac{\vec{d}\text{L}}{dt} = \frac{\vec{d}}{dt}(\vec{r} \times \vec{p}) \frac{\vec{d}r}{dt} \times \vec{p} + \vec{r} \times \frac{\vec{d}p}{dt}$$

or $$\frac{\vec{d}\text{L}}{dt} = \vec{v} \times \vec{p} + \vec{r} \times \frac{\vec{d}p}{dt}$$

or $$\frac{\vec{d}\text{L}}{dt} = \vec{r} \times \frac{\vec{d}p}{dt}$$

$$[\therefore \vec{v} \times \vec{p} = \vec{v} \times m\vec{v} = m(\vec{v} \times \vec{v}) = 0]$$

According to Nowton's second law of motion, $\frac{\vec{d}p}{dt} = \vec{\text{F}}$

$\therefore \quad \frac{\vec{d}\text{L}}{dt} = \vec{r} \times \vec{\text{F}}$ or $\frac{\vec{d}\text{L}}{dt} = \vec{\tau}$

So, the time rate of change of the angular momentum of a particle is equal to the torque acting on it. This result is the rotational analogue of the statement-"The time rate of change of the liner momentum of a particle is equal to the force acting on it"

Like all vector equations, equation (1) is equivalent to three scalar equations, namely, $\tau_x = \frac{d\text{L}_x}{dt}$, $\tau_y = \frac{d\text{L}_y}{dt}$ and $\tau_z = \frac{d\text{L}_z}{dt}$,

So, the x-component of the applied torque is given by x-component of the change with time of the angular momentum. Similar results hold for the y – and z-directions.

4.16 ROTATIONAL KINETIC ENERGY

The rotational kinetic energy of a body is the energy possessed by the body by virtue of rotational motion.

Consider a rigid body rotating with uniform angular velocity ω about a given axis. In such a rotation, every particle of the rigid body moves in a circle which lies in a plane perpendicular to the axis and has its centre on the axis. Each of n particles of the body will have the same angular velocity ω about the given axis. Let m_1, m_2, ..., m_n be the masses of the different particles constituting the body. Let r_1, r_2, ..., r_n be their respective distances from the axis of rotation. Let v_1, v_2, ..., v_n be their respective linear velocities such that

$$v_1 = r_1\,\omega,\ v_2 = r_2\,\omega,\ \ldots\ldots,\ v_n = r_n\,\omega$$

Kinetic energy of the particle of mass m_1

$$= \frac{1}{2} m_1\, v_1{}^2 = \frac{1}{2} m_1\,(r_1\,\omega)^2$$

> When a rigid body rotates about a fixed axis, every part of the body has the same regular velocity and angular acceleration. However, the linear velocities and linear acceleration are generally different.

$$= \frac{1}{2}\, m_1\, {v_1}^2\, \omega^2$$

Kinetic energy of the particle of mass m_2

$$= \frac{1}{2}\, m_2\, {v_2}^2 = \frac{1}{2}\, m_2\, (r_2\, \omega)^2$$

$$= \frac{1}{2}\, m_2\, {v_2}^2\, \omega^2$$

Kinetic energy of the particle of mass m_n

$$= \frac{1}{2}\, m_n\, {v_n}^2 = \frac{1}{2}\, m_n\, (r_n\, \omega)^2$$

$$= \frac{1}{2}\, m_n\, {r_n}^2\, \omega^2$$

Total kinetic energy of the rotating rigid body about a given axis is equal to the sum of the kinetic energies of all the constituent particles about that axis.

$$\therefore \text{ Rotational K.E.} = \frac{1}{2}\, m_1\, {r_1}^2\, \omega^2 + \frac{1}{2}\, m_2\, {r_2}^2\, \omega^2 + \ldots + \frac{1}{2}\, m_n\, {r_n}^2\, \omega^2$$

$$= \frac{1}{2}\left(m_1\, {r_1}^2 + m_2\, {r_2}^2 + \ldots + m_n\, {r_n}^2\right) \omega^2 = \frac{1}{2}\left(\sum mr^2\right) \omega^2$$

But $\sum mr^2$ = I, where I is the moment of inertia of the rigid body about the given axis.

$$\therefore \qquad \text{Rotational K.E.} = \frac{1}{2}\, \mathrm{I}\omega^2$$

When $\omega = 1$, rotational K.E. = $\frac{1}{2}$ I or I = 2 × rotational kinetic energy

Thus, the moment of inertia of a rigid body about a given axis in numerically equal to twice its rotational kinetic energy when it is rotating with unit angular velocity.

4.17 ANGULAR MOMENTUM IN TERMS OR MOMENT OF INERTIA

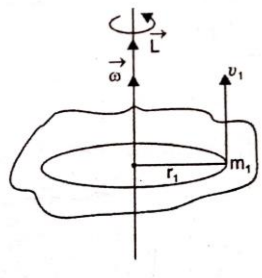


Fig. 4.22

The angular momentum of a rotating rigid body about a given axis is the sum of moments of linear momenta of the constituent particles of the body about the given axis.

Let m_1, m_2, ……, m_n be the masses of the constituent particles of the body. Let r_1, r_2, ….., r_n be their respective distances from the axis of rotation. Suppose the axis of rotation passes through the centre of mass of the body. If v_1, v_2, ..., v_n be the respective velocities of the particles, then the magnitudes of their linear momenta are m_1v_1, m_2v_2, ..., m_vv_n respectively.

Angular momentum of the body about the given axis, L = sum of the moments of momenta of the constituent particles of the body about the given axis.

$$L = (m_1v_1)\, r_1 + (m_2v_2)\, r_2 + \ldots\ldots + (m_nv_n)\, r_n$$

If ω is the angular velocity of the body about the given axis, then the angular velocity of all the constituent particles will also be ω, irrespective of their distances from the axis.

Then, $v_1 = r_1\,\omega,\ v_2 = r_2\,\omega,\ \ldots\ldots,\ v_n = r_n\,\omega$

∴ $L = (m_1v_1\,\omega)\, r_1 + (m_2v_2\,\omega)\, r_2 + \ldots\ldots + (m_nv_n\,\omega)\, r_n$

or $L = m_1{r_1}^2\,\omega + m_2{r_2}^2\,\omega + \ldots\ldots + m_n{r_n}^2\,\omega$

or $L = (m_1{r_1}^2 + m_2{r_2}^2 + \ldots\ldots + m_n{r_n}^2)\,\omega$

or $L = (\sum mr^2)\,\omega$

But $\sum mr^2 = I$

where I is the moment of inertia of the body about the given axis passing through the centre of mass of the body.

∴ $L = I\omega$

If ω = 1, then I = L

Moment of inertia of a body about an axis is numerically equal to its angular momentum about the given axis when the body is rotating with unit angular velocity.

4.18 ANGULAR MOMENTUM CONSERVATION AND THE CASE OF RIGID BODIES

Consider an isolated system on which no external forces act.

For such a system, $(\vec{F})_{external} = 0$ and $(\vec{\tau})_{total} = 0$

∴ $$\frac{d}{dt}(\vec{L})_{total} = 0$$

or $$(\vec{L})_{total} = \text{constant} \qquad \ldots(1)$$

This is law of conservation of angular momentum stated as follows :

The angular momentum of a system is conserved if no external torque acts on it. This is known as the law of conservation of angular momentum.

If $\vec{L_1}, \vec{L_2}, \vec{L_3}, \ldots\ldots, \vec{L_n}$ are the angular momenta of the constituent particles of a system, then it follows from equation (1) that

$$\boxed{\vec{L_1} + \vec{L_2} + \vec{L_3} + \ldots\ldots, + \vec{L_n} = \text{constant}}$$

It may be clearly noted that neither the magnitude nor the direction of the angular momentum changes in the absence of the external torque.

Since $\quad \vec{L_1} + \vec{I\omega}, \quad \therefore \quad \vec{I\omega} = \text{constant}$

In the case of a rotating non-rigid body, the moment of inertia may change from I to I' during rotation. In that case, the angular velocity may change from ω to ω' such that

$$I\omega = I'\omega'$$

4.19 APPLICATIONS/ILLUSTRATIONS OF THE LAW OF CONSERVATION OF ANGULAR MOMENTUM

1. The angular velocity of a planet around the Sun increases when it comes near the Sun.

When planet revolving around the Sun in an elliptical orbit comes near the Sun, the moment of inertia of the planet about the Sun decrease. In order to conserve angular momentum, the angular velocity shall increase. Similarly, when the planet is away from the Sun, there will be a decrease in the angular velocity.

2. The speed of the Inner layers of the whirlwind in a tornado is alarmingly high.

In a tornado, the moment of inertia of air will go on decreasing as the air moves towards the centre. This will be accompanied by an increase in angular velocity much that the angular momentum in conserved.

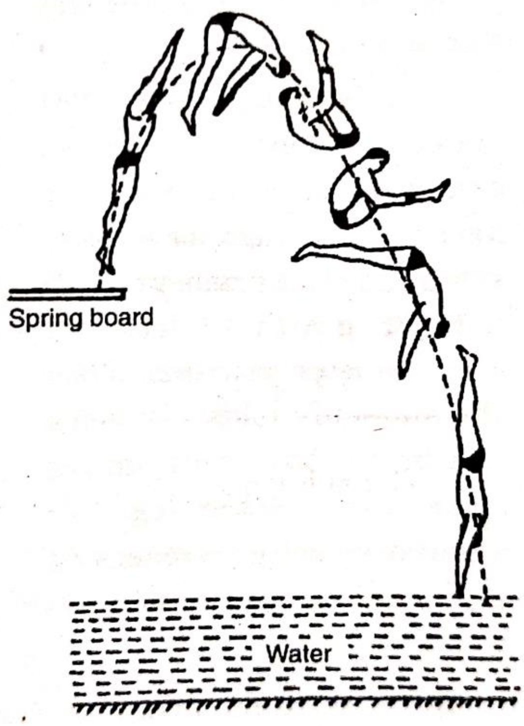


Fig. 4.23

Indian or western, classical dancers performing a pirouette on the toes of one foot display 'mastery' over the principle of conservation of angular momentum.

3. A diver jumping from a spring board performs somersaults in air.

When a diver jumps from spring board, he curls his body by rolling in his arms and legs. This decreases moment of inertia and hence increases angular velocity. He then performs some results. As the diver is about to touch the surface of water, he stretches out his limbs. By so doing he increases his moment of inertia, thereby reducing hi angular velocity.

Suppose the diver (Fig. 4.23) leaves the spring board with an angular speed ω_0 about a horizontal axis through the centre of mass. Suppose he would rotate through half a turn before he strikes the surface of water. If the diver wishes to make a one and one-half turn somersault instead, in the same time, ho must triple his angular speed. The angular momentum will be conserved because only the force of gravity is acting on the diver and this force exerts no torque about his centre of mass. Applying law of conservation of angular momentum, we find that the diver can triple his angular speed only if his rotational inertia about the horizontal axis through the centre of mass becomes one-third of the initial value. This he does by pulling in his arms and legs towards the centre of his body. The greater his initial angular speed and the more he can reduce his moment of inertia, the greater the number of revolutions he can make in a given time.

4. When a bullet is fired from a rifle, the bullet possesses not only linear velocity but also some spin about its axis. Since the angular momentum has to be conserved, therefore, the bullet *maintains its direction of motion.

5. A ballet dancer can vary her angular speed by outstretching her arms and legs.

A ballet dancer (Fig. 4.24) makes use of the law of conservation of angular momentum to vary her angular speed. Suppose a ballet dancer is rotating with her legs and arms stretched outwards. When she suddenly folds her arms and brings the stretched leg close to the other leg, her angular velocity increases on account of decrease in moment of inertia [Fig. 4.24].

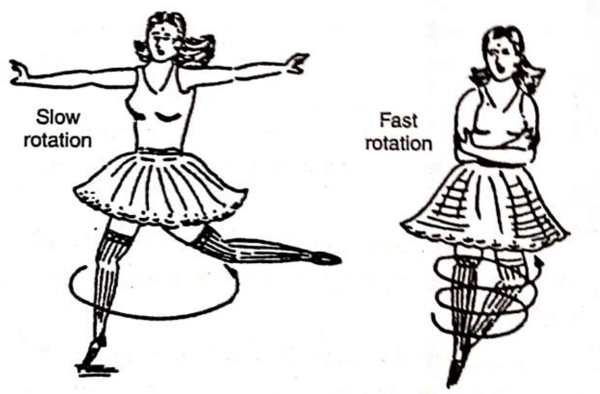


Fig. 4.24

6. A man carrying heavy weights in his hands and standing on a rotating table can vary the speed of the table.

Suppose a man is standing on a rotating table with his arms outstretched. Suppose he is holding heavy weights in his hands. When the man suddenly folds his arms, his angular velocity increases on account of the decrease in moment of inertia (Fig. 4.25).

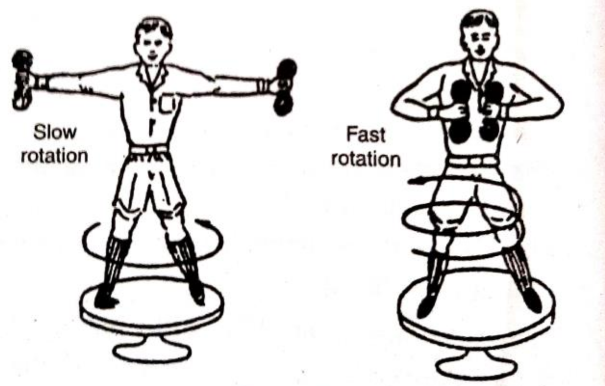


Fig. 4.25

An Important Note. In all these illustrations, the angular momentum is conserved. However, the rotational kinetic energy is not conserved.

$I_1\ \omega_1 = I_2\ \omega_2$ (law of conservation of angular momentum)

or $I_1^2\ \omega_1^2 = I_2^2\ \omega_2^2$ or $\frac{1}{2} I_1^2\ \omega_1^2 = \frac{1}{2} I_2^2\ \omega_2^2$

or $I_1(\frac{1}{2} I_1\omega_1^2) = I_2(\frac{1}{2} I_2\omega_2^2)$. If $I_2 < I_1$, then $\frac{1}{2} I_2\omega_2^2 > \frac{1}{2} I_1\omega_1^2$

So, if the moment of inertia decreases, the rotational kinetic energy increases and vice versa. This increase in kinetic energy is due to the work done (muscular effect) in decreasing the moment of inertia of the body. (Work is done because weights are moved inwardly (towards the axis) against centrifugal force acting outwardly.)

Sample Problem 4.8, The Initial angular velocity of a circular disc of mass M is ω_1. Then two small spheres of mass m are attached gently to two diametrically opposite points on the edge of the disc. What is the final angular velocity of the disc?

Solution: $L_1 = I_1\omega_1 = \frac{1}{2}MR^2\ \omega_1$; $L_1 = [\frac{1}{2}MR^2 + mR^2 + mR^2]\ \omega_1$

No external torque acts. $\therefore L_2 = L_1$

$\therefore [\frac{1}{2}M + 2m]\ R^2\ \omega_2 = \frac{1}{2}MR^2\ \omega_1$ or $\omega_2 = \left(\frac{M}{2} \times \frac{2}{M+4m}\right) \omega_1 = \frac{\mathbf{2}}{\mathbf{M+4m}}\ \boldsymbol{\omega_1}$

ELERCISES

1. A Physics teacher sits on a stool that is free to rotate nearly without friction about a vertical axis. (See Fig. 4.26) Her outstretched hands each hold a large mass so that her rotational inertia is 12.00 kg m². By pulling her arms in close to her body she is able to reduce her rotational inertia to 6.00 kg m². If her students start her spinning at 0.500 rad/s, what is her rotational speed after she draws her arms in ?

(a) (b)

Fig. 4.26

[**Ans.** 1.00 rad s^{-1}]

2. The Sun rotates around itself once in 27 days. What will be the period of its rotation if the Sun were to expand to twice its present radius ? Suppose the Sun is a sphere of uniform density. Given : moment of inertia of sphere = $\frac{2}{5}$ × (mass) × (radius)^2.

[**Ans.** 108 days]

3. Prove that for an Earth satellite, the ratio of its velocity at apogee (when farthest from the Earth) to its velocity at perigee (when nearest to Earth) is equal to the inverse ratio of its distances from apogee and perigee.

4.20 CONCEPT OF ROLLING MOTION

Rolling motion of a wheel is a combination of purely rotational motion and purely translational motion. (*a*) The purely rotational motion : all points on the wheel move with the same angular speed ω, Points on the outside edge of the wheel all move with the same linear speed $v = v_{cm}$. The linear velocities v of two Buch points, at top (T) and bottom (P) of the wheel, are shown. (b) The purely translational motion : all points on the wheel move to the right with the same linear velocity v_{cm} as the centre of the wheel. (c) The rolling motion of the wheel is the combination of Fig. 4.27 (a) and (b).

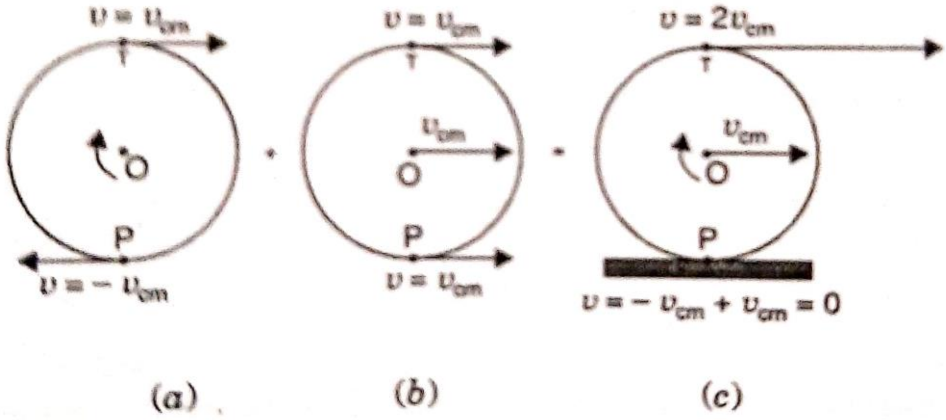


Fig. 4.27

4.21 KINETIC ENERGY OF ROLLING BODY

(i) Kinetic energy of rolling body = Translational kinetic energy + Rotational kinetic energy

$$= \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}Mv^2 + \frac{1}{2}MK^2\frac{v^2}{R^2} \qquad \boxed{= \frac{1}{2}Mv^2\left[1 + \frac{K^2}{R^2}\right]}$$

(ii) Kinetic energy of rolling body = $\frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$

$$= \frac{1}{2}MK^2\omega^2 + \frac{1}{2}MK^2\omega^2 \qquad = = \frac{1}{2}M\omega^2\,[R^2 + K^2]$$

(iii) $$\frac{\text{Rotational kinetic energy}}{\text{Translational kinetic energy}} = \frac{\frac{1}{2}I\omega^2}{\frac{1}{2}Mv^2} = \frac{MK^2\,\omega^2}{MR^2\,\omega^2} = \boxed{\frac{K^2}{R^2}}$$

(iv) $$\frac{\text{Rotational kinetic energy}}{\text{Total energy of rolling body}} = \frac{\frac{1}{2}I\omega^2}{\frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2}$$

$$= \frac{MK^2\,\omega^2}{MR^2\,\omega^2 + MK^2\,\omega^2} = \frac{K^2}{R^2 + K^2}$$

(v) $$\frac{\text{Translational kinetic energy}}{\text{Total energy of rolling body}} = \frac{\frac{1}{2}Mv^2}{\frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2}$$

$$= \frac{Mv^2}{Mv^2 + MK^2\frac{v^2}{R^2}} = \frac{1}{1 + \frac{K^2}{R^2}} = \boxed{\frac{R^2}{R^2 + K^2}}$$

Sample Problem 4.6. A solid cylinder rolls down an inclined plane. Its mass is 2 kg and radius 0.1 m. If the height of the inclined plane is 4 m, what is its rotational kinetic energy when it reaches the foot of the plane?

Solution $= \frac{1}{2} Mv^2 + \frac{1}{2} I\omega^2 = mgh$ or $= \frac{1}{2} mr^2\omega^2 + \frac{1}{2}\left(\frac{1}{2} mr^2\right)\omega^2 = mgh$

or $\frac{1}{2} mr^2\omega^2 = mgh$ or $\omega^2 = \frac{4gh}{3r^2}$

rotational K.E. $= \frac{1}{2} I\omega^2 = \frac{1}{2} \times \frac{1}{2} mr^2 \times \frac{4gh}{3r^2}$

$$= \frac{mgh}{3} = \frac{2 \times 9.8 \times 4}{3} \text{ J} = \mathbf{26.13 \text{ J}}$$

4.22 IMPORTANT TERMS

(i) Periodic motion is that motion which repeats itself after equal intervals of time. The interval of time is called the time period of periodic motion.

Examples. (i) Motion of planets around the Sun. **(ii)** Rotation of Earth about its polar axis. **(iii)** Motion of Moon around the Earth. **(iv)** Motion of Halley's comet around the Sun. **(v)** Motion of the pendulum of a wall clock. **(vi)** Motion of the balance wheel of a watch. **(vii)** Motion of the hands of a clock.

(ii) If a body moves back ond forth repeatedly about a mean position, it is said to possess oscillatory or vibratory motion.

Examples. (i) Motion of the pendulum of a wall clock. **(ii)** Vibrations of the wire of a 'sitar'. **(iii)** Vibrations of the drum of a 'tabla'. **(iv)** Oscillations of a mass suspended from a spring. **(v)** Motion of liquid in a U-tube when the liquid is once compressed in one limb and then left to itself. **(vi)** A weighted test tube floating in a liquid executes oscillatory motion when pressed down and released.

An oscillatory motion is always periodic. A periodic motion may or may not be oscillatory. So, oscillatory motion is merely a special case of periodic motion. As an example, the motion of the planets around the Sun is periodic but not oscillatory.

(iii) Harmonic oscillation is that oscillation which can be expressed in terms of single harmonic function (sine function or cosine function).

(iv) Non-harmonic oscillation is that oscillation which cannot be represented by a single harmonic function.

(iv) Any function which repeats itself after regular intervals of time is called periodic function.

4.23 SIMPLE HARMONIC MOTION

Simple harmonic motion is defined as such an oscillatory motion about a fixed point (mean position) in which the restoring force is always proportional to the displacement from that point and is always directed towards that point.

If a particle suffers a small displacement x from its mean position, then the magnitude of restoring force F is given by

$$F = - kx \qquad \ldots(1)$$

where k is known as the force constant. Its SI unit is N m^{-1}. The negative sig in equation (1) indicates that the restoring force is directed towards the mean position.

4.24 GEOMETRIC DEFINITION OF SIMPLE HARMONIC MOTION

Consider a particle moving with uniform speed along the circumference of a circle of radius 'a'. This circle is known as the reference circle while the particle is known as the reference particle or generating point.

Let P be the position of the reference particle at any time. At that time, N is the projection of the reference particle on the diameter X'OX. When the reference particle moves from X to Y, the projection N moves from X to O. When the reference particle moves from Y to X', the projection N moves from 0 to X'. Similarly, when the reference particle moves from X' to X via Y', the projection N moves from X' to X. Thus as the reference particle completes one revolution, the projection N completes one vibration about the mean position O.

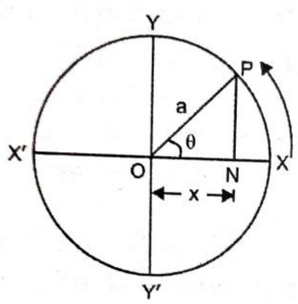


Fig. 4.28

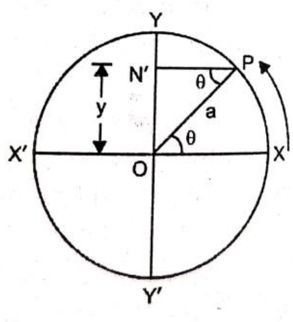


Fig. 4.29

The motion of N along X'OX is simple harmonic motion. The motion of the projection of the reference particle along any other diameter of the circle of reference will also be simple harmonic. This leads to the following geometric definition of simple harmonic motion.

Simple harmonic motion is the projection of uniform circular motion on a diameter of the circle of reference.

4.25 DISPLACEMENT IN SHM

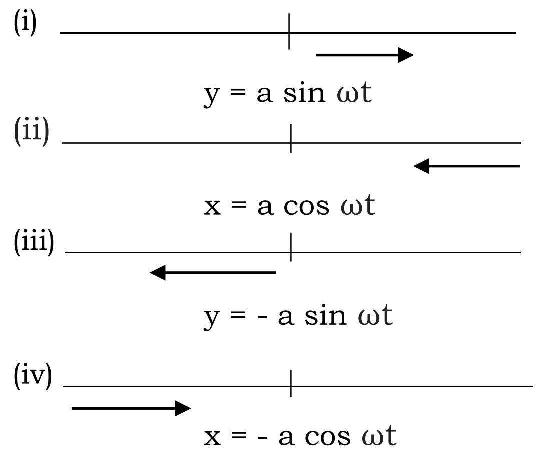
In Fig. 4.28, the magnitude of the displacement of N, from the *mean position, at any instant is given by

$$x = a \cos \theta$$

In rt. ∠d ΔONP of Fig. 4.28,

$$\cos \theta = \frac{x}{a}$$

When to use different equations?



where a is the radius of the reference circle and θ is the angle covered by the reference particle in time t. If ω be the uniform angular velocity of the reference particle, then

$$x = a \cos \omega t \qquad [\because \omega = \frac{\theta}{t}]$$

If the projection N' of the reference particle is taken on the diameter YOY', then

$$y = a \sin \theta$$

or $$y = a \sin \omega t$$

4.26 AMPLITUDE IN SHM

The quantity 'a' represents the maximum magnitude of displacement. It varies between '+a' and '-a'. The quantity 'a' is called the amplitude of the harmonic oscillation. It is also known as 'displacement amplitude'.

The amplitude of a vibrating particle is its maximum displacement from the mean position to one extreme position.

4.27 VELOCITY IN SHM

Differentiating $y = a \sin \omega t$ w.r.t. time t, we get

$$\frac{d}{dt}(y) = \frac{d}{dt}(y \sin \omega t)$$

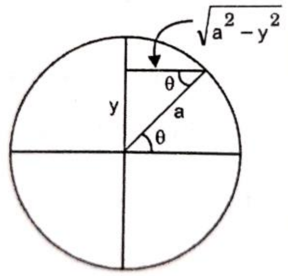


Fig. 4.31

or velocity, $v = a\,\omega \cos \omega t$ or $v = a\,\omega \cos \theta$

or $$v = a\,\omega \frac{\sqrt{a^2 - y^2}}{a}$$

or $$v = a\,\omega\sqrt{a^2 - y^2}$$

At time mean position, $y = 0$. $\therefore$ $v = a\,\omega$ (maximum value)

At time extreme position, $y = a$

$\therefore v = \omega\sqrt{a^2 - a^2}$ or $v = 0$ (minimum value)

Conclusion

A particle in SHM has maximum velocity at mean position and zero velocity at the extreme position.

4.28 ACCELERATION IN SHM

Differentiating $v = a\omega \cos \omega t$ w.r.t. time t, we get $\frac{d}{dt}(v) = \frac{d}{dt}(a\omega \sin \omega t)$

or acceleration $= a\omega \frac{d}{dt}(\sin \omega t) = -a\omega^2 \sin \omega t = -\omega^2 (a\omega \sin \omega t)$

or acceleration $= -\omega^2 y$

At the mean position, $y = 0$ $\therefore$ acceleration = zero

At the extreme position, $y = a$ $\therefore$ acceleration = $-\omega^2 a$

4.29 RESTORING FORCE AND FORCE CONSTANT

(i) Restoring force is that force which brings the system back to its original position on the removal of the external force.

Consider a particle executing simple harmonic motion. As the particle moves away from its equilibrium position, a force acts on the particle. This force is always directed towards the equilibrium position and is called restoring force. If m be the mass of the particle, then restoring force is the product of mass m and acceleration ($-\omega^2 y$).

Restoring force = $-\,m\,\omega^2 y$

At the mean position, restoring force = 0 ($\therefore y = 0$)

At the extreme position, restoring force = $-\,m\,\omega^2 a$ ($\therefore y = a$)

(ii) Force constant is restoring force per unit displacement.

Ignoring negative sign, $\frac{F}{y} = m\omega^2 = m\frac{F}{y} = k$

4.30 TIME PERIOD OR PERIODIC TIME OF SHM

Time period is defined as the time taken by the oscillating particle to complete one oscillation. It is equal to the time taken by the reference particle to complete one revolution. In one revolution, the angle covered by the reference particle is 2π radian and T is the time taken. If ω be the uniform angular velocity of the reference particle, then

$$\omega = \frac{2\pi}{T} \quad \text{or} \quad T = \frac{2\pi}{\omega}$$

4.31 FREQUENCY

It is the number of oscillations (or vibrations) completed per unit time. It is denoted by υ.

In time T second, one vibration is completed.

In 1 second, $\frac{1}{T}$ vibrations are completed.

or $$\upsilon = \frac{1}{T} \quad \text{or} \quad \upsilon T = 1$$

Also, $$\omega = \frac{2\pi}{T} = 2\pi \times \frac{1}{T} 2\pi\upsilon$$

So, equation of simple harmonic motion may also be written as under:

$$x = a \cos (2\pi\upsilon t + \phi_0)$$

the unit of υ is s^{-1} of hertz of 'cycles per second' (cps)

$$5\ s^{-1} = 5\ Hz = 5\ cps.$$

4.32 PHASE

Phase of a vibrating particle at any instant is the state of the vibrating particle regarding its displacement and direction of vibration at that particular instant.

The argument of the cosine in equation of simple harmonic motion gives the phase of oscillation at time t,

It is denoted by ϕ.

4.33 EPOCH

It is the initial phase of the vibrating particle, i.e., phase at $t = 0$.

At $t = 0, \phi = \phi_0$ $[\because \phi = \omega t + \phi_0]$

If the reference particle starts from its standard position X, then the displacement of the projection on X-axis, at any instant, is given by $x = a \cos \theta$.

If the projection is taken on Y-axis, then

$$y = a \sin \theta.$$

If instead of counting time from the instant when the reference particle crosses X-axis, it le counted from the instant when the reference particle is at A Fig. 4.32), then

$\theta = \omega t - \phi_0$ $[\angle XOA = \phi_0]$

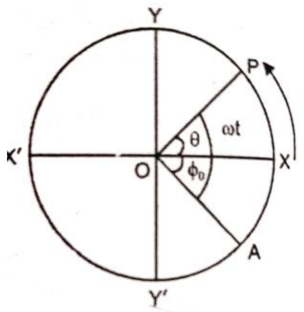


Fig. 4.32

In this case, $x = a \cos (\omega t - \phi_0)$

For projection on projection on Y-axis,

$$y = a \sin (\omega t - \phi_0)$$

The angle $(\omega t - \phi_0)$ is called the phase of the vibrating at time t.

When $t = 0$, phase $= -\phi_0$

which is the initial phase or epoch.

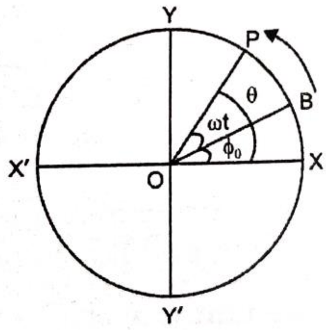


Fig. 4.33

If time is counted from the instant the reference particle is at B (Fig. 4.33), then

$$x = a \cos (\omega t + \phi_0)$$

For projection on Y-axis, $y = a \sin (\omega t - \phi_0)$

In this case, epoch $= + \phi_0$

Note. The phase of a vibrating particle changes continuously with time. But the epoch is a phase constant.

4.34 DYNAMICS OF HARMONIC OSCILLATION

In simple harmonic motion, the restoring force is directly propotional to displacement. If the displacement x is small, then the magnitude of the restoring force is given by

$$F = - kx$$

where k is the force constant. It is the force required to give unit displacement to the body.

If m be the mass of the body, then acceleration, $x = \frac{F}{m}$

[Newton's second law of motion]

or $$x = \frac{-kx}{m} = -\frac{k}{m}x$$

put $\frac{k}{m} = \omega^2$ $\quad$ x represents acceleration

where ω is a constant for the given system.

Then, $$x = -\omega^2\, x$$

or $$x + \omega^2\, x = 0$$

which is the differential equation of linear simple harmonic motion. This is the equation is which simple harmonic motion is expressed.

4.35 GRAPHICAL REPRESENTATION OF PRACTICE DISPLACEMENT, PARTICLE VELOCITY AND PARTICLE ACCELERATION

Consider the equation $y = a\sin(\omega t + \phi_0)$

If epoch ϕ_0 is assumed to be zero, then displacement $y = a \sin \omega t$

velocity $$y = a\omega \cos \omega t$$

or $$y = a\omega \sin\left(\omega t + \frac{\pi}{2}\right)$$

acceleration $$\ddot{y} = -\, a\omega^2 \sin \omega t$$

or $$\ddot{y} = -\omega^2\, (a\sin \omega t) = -\omega^2 y$$

Using these relations, we can calculate the values of displacement, velocity and acceleration for different value of *t*. These values for one complete vibration are given in the following table.

t	0	T/4	T/2	3T/4	T
Displacement y	0 (min.)	+ a (max.)	0 (min.)	- a (max.)	0 (min.)
Velocity $\dot{y}$	$a\omega$ (max.)	0 (min.)	- $a\omega$ (max.)	0 (min.)	$a\omega$ (max.)
Acceleration $\ddot{y}$	0 (min.)	- $a\omega^2$ (max.)	0 (min.)	$a\omega^2$ (max.)	0 (min.)

These results lead us to the following conclusions:

(i) All the three quantities (y, $\dot{y}$ and $\ddot{y}$) vary harmonically with time t (fig. 4.34.)

(ii) The velocity amplitude is ω times the displacement amplitude.

(iii) The acceleration amplitude is ω^2 times the displacement amplitude.

(iv) The velocity is $\frac{\pi}{2}$ ahead of the displacement is phase.

(v) The acceleration is ahead of the velocity in phase by $\frac{\pi}{2}$ on π ahead of displacement.

All the above facts have been represented graphically in Fig. 4.34.

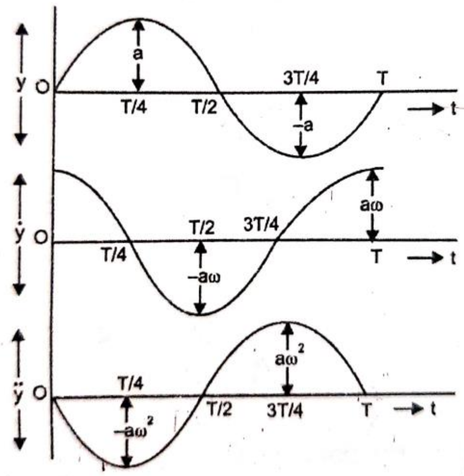


Fig. 4.34

Note. Velocity amplitude, $v_0 = \omega a = 2\pi v a$

acceleration amplitude $g_0 = \omega^2 a = 2\pi^2 v^2 a$

4.36 EXPRESSIONS FOR THE PERIOD AND FREQUENCY IN SHM

We know that $\omega^2 = \frac{k}{m}$ or $\omega = \sqrt{\frac{k}{m}}$

But $\omega = \frac{2\pi}{T}$

∴ $\frac{2\pi}{T} = \sqrt{\frac{k}{m}}$ or $\frac{T}{2\pi} = \sqrt{\frac{m}{k}}$

or $T = 2\pi\sqrt{\frac{m}{k}}$

Depending on what kind of oscillation we discuss, the quantities corresponding to m and k will go to taking different forms and names. As a general nomenclature, k is called the spring factor and m is called the inertia factor.

In general, $T = 2\pi\sqrt{\frac{\text{inertia factor}}{\text{spring factor}}}$

Frequency,

$$\nu = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{\text{spring factor}}{\text{inertia factor}}}$$

Again,

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{\text{force/displacement}}}$$

or $$T = 2\pi\sqrt{\frac{m \times \text{displacement}}{\text{force}}}$$

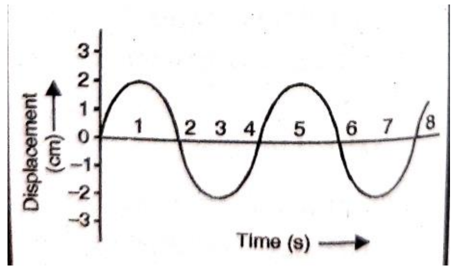
$$= 2\pi\sqrt{\frac{m \times \text{displacement}}{m \times \text{acceleration}}}$$

or $$T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

Frequency, $$\nu = \frac{1}{2\pi}\sqrt{\frac{\text{acceleration}}{\text{displacement}}}$$

CHECKPOINT

Fig. 4.35 shows the displacement-time graph of a simple harmonic oscillator. What is the amplitude, time period and initial phase of the oscillator?



Ans. 2cm, 4 s, zero.

from the origin. So, initial phase is zero.

4.37 DIFFERENTIAL EQUATION OF ANGULAR SIMPLE HARMONIC MOTION

In angular SHM, moment of restoring couple, $\tau \propto \theta$

or $$\tau = - C\,\theta$$

where C is called the torque constant. It is equal to the moment of the couple required to produce unit angular displacement. Its SI unit is N and rad^{-1}

If I be the moment of inertia of the system about the axis of rotation, then

Torque, $$\tau = I\alpha$$

where α is the angular acceleration of the system about the axis of rotation.

Now, $$\alpha = \frac{\tau}{I} = \frac{-C\theta}{I} = -\frac{C}{I}\theta$$

Put $$\frac{C}{I} = \omega^2$$

where ω is a constant of the given system.

$\therefore$ $$\alpha = \omega^2\,\theta$$

or $\frac{d^2\theta}{dt^2} = -\omega^2\,\theta$ or $\boxed{\frac{d^2\theta}{dt^2} + \omega^2\,\theta = 0}$ ...(i)

which is the differential equation of angular simple harmonic motion. As is clear from equation (i), the angular acceleration is directly proportional to angular displacement and is directed towards the means position.

Expressions for time period and frequency in angular SHM

We know that $\omega^2 = \frac{C}{I}$ or $\omega = \sqrt{\frac{C}{I}}$

or $\frac{2\pi}{T} = \sqrt{\frac{C}{I}}$ or $\frac{T}{2\pi} = \sqrt{\frac{I}{C}}$

or $T = 2\pi\sqrt{\frac{I}{C}} = 2\pi\sqrt{\frac{\text{moment of inertia}}{\text{torque contant}}}$ Frequency, $v = \frac{1}{T} = \frac{I}{2\pi}\sqrt{\frac{C}{I}}$

Simple problem 4.7. On an average, a human heart is found to beat 75 times in a minute. Calculate its beat frequency and period.

Solution: The beat frequency of hear

= 75/(1 min) = 75/(60 s)

= 1.25 s^{-1} = 1.25 Hz

Time period T = 1/(1.25 s^{-1}) = **0.8 s**

Sample Problem 4.8. A person normally weighing 60 kg stands on a platform which oscillates up and down harmonically at a frequency of 2.0 s^{-1} and an amplitude of 5.0 cm. If a machine on a platform gives the person's weight against time, deduce the maximum and minimum reading it will show. Given : g = 10 m s^{-2}.

Solution. Let P and Q be the extreme positions between which the platform vibrates. Let O be the mean position (Fig. 4.36).

Then, OP = OQ = amplitude, a = 5 cm = 0.05 m

Frequency, v = 2.00 per second

Angular frequency, $\omega = 2\,\pi v = 4\pi$ rad s^{-1}

At P or Q. the acceleration is maximum i.e. $a\omega^2$

At these positions, restoring force = $ma\omega^2$

$= 60 \times 0.05 \times (4\pi)^2$ N

= 474.1 N = 47.41 kg wt.

∴ g = 10m s^{-2} (given)

∴ 1 kg wt.= 10 N

At P, both the weight and the restoring force are directed towards the mean position. So, the effective weight is maximum at P. It is given by

$$W_1 = (60 + 47.41) \text{ kg wt.} = \mathbf{107.41 \text{ kg wt.}}$$

At Q, the weight and the restoring force are opposed to each other. So, the effective weight is minimum and is given by

$$W_2 = (60 - 47.41) \text{ kg wt.} = \mathbf{12.59 \text{ kg wt.}}$$

EXERCISES

1. The displacement x (in metre) of an oscillating particle varies with time t (in second according to the equation

   $$x = 0.02 \text{ Cos}\left[0.5\pi t + \frac{\pi}{3}\right]$$

   Calculate (*a*) amplitude of oscillation (*b*) time period of oscillation (*c*) maximum velocity of particle (d) maximum acceleration of particle.

   [**Ans.** (*a*) 0.02 m (b) 4 s (c) $3.142 \times 10 \text{ ms}^{-1}$ (d) $4.94 \times 10^{-2} \text{ ms}^{-2}$]

2. The vertical motion of a huge piston in a machine is approximately simple harmonic with a frequency of 0.50/s. A block of 10 kg is placed on the piston. What is the maximum amplitude of the piston's SHM for the block and the piston to remain together?

   [**Ans.** 0.99m]

3. A horizontal platform with an object placed on it is executing SHM in the vertical direction. The amplitude of oscillation is 2.5×10^{-2} m. What must be the least period of these oscillations so that the object is not detached from the platform?
   Take g = 10 m s^{-2}. [**Ans.** 0.3142s]

4.38 MOTION OF CANTILEVER

A cantilever is a horizontal beam whose one end is fixed to a rigid support and the other end is free. Consider the free and the cantilever loaded with mass m and depressed downwards by a distance y_0 from its original position. According to Hooke's law, the restoring force is proportional to displacement. This restoring force is opposite to displacement.

$$F = -ky_0$$

Here k is a constant whose value depends on the elastic properties of the cantilever. This restoring forces balances the weight mg of mass m.

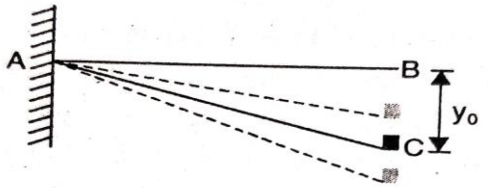


Fig. 4.37

$$mg = ky_0 \text{ or } \frac{m}{k} = \frac{y_0}{g}$$

Let the mass m be pulled down further through a small distance y and released. Now, a restoring force proportional to displacement from equilibrium position AC will produce simple harmonic motion. The time period of simple harmonic motion is given by

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{y_0}{g}}$$

Frequency, $$v = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

4.39 FREE VIBRATIONS

(a) Definition. When a body oscillates with its own natural frequency, it is said to execute free oscillations.

(b) Explanation. The frequency and the time period of free oscillations depend only on the dimensions of the body and the force constant, i.e., the inertia factor and the spring factor. The frequency of free oscillations (or natural oscillations) is given by

$$v_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

(c) Examples. (i) When a stretched string is plucked, it executes free vibrations. **(ii)** When air is blown gently across the mouth of a test tube or a bottle, the free oscillations are produced. **(iii)** When a tuning fork is struck against a rubber pad, the prongs begin to execute free oscillations.

4.40 FORCED VIBRATIONS

(a) Definition. When a body is maintained in a state of oscillation by a strong periodic force of frequency other than the natural frequency of the body, the oscillations are called forced vibrations.

(b) Explanation. The frequency of forced oscillations is different from the natural frequency of the body. It is equal to the frequency of the applied force.

Let an external periodic force of frequency v be applied to a body A of natural frequency v_0. The body A will start oscillating with frequency v and not v_0. The external force is called the 'driver' while the body A is called the 'driven oscillator'.

The amplitude of the forced oscillations is determined by the difference between the frequency of the applied force and the natural frequency. If the difference in frequencies is large, then the amplitude will be small. It is possible to obtain forced oscillations of larger amplitude by applying a series of small forces at proper frequency as in the case of 'swing'.

(c) Examples. (i) Press the stem of a vibrating tuning fork against the top of a tabla. The tabla will suffer forced oscillations. **(ii)** Hold the bob of a simple pendulum and give it any number of oscillations in unit time. **(iii)** The sound boards of stringed musical instruments suffer forced oscillations.

4.41 RESONANT VIBRATIONS AND RESONANCE

(a) Definition. When a body is maintained in a state of oscillation by periodic force having the same natural frequency as that of the body, the oscillations are called resonant or sympathetic oscillations. Resonant oscillations are merely a special case of forced oscillations. The phenomenon of producing resonant oscillations is termed as resonance.

The phenomenon of producing oscillatory motion in a system by an external periodic force having the same frequency as that of the natural frequency of the system is called **resonance.**

(b) Explanation. The amplitude of the forced oscillations depends on the difference between the natural frequency v_0 of the body and the frequency v of the applied force. The amplitude becomes larger as the difference between the frequencies decreases. So, the amplitude becomes maximum when the two frequencies are exactly equal to each other. These facts are graphically represented in Fig. 4.38.

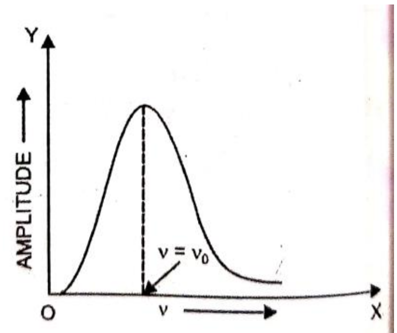


Fig. 4.38

(c) Examples. (i) All mechanical structures have one or more natural frequencies, and if a structure is subjected to a strong external periodic driving force that matches one of these frequencies, the resulting oscillations of the structure may rupture it. The Tacoma Narrows Bridge at Puget Sound, Washington USA was opened on July 1, 1940. Four months later, winds produced a pulsating resultant force in resonance with the natural frequency of the structure. This caused a steady increase in the amplitude of oscillations until the bridge collapsed.

(ii) Aircraft designers make sure that none of the natural frequencies at which a wing can oscillate match the frequency of the engines in flight.

(iii) Resonance can cause vast devastation in an earthquake. If the natural frequency of a building matches the frequency of the periodic oscillations present within the Earth, then the building will begin to vibrate with large amplitude thereby damaging itself.

It is interesting to note that sometimes, in an earthquake, short and tall structures remain unaffected while the medium height structures fall down. This happens because the natural frequencies of the short structures happens to be higher and those of taller structures lower than the frequency of the seismic wages.

(iv) Soldiers are asked to break step while crossing a bridge. If the soldiers march in step, there is a possibility that the frequency of the foot steps may match the natural frequency of the bridge. Due to resonance, the bridge may the start oscillating violently, thereby damaging itselg.

Tacoma Narrows Bridge was known during its short life as Galloping Gertie because it oscillated in the wind. It took a wind of only 42 mph to make it collapse.

SUMMARY

- Moment of inertia of a body about a fixed axis is the sum of the products of the masses of all the constituent particles of the body and the squares of their respective distances from the axis of rotation.
- Moment of inertia of a ring of mass M and radius R about an axis passing through the centre of the ring and perpendicular to the plane of the ring is MR^2.
- Moment of inertia of a disc of mass M and radius R about an axis passing through the centre of the disc and perpendicular to the plane of the disc is $\frac{1}{2}MR^2$.

- Moment of inertia of a solid sphere about its diameter is $\frac{2}{5}MR^2$, where M is the mass of the solid sphere and R is the radius of the sphere.
- The mgnitude of torque is equal to the product of the magnitude of force and the perpendicular distance of the line of action of force from the axis of rotation.
- Work done in rotational motion is the product of torque and angular displacement.
- Power in rotational motion is the product of torque and angular velocity.
- The magnitude of the angular momentum is the product of the magnitude of momentum and the perpendicular distance of the line of action of momentum from the axis.
- Angular momentum is twice the product of mass and areal velocity.
- The time rate of change of angular momentum is equal to torque.
- The product of moment of inertia and angular velocity gives angular momentum.
- If no external torque acts on a system, its angular momentum is conserved.
- The kinetic energy of a rolling body is the sum of translational and rotational kinetic energies of the body.
- In simple harmonic motion, restoring force is proportional to displacement and is directed towards the mean position.
- When a body oscillates with its own natural frequency, the vibrations are said to be free vibrations.
- When a body is maintained in a state of vibration by a periodic force of frequency other than the natural frequency of the body, the vibrations are called forced vibrations.
- When a body is maintained in a state of vibration by periodic force having the same natural frequency as that of the body, the vibrations are called resonant vibrations.

TEST YOURSELF

1. Define moment of inertia. Give its SI unit and dimensional formula.
2. Derive expressions for the moment of inertia of a ring about four different axes.
3. Derive formula for the moment of inertia of a disc about four different axes.
4. Calculate the moment of inertia of a solid sphere about one of its diameters.
5. What is torque ? Discuss the dependence of torque on lever arm.
6. What is angular momentum ? Discuss angular momentum in vector notation.
7. Prove that the time rate of change of angular momentum is equal to the torque.

8. Derive formula for the rotational kinetic energy of a rigid body.
9. Prove that the angular momentum of a rigid body is the product of moment of inertia and angular velocity.
10. Discuss the principle of conservation of angular momentum.
11. Give four illustrations of the principle of conservation of angular momentum.
12. Calculate the kinetic energy of a rolling body.
13. Derive formulae for the following for the case of a body performing simple harmonic motion:
    (i) Displacement (ii) Velocity
    (iii) Acceleration (iv) Time period
    (v) Frequency
14. What do you understand by the following ?
    (i) Free vibrations (ii) Forced vibrations
    (iii) Resonant vibrations.

SECTION - C

5 HEAT: TEMPERATURE AND ITS MEASUREMENT

LEARNING OBJECTIVES

- Concept of heat and temperature on the basis of kinetic energy of molecules.
- Units of heat.
- Basic principles of measurement of temperature.
- Thermoelectricity.
- Seebeck effect.
- Variation of seebeck emf with temperature: Thermocouple : Thermo emf: Neutral and inversion temperatures.
- Thermoelectric thermometer.
- Bimetallic thermometer.
- Platinum resistance thermometers.
- Pyrometry
- Radiation pyrometers.
- Mercury thermometer or clinical thermometer
- Constant volume gas thermometer.
- Criteria for selection of thermometers.

5.1 CONCEPT OF HEAT AND TEMPERATURE ON THE BASIS OF KINETIC ENERGY OF MOLECULES

Heat is a form of energy that produces in us the sensation of warmth. The temperature is the degree of warmth or hotness. The greater the energy of the gas due to random motion of its molecules, the higher is the temperature of the gas. A higher temperature means greater disorderly motion of the molecules. As compared to gases, the molecular motion in liquids exists to a lesser extent. In solids, the atoms vibrate back and forth about their equilibrium positions in different directions. In a solid, the vibrating atoms possess both kinetic and potential energies. In a gas, the interatomic forces are negligibly small. So, we can say that the molecules of a gas possess only the kinetic energy. The total sum of all the energies of the molecules of a system is known as the internal energy of the system. This internal energy increases with the temperature of the body. So, internal energy is also called thermal energy or heat energy. When two systems of different thermal energies are brought in thermal contact, transfer of energy takes place by molecular collisions from the system of higher thermal energy to that of lower thermal energy. The transfer of energy continues until the average kinetic energy per molecule in each system is the same. When the average kinetic energy per molecule in the two systems is the same, then the two systems are in thermal equilibrium. The energy transfer affected due to molecular collisions until the thermal equilibrium is reached is called heat.

5.2 UNITS OF HEAT

In SI, the unit of heat is joule (J). Another commonly used unit of heat is calorie (abbreviated as cal). It is defined as under:

One calorie of heat is the quantity of heat energy which must be supplied to one gram of water (under one atmospheric pressure) to change its temperature from 14°C to 15°C.

$$1 \text{ cal} = 4.186 \text{ J}$$

A bigger unit of heat (in MKS units) is known as kilocalorie (kcal).

$$1 \text{ kcal} = 10^3 \text{ cal}$$

In FPS system, the unit of heat is British thermal unit (Btu).

1 Btu is the quantity of heat required to raise the temperature of one pound of water from 63°F to 64°F.

$$1 \text{ Btu} = 252 \text{ cal} = 0.252 \text{ kcal.}$$

5.3 BASIC PRINCIPLES OF MEASUREMENT OF TEMPERATURE

For getting the numerical value of the temperature of a body, we experimentally measure some physical property of a substance which varies in a regular manner with temperature. As an example, there are many substances whose thermal expansion is regular and reproducible. So, this property can be used for the measurement of temperature. The device used to measure temperature is called a thermometer. The property used for temperature measurement is known as thermometric property. The thermal expansion of a substance, variation of electrical resistance with temperature, energy radiated by a body at different temperatures, generation of thermo emf when two junctions are maintained at different temperatures etc. are some of the thermometric properties. We have different types of thermometers based on different thermometric properties.

5.4 THERMOELECTRICITY

The phenomenon of production of electricity with the help of heat is called **thermoelectricity** and this effect is called **thermoelectric effect.**

This effect comprises of three related effects: Seebeck effect, Peltier effect and Thomson effect.

An arrangement of two wires of different materials joined at their ends to form junctions is called thermocouple.

5.5 SEEBECK EFFECT

This effect was discovered by a German physicist Thomas Johann Seebeck.

In order to understand Seebeck effect, consider a closed circuit consisting of two different metals Cu and Fe. A sensitive galvanometer G is introduced as shown in Fig. 5.1. When one of the junctions is kept hot and the other cold, a current begins to flow from Cu to fe through the hot junction and from Fe to Cu through the cold junction. This current is called **thermoelectric current.** The existence of current implies that there is change its temperature an emf in the circuit. This emf is known as **thermoelectric emf.** The arrangement is called **thermoelectric couple or thermocouple.**

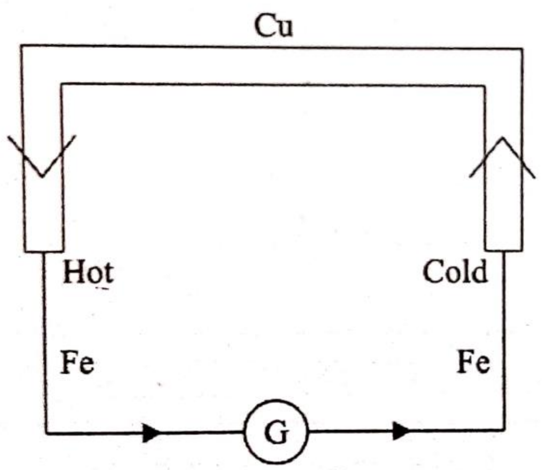


Fig. 5.1. Cu-Fe thermocouple

The phenomenon of generation of an electric current in a thermocouple by keeping its junctions at different temperatures is called **Seebeck effect** or **Thermoelectric effect.**

5.6 VARIATION OF SEEBECK EMF WITH TEMPERATURE; THERMOCOUPLE; THERMO EMF; NEUTRAL AND INVERSION TEMPERATURE

Consider a copper-iron thermocouple. Let one junction of this thermocouple be put in hot oil bath. A thermometer and a stirrer are put in the hot oil bath as shown in Fig. 5.2. The other junction is kept in ice. A sensitive galvanometer G is introduced in the circuit as shown. The deflection of the galvanometer measures the thermo emf. It is measured for different temperatures of the hot junction. A graph is plotted between the temperature *t* of the hot junction and the thermo emf.

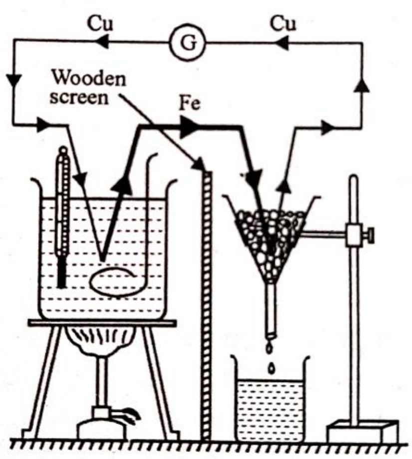


Fig. 5.2. Experimental arrangement

It is clear from the curve that thermo emf increases with an increase in the temperature of the hot junction, becomes maximum for a particular temperature t_n and then again starts decreasing.

The temperature t_n of the hot junction at which the thermo emf becomes maximum is called the **neutral temperature.** It may also be defined as that temperature of the hot Junction at which maximum current flows in a thermocouple. It is independent of the Temp of Hot Junction temperature of the cold junction. It depends only upon the nature of the metals constituting the thermocouple.

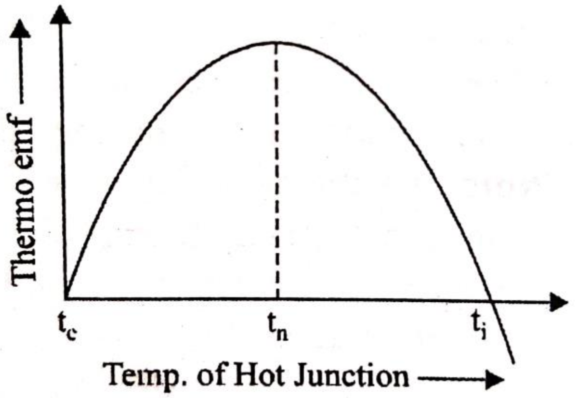


Fig. 5.3. Variation of thermo emf with temperature

The neutral temperature for a copper-iron thermocouple is 275°C i.e., 548 K.

The temperature t_i for which the thermo emf becomes zero and then changes direction (becomes inverted) is called the **temperature of inversion.** It depends upon the temperature of the cold junction. It is as much above the neutral temperature t_n as the temperature of cold junction lies below the neutral temperature.

$$\therefore \qquad t_n - t_c = t_t - t_n \text{ or } 2t_n = t_i + t_c$$

or
$$t_n = \frac{t_i + t_c}{2}$$

So, the neutral temperature is the arithmetic mean of the temperature of inversion and temperature of cold junction.

5.7 THERMOELECTRIC THERMOMETER

Principle. It is based upon Seebeck effect.

We know that if a graph be plotted between the temperature of hot junction (other being kept at 0°C) and the thermo emf developed, it is practically a straight line for a major portion before the neutral temperature is reached.

Construction. For preparing a thermoelectric thermometer, two suitable dissimilar metals are chosen after taking into consideration the range of temperatures to be measured and the accuracy required. This is due to the fact that the thermo emf generated depends upon the nature of the metals used and also on the difference of temperatures of the two junctions.

The wires constituting the thermocouple are welded together at one end. This end forms the hot junction. Portions of the wire near the hot junction are properly insulated from each other by enclosing them in a hard glass capillary tube C.T. The wires are passed through mica discs D, D. These discs are fitted one above the other in a long thin porcelain tube T. The ends of the two wires are connected to terminals T_1 and T_2 which are further connected to compensating leads L_1, L_2 of the same materials as constitute the thermocouple. The leads L_1 and L_2 are welded together at the other end. This end forms the cold junction. It is immersed in ice to keep it at 0°C (or 273 K). Note that the cold junction has been shifted to a convenient place.

A galvanometer G or a millivoltmeter is connected in the circuit. Nowadays, electronic digital voltmeters are commonly used. Such voltmeters have a high internal resistance of the order of 100 mega ohm. So, they do not draw appreciable current. This increases the efficiency of the measurement. The accuracy is also large because the least count of electronic digital voltmeters is less than 10^{-7} volt. If a digital voltmeter is used with a copper-constantan thermocouple, then a temperature difference of 2.5×10^{-3}°C can be easily measured.

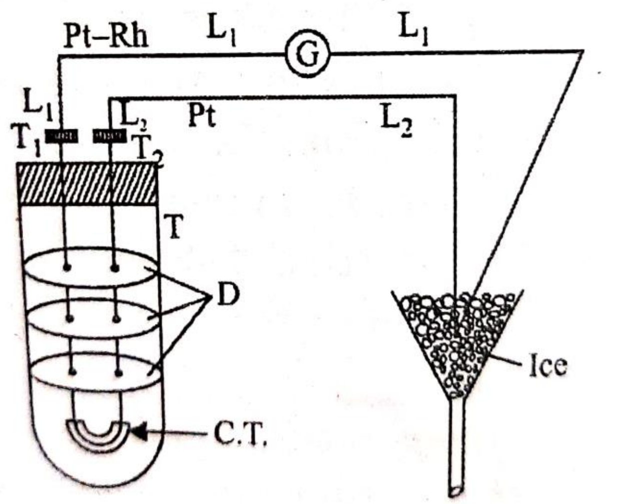


Fig. 5.4. Thermoelectric thermometer

Calibration of the galvanometer. The porcelain tube containing one junction of the thermocouple is placed in a furnace whose temperature can be varied and is also known. The other junction is kept at 0°C. Thermo emf developed is measured by a sensitive potentiometer and the galvanometer is calibrated so as to read the temperature directly.

Measurement of unknown temperature. The porcelain tube containing one junction is put in the furnace where temperature is to be measured. The other junction is kept at 0°C. The temperature in directly read on the galvanometer scale.

For different ranges of temperatures, different thermocouples are used as discussed below:

(i) Copper with cold-iron alloy is used between 1 K and 50 K.
(ii) Copper-constantan thermocouple gives a large thermo emf of 48 μ V to 60 μV per degree temperature difference between its two junction. However it gets oxidised when heated to a temperature higher than 400 K. So, this thermocouple in generally used for measuring temperatures from 50 K to 400 K.
(iii) Iron-nickel thermocouple is used from 400 K to 900 K.
(iv) Platinum und platinum-iridium alloy thermocouple is used from 900 K to 1300 K.
(v) Platinum and platinum-rhodium alloy thermocouple is used from 1300 K to 2000 K.

Advantages

(i) It has a very wide range of measurement from - 200°C to 1400°C.
(ii) It is very sensitive and accurate. It can measure temperature correctly up to 0.05°C.
(iii) It is very cheap and simple in construction.
(iv) It can measure the temperature at a specific point.
(v) Its thermal capacity is low. So, it can be used to measure rapidly varying temperatures, because the hot junction will quickly acquire the temperature of the body in contact with which it has been placed.
(vi) It can be used to determine the temperature of upper region of the atmosphere of the temperature at the top of a mountain. In such a case, one junction is set up in a balloon and the other is kept at ground at a constant temperature.
(vii) Temperatures of very hot bodies and distant bodies can be measured by using radiation pyrometer which employs a thermoelectric thermometer.

Disadvantages

(i) It is difficult to maintain the temperature of cold junction constant for long time.
(ii) Stray emfs do come in due to different temperatures of the various parts of the thermocouple.
(iii) The presence of the neutral temperature restricts their use because the behaviour of the thermocouple is linear only upto neutral temperature.
(iv) Since there is no theoretical formula, therefore, each thermometer has to be calibrated separately.

5.8 BIMETALLIC THERMOMETER

Construction. Bimetallic thermometer consists of a compound bar. This is made of two thin strips, of different metals (usually brass and iron), bonded together at a certain temperature, say T_0. The bar remains flat (unbent) at temperature T_0, as shown in Fig. 5.5(*a*). When temperature rises ($T > T_0$), the metal with larger coefficient of expansion expands more. However, the two strips remain in perfect contact with each other because they are welded together. Consequently, the bar gets bent in an arc with the metal of higher coefficient of expansion (brass) forming the convex side. This is shown in Fig. 5.5(*b*). If temperature T is less than T_0, then the metal with higher coefficient of expansion forms the concave side as shown in Fig. 5.5(*c*).

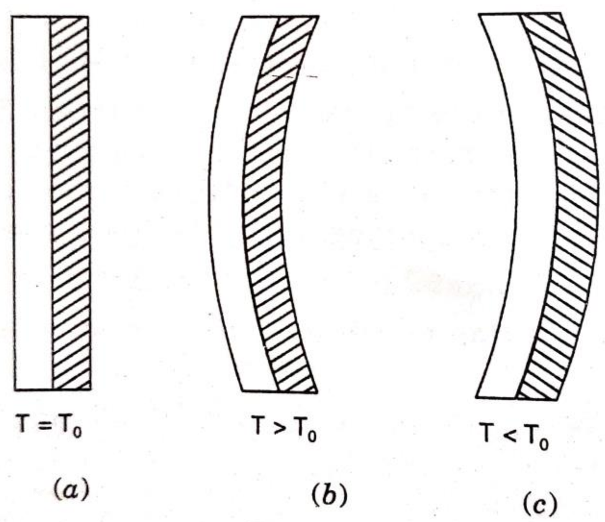


Fig. 5.5

The compound bar (bimetallic strip) is wound round in a spiral form with the metal of larger coefficient of expansion on the outer side. While one end of the strip is fixed, the other end is attached to the spindle of a pointer. The pointer can move over a scale calibrated in centigrade degrees.

Fig. 5.6

Measurement of Temperature. The fixed end of the bimetallic strip is brought in contact with the hot body whose temperature is to be measured. The strip absorbs heat from the hot body. It expands with its length on the outer side increasing more as compared to the inner strip. The coil tends to curl in the clockwise direction and the poionter moves forward on the scale with temperature.

Range of Temperature. The bimetallic thermometers have a small range measurement of temperature. This range is from - 50°C to 500°C.

5.9 PLATINUM RESISTANCE THERMOMETERS

The fact that the electrical resistance of a metal wire increases uniformly over a fairly wide range of temperature has been made use of in electrical resistance thermometers. The variation of the resistance of a metal wire with temperature may be represented by the following approximate relation.

$$R_t = R_0\,(1 + \alpha t) \qquad \ldots(i)$$

Here, R_t is the resistance at t°C, R_0 is the resistance at 0°C and α is the temperature coefficient of resistance. The value of α depends upon the nature material of the wire.

The relation (1) was first suggested by Clausius. This relation does not represent observations accurately even at ordinary temperatures. But Callendar was able to show that pure platinum wire always possesses the same resistance at the same temperature. The variation of resistance of platinum wire with temperature can be represented fairly accurately by the following relation.

$$R_t = R_0\,(1 + \alpha t + \beta t^2)$$

Here, α and β are constants.

In the year 1871, Siemens was the first to construct a platinum thermometer. But the modern version of platinum thermometer is based on improvements by Callendar and Griffiths. In **the platinum thermometer,** a well-annealed pure platinum wire is doubled on itself to avoid induction effects. The wire is wound on a thin flat strip of mica. It is placed at the bottom of a hard glass or glazed porcelain tube provided with an ebonite cap. The free ends of the wire are attached to long leads which pass through holes in several mica sheets. If we have to work upto 700°C, then copper leads can be used and the tube can be of glass. For temperatures higher than 700°C, platinum leads are used and the tube is made of porcelain.

Advantages

(i) It has a wide range from 200°C to 1200°C.

(ii) Temperature can be determined with an accuracy of 0.01°C upto 600°C and with an accuracy of 0.1°C upto 1200°C.

Disadvantages

(i) The bulb of the thermometer has low thermal conductivity and large thermal capacity.
(ii) Beyond 1000°C, there is a danger of contamination by insulating materials.

Sample Problem 5.1. The electrical resistance in ohm of a certain thermometer varies with temperature according to the approximate law : R = R_0 [1 + 5 × 10^{-3} (T-T_0)]

The resistance is 101.6 Ω at the triple-point of water (273.16 K) and 165.5 Ω at the normal melting point of lead (600.5 K). What is the temperature when the resistance is 123.4 Ω ?

Solution: If R_0 is the resistance at temperature T_0, then

$$101.6 = R_0 [1 + 0.005 (273.16 - T_0] \quad \ldots(i)$$

$$165.5 = R_0 [1 + 0.005 (600.5 - T_0] \quad \ldots(ii)$$

Dividing and simplifying, $T_0 = - 49.67$ K

Putting the value of T_0 in equation (i), we get R0 = 38.87

Substituting value in the given equation, we get T = 385.27 K

5.10 PYROMETRY

It is that branch of Physics which deals with the measurement of relined to high temperature. Instruments or devices used to measure high temperature are called pyrometer.

Resistance thermometers and thermoelectric thermometers are called pyrometers because these can be employed to measure temperatures upto 1900 K. At higher temperatures, most of these would melt. So, these cannot be employed to measure temperatures higher than 1000 K.

5.11 RADIATION PYROMETERS

According to Stefan's law, the thermal radiation from a black body depends upon the temperature of the black body. So, thermal radiation can be used to measure high temperature. The instruments or devices based on this principle are called radiation pyrometers. The radiation pyrometers are superior to other pyrometers.

These can be employed to measure the temperature of distant objects like sun. When radiation pyrometer is used to measure the temperature of the sun, it is called **pyrheliometer.**

It may be pointed out that the radiation pyrometers cannot be used to measure a temperature lower than 900 K. This is because the radiation emitted at such a temperature will be too weak to be accurately measured. Moreover, if the hot body is not a black body, then the measured temperature will be less than the temperature of the hot body. This is due to the fact that the working of the radiation pyrometer is based on laws which are true only for black body radiation. Another drawback of radiation pyrometers is that they cannot measure the temperature in the interior of a hot body. They are suitable only for the measurement of surface temperatures

The radiation pyrometers are of two types:

(i) Total radiation pyrometers. These measure the total radiation emitted by the body. The temperature is calculated with the help of Stefan's law.

(ii) Optical or spectral pyrometers. These compare the intensity of radiation of certain wavelength emitted by the body with that of the intensity of radiation of some wavelength emitted by standard lamp to obtain the condition of equal brightness.

The spectral radiation pyrometers are of two types:

(a) The disappearing filament optical radiation pyrometer. In this pyrometer, the Intensity of the standard lamp is varied until its intensity I equal to the intensity of the radiation emitted by the body whose temperature is to be measured.

(b) The polarising optical radiation pyrometer. In this pyrometer, the intensity of the standard lamp is kept constant and the intensity of the radiation emitted by the body is varied by means of a polarising device until the intensity of the radiation is equal to that of the standard lamp.

Fery's total radiation pyrometer. It consists of a concave mirror M made of copper and plated with nickel on the front surface. The mirror has a small circular hole at the centre at which an eye-piece E is fitted. A diaphragm D is placed at the focus of M. The diaphragm D consists of two semi-circular mirrors inclined to each other at an angle of nearly 5° with a central hole of nearly 1.5 mm to allow the radiation to pass through it. A metallic strip S is placed behind D. The surface of S which faces the diaphragm is blackened. To the other surface of S, one junction of a thermocouple TT is connected. The other junction of the thermocouple is kept at 0°C. The thermocouple is connected to a sensitive calibrated galvanometer G. A screen R is used to protect the strip from direct radiation. The position of the mirror M can be adjusted by a screw P.

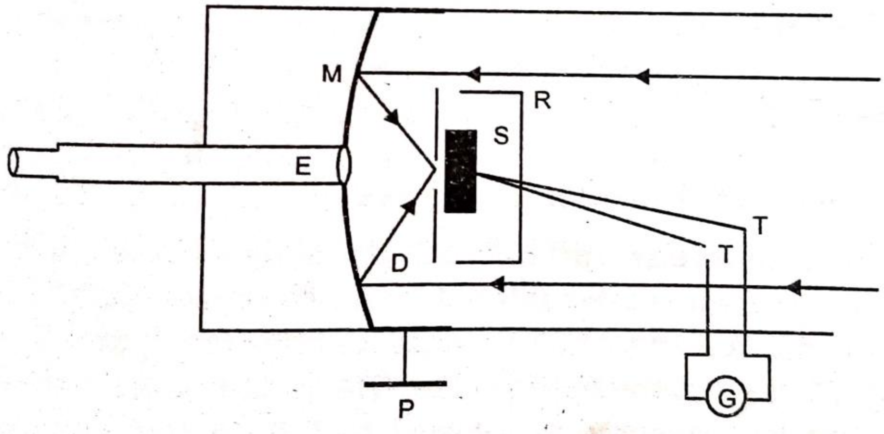


Fig. 5.7. Total radiation pyrometer

Working. Radiation from the body whose temperature is to be measured is made incident on the mirror. The focussed beam falls on the strip S through a diaphragm D. When the diaphragm is viewed through the eye-piece E, it appears circular as shown in Fig. 5.8, if the focussing is perfect. But if it is not so, then the diaphragm will appear as shown in Fig. 5.9. When this is so, the screw P is moved till the diaphragm appears as shown in Fig. 5.8. When this has been achieved, the focusing perfect. There will be a rise in the temperature of the strip S due to the focussed beam. The galvanometer G will show a deflection. The deflection of the galvanometer is a direct measure of the temperature of the body.

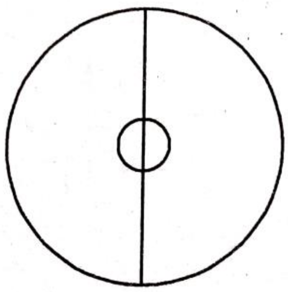


Fig. 5.8

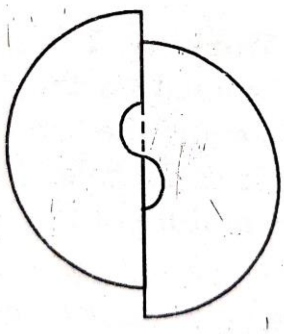


Fig. 5.9. Diaphragm

Determination of the temperatu

Let T = temperature of the body

T_0 = temperature of the strip

θ = reading of the galvanometer

Applying Stefan's of law,

$\theta \propto (T^4 - T_0^4)$ or $\theta = a\,(T^4 - T_0^4)$, where '$a$' is a constant.

The pyrometer is calibrated by using radiation from a black body maintained at different known temperatures. Thus, the galvanometer can be calibrated directly in degrees Kelvin or a calibration curve can be plotted between the galvanometer deflection and the temperature.

If the temperature to be determined is extremely high, then the intensity of the radiation coming from the source is very large. In such a case, we use **rotating sector device**. This helps to reduce the intensity of radiation. This device is an opaque disc from which a sector making an angle θ° at the centre is cut off. It is rotated about an axis of the pyrometer tube. In this case, only a fraction of the incident radiation enters the pyrometer per revolution. It is obvious that the temperature T_1 measured by the instrument will be less than the actual temperature T of the body.

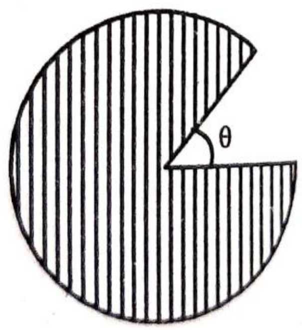


Fig. 5.10. Rotating sector device

Applying Stefan's law,

$$\frac{T_1^4}{T^4} = \frac{\theta}{360} \text{ or } T = T_1 \left(\frac{360}{\theta}\right)^{1/4}$$

Where θ is the angle of the sector in degrees.

Disappearing filament optical radiation pyrometer. It consists of a telescope whose cross wires are replaced by an electrical lamp. The lamp has a filament F which can be heated by a suitable current. D_1 and D_2 are the two diaphragms which restrict the field of view. The radiation from the body whose temperature is to be measured is allowed to fall on the objective O of the telescope through a small circular hole H in a shield R. G is red glass filter. It allows only a narrow band of rays (in the red region) to pass through it.

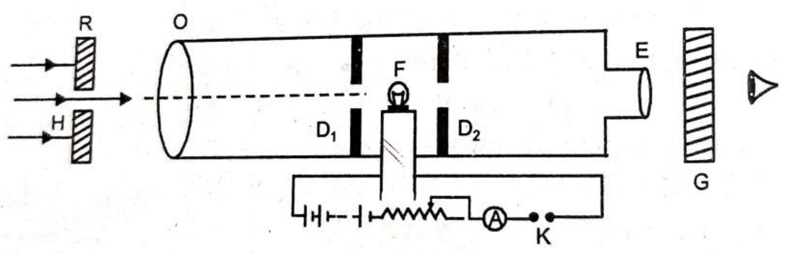


Fig. 5.11. Optical radiation pyrometer

Working. The eye-piece is focused on the filament of the lamp. The hole is focussed on the filament by adjusting the telescope. The current in the filament of the lamp is so adjusted that it becomes invisible against the bright image of the hole. When this is so, the temperature of the body is equal to the temperature of the filament.

The pyrometer is pre-calibrated by taking hot bodies at different know temperatures and noting the corresponding currents passed through the filament for equal brightness. The ammeter can be calibrated directly in degrees kelvin or a calibration curve can be plotted between the temperature and the current.

5.12 MERCURY THERMOMETER OR CLINICAL TERMOMETER

Gabriel Fahrenheit was the first to use mercury as thermometric substance.

This thermometer is a short-range thermometer which covers a few degrees on either side of the average temperature of the human body, i.e., 98.4°F. This thermometer is generally used by doctors.

A mercury thermometer consists of a thick-walled glass capillary tube of uniform and fine bore. A thin- walled cylindrical bulb is provided at one end of the capillary tube. The capillary tube is sealed at the top. Pure and dry mercury fills the bulb and a part of the tube. The stem is graduated in degrees. Just above the bulb, there is a small constriction in the stem. When the temperature rises, mercury easily passes through this constriction. But the reverse is not true, i.e., when temperature falls, mercury cannot pass this constriction. The front part of the thermometer is made prismatic in shape. This gives a magnified view of the scale and thread when observed from a suitable angle.

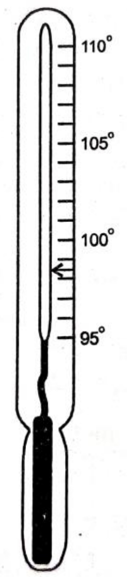


Fig. 5.12. Mercury thermometer

A small red or black arrow is marked on the stem at 98.4°F which is the normal temperature of the human body.

To make a mercury thermometer, a thick-walled glass capillary tube of uniform and fine bore is taken. It is thoroughly cleaned. A small cylindrical or spherical bulb is blown at one end of this tube. A funnel containing pure and dry mercury is placed at the upper end of the tube. The bulb is now heated. The air inside the tube expands and moves out of the tube. As a result of this, some mercury enters the tube. The bulb and a part of the tube are filled with mercury by alternate heating and cooling. After having done this, the upper end of the tube is sealed. Now, the tube is said to be hermetically sealed. Before marking the lower and upper fixed points, the tube is allowed to cool for many months.

In order to determine the temperature of the human body, the bulb of the thermometer is placed under either the tongue or the armpit. **This ensures good thermal contact between the body and the thermometer.** The thermometer picks up the temperature of the body in about two minutes time. As the mercury suffers thermal expansion, it passes the constriction without any difficulty. After the thermometer is taken out, there is no need to hurry while reading it. This is because the mercury will not, by itself, fall. After having taken the reading, the thermometer is vigorously shaken so that the mercury again passes through the constriction and enters the bulb.

Advantages of Mercury Thermometer

1. Mercury freezes at - 39°C and boils at 357°C. Moreover, it has uniform coefficient of expansion over a wide range of temperature. So, the range of the mercury thermometer is quite wide.

2. Since mercury is a good conductor of heat therefore it quickly attains the required temperature.

3. Mercury does not wet glass. So, the rise and fall of mercury in the tube is clean and smooth.

4. Mercury has low specific heat. So, it requires only a very small amount of heat to attain the required temperature.

5. Since mercury is bright and opaque therefore it can be easily seen.

Errors of a Mercury Thermometer

An ordinary mercury thermometer is not reliable for accurate work. This is because it possesses some errors. Some of these errors are given below:

1. Glass takes many years to regain its original length after it has been heated once. The effect of slow contraction of glass is that the zero point is raised slightly. Similarly, the upper fixed point also varies.

2. When mercury thermometer is used, a good part of the stem projects outside the body whose temperature is to be measured. This introduces an errors.

3. The bore of the thermometer may not be uniform. So, the rise of mercury thread will not be uniform.

5.13 CONSTANT VOLUME GAS THERMOMETER

At any temperature, the pressure of a gas depends upon its volume. If the volume is kept constant, the pressure depends upon the temperature and increases steadily with rising temperature. The constant volume gas thermometer uses the '**pressure at constant volume**' as the thermometric property.

The constant volume gas thermometer is shown diagrammatically in Fig. 5.13. It consists essentially of a bulb C of glass, glazed porcelain, fused quartz,

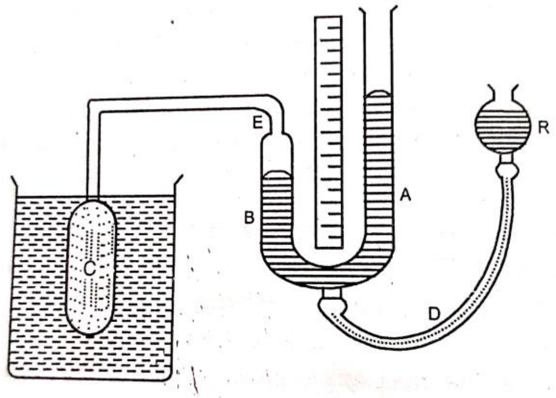


Fig. 5.13. Constant volume gas thermometer

platinum or platinum-iridium (depending upon the temperature range over which it is to be used). The bulb is connected by a capillary tube to a mercury pressure gauge such as an open manometer. The bulb is immersed in the system whose temperature is to be measured. The bulb contains some gas such as helium or hydrogen or nitrogen or even air. The mercury reservoir R is so adjusted that the mercury in the branch B of the U-tube is at a fixed reference mark E to keep the confined gas at a constant volume. Then, we read the height of the mercury in the A branch. The pressure of the confined gas is the difference of the heights of the mercury columns (times ρg) plus the atmospheric pressure (as indicated by the barometer reading). In actual practice, we have to apply corrections for the small volume change owing to slight contraction or expansion of the bulb. We have also to consider the fact that not all the confined gas has been immersed in the bath. Assume that these and other possible corrections have been made. If P be the corrected pressure at the temperature of the bath, then the temperature of the bath is given by

$$T = 273.16\ K\frac{P}{P_0} \text{ (at constant volume)}$$

Sample Problem 5.2. A constant volume gas thermometer using helium records a pressure of 20.0 kPa at the triple-point of water (= 273.16 K) and pressure of 14.3 kPa at the temperature of 'dry ice' (solid Co_2). What is the temperature of dry ice'?

Solution. P_{tr} = 20.0 kPa, T_{tr} = 273.16 K

If T be the temperature of the dry ice, then $T = \frac{P}{P_0} \times T_{tr} = \frac{143 \times 273.16}{20.0}$ K = 195.31 K

Temperature of dry ice on celsius scale = (195.31 - 273.15)°C = **- 77.84°C**

5.14 CRITERIA FOR SELECTION OF THERMOMETERS

The following criteria are considered while selecting a thermometer for particular temperature measurement.

1. Temperature Range. The thermometer selected for temperature measurement should be able to work in the temperature range to be measured.

2. Accuracy. The selected thermometer should have sufficient accuracy.

3. Response time. The selected thermometer should have small response time. It should have the capacity to quickly attain the temperature of the body.

4. Accessibility. The selected thermometer should be such that it can be put close to the body whose temperature has to be measured. Of course, this criterion is overlooked in the case of pyrometers.

5. Ruggedness. The selected thermometer should be such that it is not disturbed by the surroundings of the body.

SUMMARY

- Internal energy of a system is the sum of all the energies of the molecules of a system.
- The SI unit of heat is joule.
- Thermoelectric thermometer is based upon Seebeck effect.
- Different metals have different coefficients of thermal expansion. This is the basis of bimetallic thermometer.
- Platinum resistance thermometer is based on the variation of resistance with temperature.
- Pyrometry is that branch of Physics which deals with the measurement of high temperature.

- Fery's total radiation pyrometer and optical radiation pyrometer are important pyrometer.
- Clinical thermometer is known as mercury thermometer because it uses mercury as thermometric substance.
- The working relation for constant volume gas thermometer is:

$$T = 273.16\text{ K}\,\frac{P}{P_0}\text{ (at constant volume)}$$

- Temperature range, accuracy, response time, accessibility etc. are some important criteria for selection of thermometer.

TEST YOURSELF

1. What do you mean by internal energy of the system?
2. What are the different units of heat ?
3. What are the basic principles of measurement of heat ?
4. What do you understand by Seebeck effect?
5. Give the principle and construction of thermoelectric thermometer.
6. How can you measure unknown temperature with the help of thermoelectric thermometer ?
7. Give the construction of bimetallic thermometer. How can you measure temperature with the help of bimetallic thermometer ?
8. Discuss the principle, construction, advantages and disadvantages of platinum resistance thermometer.
9. What is a radiation pyrometer ? Explain two types of radiation pyrometers.
10. Write a note on clinical thermometer.
11. Write a note on constant volume gas thermometer.
12. What are the criteria for the selection of thermometers ?

6 EXPANSION OF SOLIDS

LEARNING OBJECTIVES

- Thermal expansion of solids.
- Coefficient of linear expansion.
- Coefficient of surface expansion.
- Coefficient of cubical expansion.
- Relationship between the three coefficients of expansion of a body.
- Thermal stress.
- Application of thermal stress.
- Consequences of the expansion of solids.
- Effect of temperature on the density of solids and liquids.

6.1 THERMAL EXPANSION OF SOLIDS

Practically, every solid expands on heating, more or less, depending upon its nature. In one dimension only, we see that a particular length increases due to heat. This is termed as **linear expansion.** In two dimensions, a particular area increases to a new value on heating. This phenomenon is termed as **superficial expansion**. When we consider the whole volume, in three dimensions, we see that the volume increases on heating. This is known as the **cubical expansion** of the substance.

6.2 COEFFECIENT OF LINEAR EXPANSION

When the length of a solid increases on heating, the thermal expansion is called linear expansion.

Consider a solid in the form of a rod. Let its length be L_0. Let the temperature of the rod be increased by a small amount ΔT. Let its length increase to L. Then, the linear expansion ΔL is $(L - L_0)$. It is observed that the linear expansion is proportional to original length L_0 and the rise in temperature ΔT.

$\therefore \quad \Delta L = L - L_0 = L_0 \alpha \Delta T \quad$ or $\quad \boxed{L = L_0 (1 + \alpha \Delta T)}$

The constant of proportionality a is called coefficient of linear expansion of the material of the rod. Its value depends upon the nature of material of the rod and temperature.

Now, $\alpha = \frac{L - L_0}{L_0 \Delta T} = \frac{\Delta L}{L_0 \Delta T}$

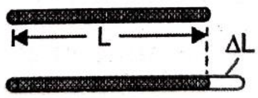


Fig. 6.1. Linear expansion

In words, coefficient of linear expansion

$$= \frac{\text{Change in Length}}{\text{Original length} \times \text{Change in temperature}}$$

The **coefficient of linear expansion** of the material of a rod is defined as the change in length per unit length per unit change in temperature.

The SI unit of α is K^{-1}. It may even be written as $°C^{-1}$, because the size of degree is same on celsius and absolute scales.

Ideally speaking, the coefficient of linear expansion is defined in the temperature interval 0°C to 1°C. Since the value of α is very small therefore we generally find the length L_1 of the rod at some low temperature T_1 and the length L_2 at some higher temperature T_2. The coefficient of linear expansion is then given by

$$\alpha = \frac{L_2 - L_1}{L_1(T_2 - T_1)} \quad \text{or} \quad L_2 = L_1[1 + \alpha(T_2 - T_1)]$$

For most of the solids, value of a lies between 10^{-6} to 10^{-5} K^{-1} in the temperature range 0°C to 100°C. Typical value for iron is $\alpha = 0.000012\ K^{-1}$. The value of a for ionic solids is more than that for non-ionic solids.

6.3 COEFFICIENT OF SURFACE EXPANSION

When the area of a solid increases on heating, the thermal expansion is called surface expansion or area expansion.

Let S_0 be the surface area of a solid sheet. Let the temperature of the sheet be raised by a small amount ΔT. Let the surface area increase to S. If ΔS be the increase in surface area, then

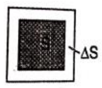


Fig. 6.2. Superficial expansion

$$\Delta S = S - S_0$$

Then $\Delta S = S - S_0 = S_0 \beta \Delta T$

where β is called coefficient of surface expansion of the body.

Now, $\boxed{S = S_0 (1 + \beta \Delta T)}$

The value of β depends upon the nature of material and temperature.

Now, $\beta = \frac{S - S_0}{S_0 \Delta T} = \frac{\Delta S}{S_0 \Delta T}$

In words, coefficient of surface expansion

$$= \frac{\text{Change in Length}}{\text{Original length} \times \text{Change in temperature}}$$

The coefficient of surface expansion of the material of a body is defined as the change in surface area per unit surface area per unit change in temperature.

The SI unit of β is K^{-1}

If S_1 and S_2 are the surface areas of the solid sheet at temperatures T_1°C and T_2°C respectively, then

$$\beta = \frac{S_2 - S_1}{S_1(T_2 - T_1)} = \text{or } S_2 = S_1[1 + \beta\,(T_2 - T_1)]$$

6.4 COEFFICIENT OF CUBICAL EXPANSION

When the volume of a solid increases on heating, the thermal expansion is called cubical expansion or volume expansion.

Consider a solid body of volume V_0. Suppose its volume increases to V when its temperature is raised by a small amount ΔT. Then the volume expansion ΔV is $(V - V_0)$. It is observed that the volume expansion is proportional to original volume V_0 and the rise in temperature ΔT.

$$\Delta V = V - V_0 = V_0\,\gamma\Delta T \quad \text{or} \quad \boxed{V = V_0(1 + \gamma\Delta T)}$$

γ is called coefficient of cubical expansion of the material of the solid. It depends upon the nature of material of the solid and temperature.

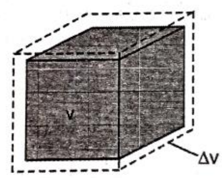


Fig. 6.3. Cubical expansion

Now, $$\gamma = \frac{V - V_0}{V_0\Delta T} = \frac{\Delta V}{V_0\Delta T}$$

In words, coefficient of cubical expansion

$$= \frac{\text{Change in Length}}{\text{Original length} \times \text{Change in temperature}}$$

The coefficient of cubical expansion of the material of a body is defined as the change in volume per unit volume per unit change in temperature.

The SI unit of γ is K^{-1}.

If V_1 and V_2 be the volumes of a solid body at temperatures T_1°C and T_2°C respectively, then

$$\gamma = \frac{V_2 - V_1}{V_1(T_2 - T_1)} = \text{or } V_2 = V_1[1 + \gamma\,(T_2 - T_1)]$$

6.5 RELATIONSHIP BETWEEN THE THREE COEFFICIENTS OF EXPANSION OF A BODY

Consider a solid cube of each side L_0 and volume V_0 (= L_0^3). Let S_0 (= L_0^2) be the surface area of each face of the cube. Let the temperature of the cube be raised by a small amount at such that the length of each side becomes L. Area of each face will be S such that $S = L^2$ Similarly, volume of cube will be V such that $V = L^3$.

Relation between α and β

$$L = L_0(1 + \alpha\Delta T) \qquad \ldots(1)$$

Squaring, $L^2 = L_0^2(1 + \alpha\Delta T)^2$

$$S = S_0(1 + \alpha\Delta T)^2 \quad \text{or} \quad S = S_0(1 + 2\alpha\Delta T + \alpha^2\Delta T^2)$$

Since α is very small therefore the term $\alpha^2\Delta T^2$can be neglected.

$$\therefore \quad S = S_0(1 + 2\alpha\Delta T)$$

Comparing with $S = S_0(1 + \beta\Delta T)$, we get

$$\boxed{\beta = 2\alpha} \qquad \ldots(2)$$

Relation between α and γ

Taking the cube of equation (1), we get

$$L^3 = L_0^3(1 + \alpha\Delta T)^3$$

$$V = V_0(1 + 3\alpha\Delta T + 3\alpha^2\Delta T^2 + \alpha^3\Delta T^3)$$

Since α is very small therefore the terms $\alpha^2\Delta T^2$ and $\alpha^3\Delta T^3$ can be neglected.

$$\therefore \quad V = V_0(1 + 3\alpha\Delta T)$$

Comparing with $V = V_0(1 + \gamma\Delta T)$, we get

$$\gamma = 3\alpha \qquad \ldots(3)$$

Combining (2) and (3) together, we obtain the result

$$\boxed{\frac{\alpha}{1} = \frac{\beta}{2} = \frac{\gamma}{3}} \quad \text{or} \quad \boxed{6\alpha = 3\beta = 2\gamma}$$

This is the relation between all the three coefficients of thermal expansion of the material of a body.

An anisotropic substance is a substance which does not exhibit the same properties in all directions.

For anisotropic solid,

$\gamma = \alpha_1 + \alpha_2 + \alpha_3$.

Here α_1, α_2, α_3 are coefficients of linear expansions in three mutually perpendicular directions.

6.6 THERMAL STRESS

Even a small thermal expansion in solids produces a tremendous force. What happens by preventing the thermal expansion of a rod by fixing its ends rigidly? Clearly, the rod acquires a compressive strain due to the external forces provided by the rigid support at the ends. The corresponding stress set up in the rod is called **thermal stress.**

Illustration. Consider a steel rail of length 5 m and cross-sectional area 40 cm^2. Suppose it is prevented from expanding while the temperature rises by 10°C. For steel, $\alpha = 1.2 \times 10^{-5}\ K^{-1}$ and $Y = 2 \times 10^{11}\ Nm^{-2}$.

Compressive strain, $\frac{\Delta l}{l} = \alpha\Delta T = (1.2 \times 10^{-5}) \times 10 = 1.2 \times 10^{-4}$

Thermal stress $= \frac{\Delta F}{A} = Y\left(\frac{\Delta l}{l}\right) = 2 \times 10^{11} \times 1.2 \times 10^{-4}$

$= 2.4 \times 10^{-7}\ Nm^{-2}$

or $\Delta F = AY\left(\frac{\Delta l}{l}\right) = 40 \times 10^{-4} \times 2.4 \times 10^{-7}\ N \simeq 10^5\ N$

If two such steel rails fixed at their outer ends are in contact at their inner ends, a force of this magnitude can easily bend the rails. It is precisely for this reason that the rails over which the trains pass have some space between them.

6.7 APPLICATION OF THERMAL STRESS

1. **Fixing of Iron Rims on Wooden Wheels**

   The blacksmith makes the iron rim slightly smaller than the wooden wheel (of the bullock cart or tonga) on which the rim is to be fixed. To fix the iron rim on the wooden wheel, the rim is heated till it expands to a diameter more than the diameter of the wooden wheel. The hot rim is quickly made to slide over the wooden wheel. Water is poured over the rim to cool it. On cooling, the rim contracts and grips the wheel very tightly.

2. **Tight Riveting of Metal Plates**

   Parts of many metallic structures are not riveted by welding process. Hot rivets are passed through the holes in the two steel plates to be held together tightly. The hot rivet is hammered. As the rivet cools, it contracts and holds the two plates very tightly. Riveting is extensively used in girders, beams, boilers, ship plates, etc.

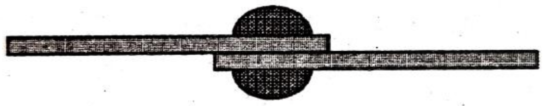


Fig. 6.4

3. Opening of Bottle Cap

When hot water is poured on the metal cap of a bottle, the cap expands. So, the cap can be removed easily.

4. Bimetallic Strips Used as Thermostats

A bimetallic strip is made by riveting two metal strips, usually one of brass and the other of iron, as shown. Brass expands more than iron when heated. So, the bimetallic strip bends with brass on the outer side because it is longer now as shown in Fig. 6.5(ii).

Bimetallic strips are used in making thermoswitches (also called thermostats).

A thermoswitch is used in automatic switching on and off of the electric supply to any electrical appliance (electric iron, geyser, electric oven, refrigerator, etc.)

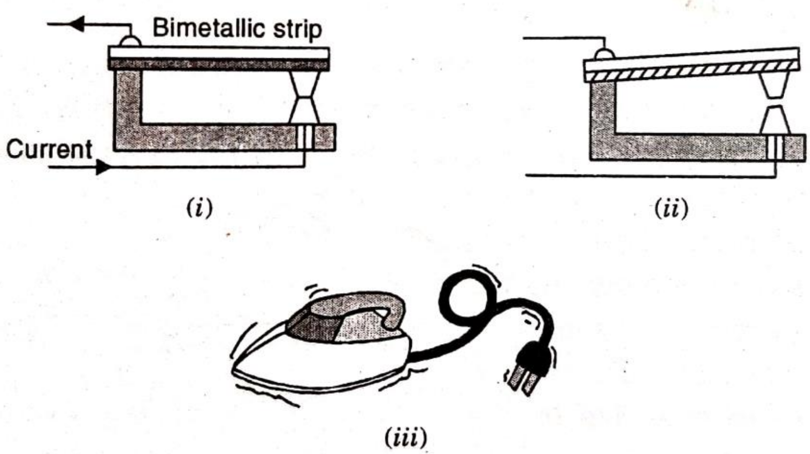


Fig. 6.5 (i) When, cool, contact closed (ii) when hot, contact open. (iii) thermo switch is electric iron

The use of bimetallic thermoswitch in an automatic electric iron is shown in Fig. 6.5.

Fig. 6.6 shows the use of bimetallic strip in fire alarm.

Due to different coefficients of thermal expansion of the two elements of the bimetallic strip, the strip bends outwards on heating. On cooling, the strip bends inwards. When the desired high temperature is attained, the outward bending of the strip is sufficient to switch off the circuit. On cooling, the position of the strip is such as to switch on the electric appliance.

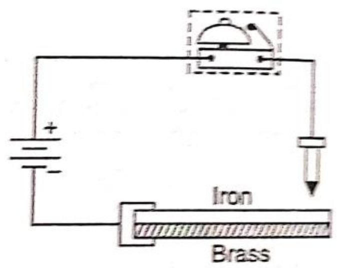


Fig. 6.6

6.8 CONSEQUENCES OF THE EXPANSION OF SOLIDS

1. Gaps Between Rails of Railway Tracks

Railway tracks consist of long steel rails (each nearly 13 m long). These steel rails are laid end to end but a small gap is left between the rails. This gap is left between the rails to provide scope for expansion during summer. In the absence of these gaps, the strains due to tendency for thermal expansion would cause the rails to bend or buckle.

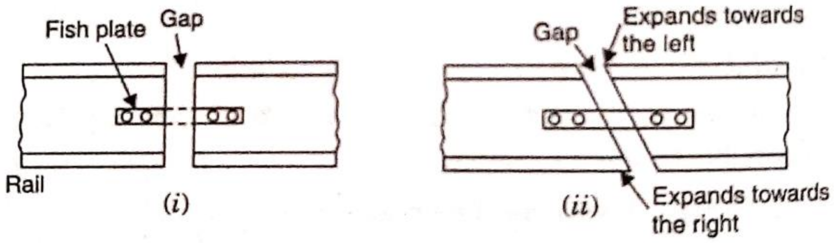


Fig. 6.7

The rails are joined by fish plates bolted to rails. The holes in the fish plates are oval-shaped. This permits movements of the section of the rail in the direction of length only.

The tracks used these days are continuous lengths of a few kilometre welded together. The ends of these rails are in the form of wedges. This enables the ends to slide past each other during expansion.

2. Loops in Pipes

In factories, long metal pipes are used for transporting oil, hot water and steam. These are provided with loops at regular intervals. Any expansion of contraction due to change in temperature of fluid or pipe is absorbed by the loop. When the temperature rises, the pipeline expands. This causes a slight increase in the size of the loop. When the temperature falls, the pipeline contracts. There is a slight decrease in the size of the loop.

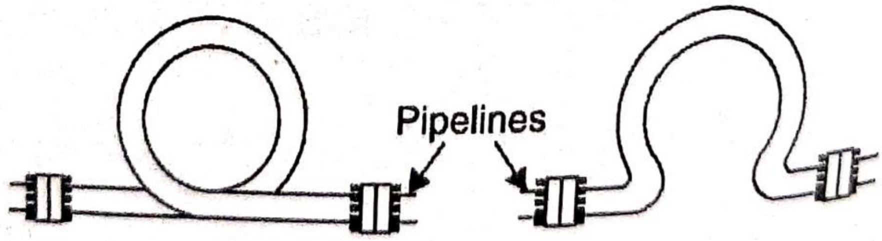


Fig. 6.8

In the absence of loops, a large strain is developed due to change in temperature. This can damage the pipeline.

3. Sag in Telephone Wires

During summer, the telephone wires become loose and begin to sag. This is due to thermal expansion. But, in winter, the telephone wires tighten. This is due to thermal contraction.

While laying the telephone lines in summer, these are not tightened much. A scope is left for contraction during winter. If this is not done, then the thermal contraction during winter would cause severe strain. This can result in snapping of the wires or uprooting of the poles.

4. Cracking of Glass Tumbler

When boiling tea is poured in a thick-walled glass tumbler, it cracks or breaks. This is because the outer surface of the tumbler is at a lower temperature while the inner surface of the tumbler is at a higher temperature. Note that glass is a bad conductor of heat. So, heat cannot quickly flow from inner surface to outer surface to equalise temperature.

Due to higher temperature, the inner surface expands. This thermal expansion causes large strain due to which the tumbler cracks.

Under identical conditions, a pyrex glass beaker shall not crack. This is because of low value ($3 \times 10^{-6}{}^{\circ}C^{-1}$) of a for pyrex as compared to glass ($\alpha = 90 \times 10^{-6}{}^{\circ}C^{-1}$).

5. Rollers Under Free end of Bridges

Bridges are made by welding together of steel girders and bars. If both ends of a bridge are fixed, then the thermal expansion of the bridge in summer can cause large strains. This can damage the bridge. In order to prevent this damage, one end of the bridge is fixed and the other end is placed on rollers. So, the bridge is free to expand the rail in the in summer and contract in winter.

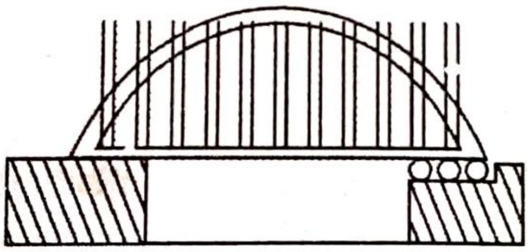


Fig. 6.9

6.9 EFFECT OF TEMPERATURE ON THE DENSITY OF SOLIDS AND LIQUIDS

We know that $V = V_0(1 + \gamma\Delta T)$

If M be the mass of a given solid or liquid, then

$$V_0 = \frac{M}{\rho_0} \text{ and } V = \frac{M}{\rho}$$

where ρ_0 and ρ are the density at temperature T and T + ΔT respectively.

$$\therefore \qquad \frac{M}{\rho} = \frac{M}{\rho_0}(1 + \gamma\Delta T) \qquad \text{or} \qquad \rho = \frac{\rho_0}{1+\gamma\Delta T} = \rho_0(1 + \gamma\Delta T)^{-1}$$

Using Binomial theorem and neglecting squares and higher powers, we get

$$\rho = \rho_0(1 + \gamma\Delta T)$$

clearly, $\rho < \rho_0$. So, density decrease with rise in temperature.

Sample Problem 6.1. A brass disc at 20°C has a diameter of 30 cm and a hole cut in its centre is 10 cm in diameter. Calculate the diameter of the hole when the temperature of the disc is raised through 50°c. Coefficient of linear expansion of brass = 1.8 × 10^{-5} per °C.

Solution. When the disc suffers thermal expansion, the circumference of the hole also increases.

Length of circumference of hole at 20°C, $L_1 = 10\pi$ cm

Length of circumference of hole at 70°C, $L_2 = \pi D$ cm

Rise of temperature = $t_1 - t_2$ = (70 - 20)°C - 50°C

$$L_2 = L_1[1 + \alpha(t_1 - t_2)]$$

or $\pi D = 10\pi\ (1 + 1.8 \times 1.^{-5} \times 50)$ cm

or $D = 10\ (1 + 0.0009)$ cm = **10.009 cm**

Sample Problem 6.2. A brass rod at 30°C is observed to be a metre long when measured by a steel scale which is correct at 0°C. Find the correct length of the rod at O°C. Given: a for steel = 0.000012 per °C and a for brass = 0.000019 per °C.

Solution. Since the scale is correct at 0°C therefore each division of scale is 1 cm. At 30°C, each cm division becomes (1 + 0.000012 × 30) cm, i.e., 1.00036 cm.

True length of steel scale at 30°C = 100 × 1.00036 cm = 100.036 cm

Length of brass rod at 30°C = 100.036 cm

If L_0 is the length of the brass rod at 0°C, then

$$L_0\ (1 + 0.000019 \times 30) = 100.036$$

or $L_0 = \frac{100.036}{1.00057}$cm = **99.98cm**

EXERCISES

1. What space must be left between two iron rails 20 m long so as to allow for expansion up to 40°C? The rails are laid at 15°C and coefficient of linear expansion for iron is 0.000012/°C

[**Ans.** 6×10^{-3} m]

2. A metal rod measures 50 cm in length at 20°C. When it is heated to 95°C, the length becomes 50.06 cm. What is the coefficient of linear expansion of rod? What will be the length of the rod at -20°C?

[**Ans.** 1.6×10^{-5}/°C, 49.968 cm]

3. The length of steel span of a bridge is 0.5 km and it has to withstand temperatures from 5°C to 50°C. What allowance should be kept for its expansion? Given: α for steel - 10^{-5}/°C.

[**Ans.** 0.225 m]

4. The original area of a metal plate is 100 cm^2 at 20°C. If β for the metal is 0.000032/°C then what is the area of the plate at 200°C?

[**Ans.** 100.576 cm^2]

5. A sheet of brass is 50 cm long and 10 cm broad at 20°C. If the surface area at 120°C is 500.1 cm^2, calculate the coefficients of superficial and linear expansion.

[**Ans.** 2×10^{-6}/°C, 1×10^{-6}/°C]

6. A solid occupies 1000 cm^3 at 20°C. Its volume becomes 1016.2 cm^3 at 320°C. What are the values of coefficients of cubical and linear expansion? [**Ans.** 5.4×10^{-5}/°C, 1.8×10^{-5}/°C]

7. An iron sphere has a radius of 10 cm at a temperature of 0°c, Calculate the change in volume of the sphere if it is heated to 100°C. Given α_{Fe}. = 1.1×10^{-6}/°C.

[**Ans.** 1.383 cm^3]

8. The volume of a metal sphere is increased by 1% of its original volume when it is heated from 320 K to 522 K. calculate the coefficients of linear, superficial and cubical expansion of the metal.

[**Ans.** 1.65×10^{-5}/°C, 3.30×10^{-5}/°C, 4.95×10^{-5}/°C,]

SUMMARY

- When the length of a solid increases on heating, the thermal expansion is called linear expansion.
- When the area of a solid increases on heating, the thermal expansion is called surface expansion.
- When the volume of a solid increases on heating, the thermal expansion is called volume expansion.
- The coefficient of linear expansion α is the change in length per unit length per unit change in temperature.
- The coefficient of surface expansion is the change in surface area per unit surface area per unit change in temperature.
- The coefficient of cubical expansion γ is the change in volume per unit volume per unit change of temperature.
- $\gamma = 3\alpha$, $\beta = 2\alpha$.
- Thermal stress is the product of Young's modulus of elasticity and longitudinal strain.

TEST YOURSELF

1. What is linear expansion? What is coefficient of linear expansion?
2. What is coefficient of surface expansion?
3. What is volume expansion? What is coefficient of volume expansion?
4. Show that the coefficient of surface expansion in twice the coefficient of linear expansion.
5. Prove that the coefficient of volume expansion is thrice the coefficient of linear expansion.
6. What is thermal stress? Discuss three important applications of thermal stress.

7 HEAT TRANSFER

LEARNING OBJECTIVES

- Three modes of transfer of heat.
- Conduction.
- Units of K.
- Dimensional formula of K.
- A good conductor of heat is also a good conductor of electricity.
- Applications of thermal conductivity to everyday life.
- Determination of thermal conductivity of a good conductor by Searles's method
- Lee's method for the determination of thermal conductivity of a bad conductor.
- Conduction of heat through compound media.
- Convection.
- Phenomena based on convection.
- Heat radiation.
- Characteristics of heat radiation.
- Prevost's theory of heat exchanges.
- Absorbtance, reflectance and transmittance.
- Emissivity and absorbtivity.
- Black body.
- Kirchhoff's law.
- Applications of Kirchhoff's law.
- Stefan's law of black body radiation.
- Deduction of Newton's law of cooling from Stefan's law.
- Distribution of energy in the spectrum of black body radiation.

7.1 THREE MODES OF TRANSFER OF HEAT

Heat is energy transferred from one system to another or from one part of a system to another part, arising due to temperature difference. What are the different ways by which this energy transfer takes place ? There are three distinct modes of heat transfer : conduction, convection and radiation (Fig. 7.1).

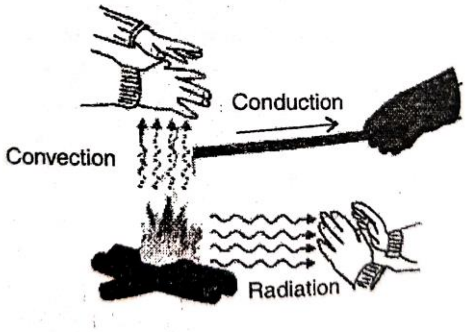


Fig. 7.1

7.2 CONDUCTION

(i) Conduction is the mechanism of transfer of heat between two adjustment parts of a body because of their temperature difference, without the actual movement of the particles from their equilibrium positions.

Suppose one end of a metallic rod is put in a flame. The other end of the rod will soon feel so hot that you cannot hold it by your bare hands. Here heat transfer takes place by conduction from the hot end of the rod though its different parts to the other end.

The ability to conduct heat differs widely from substance to substance. Gases are poor thermal conductors. Liquids have conductivities intermediate between solids and gases.

(ii) Quantitative description of heat flow. Consider a metal rod of length L and uniform cross-sectional area A with its two ends maintained at temperatures θ_1 and θ_2 such that $\theta_1 > \theta_2$ (Fig. 7.2).

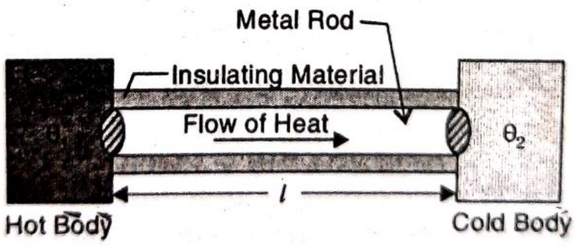


Fig. 7.2

It has been experimentally observed that the amount of heat (Q) following from hot face to the cold face is:

(a) directly proportional to the cross-sectional area S

$$Q \propto A \qquad \ldots(i)$$

(b) directly proportional to the temperature difference (θ_1 - θ_2) between the hot and cold faces

$$Q \propto (\theta_1 - \theta_2) \qquad \ldots(ii)$$

(c) directly proportional to the time t for which the heat flows.

$$Q \propto t \qquad \ldots(iii)$$

(d) inversely proportional to the distance between the hot and cold faces.

$$Q \propto \frac{1}{l} \qquad \ldots(iv)$$

Combining factors (i), (ii), (ii) and (iv), we get

$$Q \propto \frac{A\,(\theta_1 - \theta_2)t}{l}$$

or

$$Q \propto \frac{KA\,(\theta_1 - \theta_2)t}{l} \qquad \ldots(v)$$

Here, K is a constant of proportionality called coefficient of thermal conductivity of the material of the block. Its value depends upon the nature of material of the block.

(iii) Heat current. When the steady state is reached, the temperature of the rod decreases uniformly with distance from the hot end to the cold end. The hot body supplies heat at a constant rate which transfers trough the bar and is given out at the same rate to the cold body. The rate of flow of heat is called heat current. It is denoted by H.

$$H = \frac{Q}{t} = \frac{KA\,(\theta_1 - \theta_2)t}{l} \qquad \ldots(vi)$$

So, the heat current is proportional to the temperature difference (θ_1 - θ_2) and the area of cross-section A and is inversely proportional to the length l.

(iv) Coefficient of thermal conductivity. If A = 1, l = 1, t = 1 and (θ_1 - θ_2) = 1, then K = Q.

This leads us to the following definition of coefficient of thermal conductivity.

The coefficient of thermal conductivity of a material is defined as the quantity of heat flowing per second through a rod (or slab or block) of that material having unit length and unit cross-sectional area in the steady state when the difference of temperature between two ends of the rod (or slab or block) is 1°C or 1 K and the flow of heat is perpendicular to the end-faces of the rod (or slab or block).

The coefficient of thermal conductivity may also be defined in terms of 'unit cube'

The coefficient of thermal conductivity of a material is the quantity of heat flowing per second across the opposite faces of a unit cube, made of that material, when the opposite faces are maintained at a temperature difference of 1°C or 1 K.

(v) Concept of temperature gradient. Temperature gradient is defined as the rate of change of temperature with distance between two isothermal surfaces.

In equation (v), $\frac{(\theta_1 - \theta_2)}{l}$ gives the rate of change of temperature with distance.

It is called temperature gradient. Let it be denoted by $\frac{d\theta}{dl}$ Here negative sign indicates the decrease of temperature with distance.

Rewriting equation (v) in terms of temperature gradient, we get

$$Q = -KA\frac{d\theta}{dl}t$$

If A = 1, $-\frac{d\theta}{dl} = 1$ and $t = 1$, then K = Q.

The **coefficient of thermal conductivity** of a material is the rate of flow of heat energy through a rod, made of that material, of unit cross-section area under a unit temperature gradient, the flow of heat being normal to the cross- sectional area.

7.3 UNITS OF K

We know that $Q = \frac{KA\,(\theta_1 - \theta_2)t}{l}$ or $A = \frac{Ql}{A\,(\theta_1 - \theta_2)t}$

In cgs system, Q, l, A, $(\theta_1 - \theta_2)$ and t are measured in cal, cm, cm^2, °C and s respectively. So, the cgs unit of K is cal cm^{-1} °C^{-1} s^{-1}.

In SI, Q, l, A, $(\theta_1 - \theta_2)$ and t are measured in joule, m, m^2, K and s respectively.

So, the SI unit of K is Jm^{-1} K^{-1} s^{-1} or W m^{-1} K^{-1}.

7.4 DIMENSIONAL FORMULA OF K

$$[K] = \frac{[Q]\,[l]}{[A](\theta_1 - \theta_2)[t]} = \frac{[ML^2T^2][L]}{[L^2][K][T]} = [ML\,T^{-3}\,K^{-1}]$$

Table 7.1 Thermal conductivities (K)

		$W\ m^{-1}\ K^{-1}$	$cal\ s^{-1}\ cm^{-1}\ (°C)^{-1}$
Metals	Aluminum	205	0.49
	Brass	109	0.26
	Copper	385	0.92
	Lead	34.7	0.083
	Mercury	8.3	0.20
	Silver	406	0.97
	Steel	50.2	0.12
	Insulating brick	0.15	0.00035
	Red brick	0.6	0.0015
	Concrete	0.8	0.002
	Cork	0.04	0.0001
	Felt	0.04	0.0001
	Glass	0.8	0.002
	Ice	1.6	0.004
	Rock wool	0.04	0.0001
	Styrofoam	0.01	0.00002
	Wood	0.12-0.04	0.0003-0.0001
	Air	0.024	0.000057
	Argon	0.016	0.000039
	Helium	0.14	0.00034
	Hydrogen	0.14	0.00033
	Oxygen	0.023	0.000056

Generally, metals are good conductors of heat, silver being the best. Non-metals like wood, glass are poor conductors with low value of K.

THERMAL RESISTANCE

We know that = $\frac{Q}{t} = \frac{KA\,(\theta_1 - \theta_2)}{d}$ or $\frac{Q}{t} = \frac{\theta_1 - \theta_2}{d/KA}$

Comparing with $I = \frac{V}{R}$ (OHM'S LAW), we find that $\frac{d}{KA}$ plays the same role in 'heat' as is played by R in current electricity. So, $\frac{d}{KA}$ is called thermal resistance.

7.5 A GOOD CONDUCTOR OF HEAT IF ALSO A GOOD A GOOD CONDUCTOR OF ELECTRICITY

The conduction of both heat and electricity is due to the movement of free electrons. So, it is no surprise that there must be some relationship between thermal conduction and electrical conduction. In the year 1853

Widemann and Franz established a relation between thermal conductivity K and electrical conductivity. According to **Wiedmann-Franz law,** the ratio of thermal and electrical conductivities is proportional to the absolute temperature T of the metal.

$$\therefore \quad \frac{K}{\sigma} \propto T \quad \text{or} \quad \frac{K}{\sigma T} = \text{constant}$$

7.6 APPLICATIONS OF THERMAL CONDUCTIVITY TO EVERYDAY LIFE

1. **In winter, a brass knob appears colder than a wooden knob**. It is due to the reason that the thermal conductivity of brass (K = 109 Wm^{-1} K^{-1}) is more than that of wood (K = 0.12 Wm^{-1} K^{-1})). When the brass knob is touched, the heat energy is quickly conducted away from the hand. On the other hand, when the wooden knob is touched, the flow of heat energy from the hand is extremely slow. Thus, a brass knob appears colder than a wooden knob although both may be at the same temperature.
2. **Woollen clothes keep us warm.** This is because wool contains air in its pores. Air (K = 0.024 Wm^{-1} K^{-1})) is a bad conductor of heat. In fact, wool (K = 0.01 Wm^{-1} K^{-1})) is also a bad conductor of heat. Both the air and the wool do not permit heat to be conducted away from the body. So, the woollen clothes keep us warm.
3. **A new quilt is warmer than old quilt.** This is because new quilt contains more air in its pores as compared to old quilt. Since air is a bad conductor of heat therefore the heat is not conducted away from the body.
4. **Cooking utensils are provided with wooden handles.** This is because wood is a bad conductor of heat. So, the wooden handle would not permit heat to be conducted from hot utensil to hand. Thus, the hot cooking utensil can be easily held in hand through the wooden handle.
5. **Cooking utensils are made of aluminium and brass.** This is because aluminium and brass are good conductors of heat. They rapidly absorb heat from the fire and supply it to the food to be cooked.
6. **Ice is covered in gunny bags to prevent melting of ice.** This is because of the fact that gunny bags are bad conductors of heat. The pores of gunny bags contain air which is also a bad conductor of heat.
7. **The double-walled houses of ice made by Eskimos are warn from inside.** This is because the air within the walls does not allow heat to be conducted away to the outside air.
8. **Two thin blankets are warmer than a single thick blanket.** This is because the two thin blankets enclose a layer of air between them. Since air is a bad conductor of heat therefore the conduction of heat is prevented.

9. **When a metallic wire gauge is placed over the flame of a bunsen burner the flame does not so beyond the gauge.** This is due to the fact that the wire gauge, being a good conductor, absorbs the heat of the flame. Davy's safety lamp has been designed on this principle. In mines, a gauge is placed around the flame of the lamp. Since gauge is a good conductor therefore it absorbs heat of the flame. As a result of this, the temperature outside the gauge does not become high. Thus, the gases outside the gauge do not catch fire.
10. **A hot liquid remains hot and a cold liquid remains cold in a thermos flask.** This is due to the fact that a vacuum is created between the two walls of the thermos flask. Heat can neither flow from inside the flask to outside nor from outside air to the liquid inside the flask. Note that, in this way, loss of heat by conduction and convection has been minimised. To minimise loss of heat by radiation, the surface is made shining.
11. **Body of refrigerator is made of 8 cm thick insulated walls of fibre glass.** This is done to minimise chances of heat flowing into the refrigerator.

7.7 DETERMINATION OF THERMAL CONDUCTIVITY OF A GOOD CONDUCTOR BY SEARLE'S METHOD

Fig. 7.3 shows Searle's apparatus for the determination of thermal conductivity or a good conductor. A cylindrical bar of the material whose thermal conductivity is to be determined is taken. One end of the cylindrical bar is enclosed in a steam chamber C. Nearly one-fourth of the length of the bar at the other end is closely wound by a spiral of thin-walled copper tubing. The bar is covered with felt to minimise heat losses at the surface due to convection in air. Moreover, the apparatus is enclosed in a wooden box. Two thermometers T_1 and T_2 are placed in small cavities made in the bar at a known distance x apart. When steam is passed in the steam chamber, one end of the bar is heated up. The other end of the bar is cooled by circulating cold water through the copper tubing. Thermometers T_3 and T_4 are used to measure respectively the temperatures of water entering the copper tubing and coming out of it.

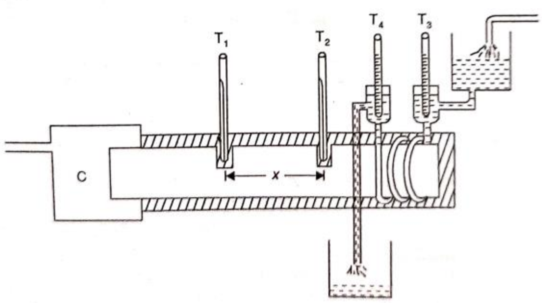


Fig. 7.3

When the cylindrical bar reaches steady state, the temperatures indicated by the four thermometers become constant. Let the thermometers T_1, T_2, T_3 and T_4 indicate temperatures θ_1, θ_2, θ_3, and θ_4 respectively. If A represents the cross-sectional area of the bar and K the thermal conductivity, then the rate of flow of heat q across the rod is given by

$$q = \frac{KA\,(\theta_1 - \theta_2)}{x} \qquad \ldots(i)$$

If m be the mass of water flowing through the copper tubing per second, then

$$q = m(\theta_3 - \theta_4) \qquad \ldots(ii)$$

Here, specific heat of water is taken as 1 cal g^{-1} °C^{-1}.

It follows from (1) and (2) that

$$\frac{KA\,(\theta_1 - \theta_2)}{x} = m(\theta_3 - \theta_4)$$

or
$$K = \frac{m(\theta_1 - \theta_2)x}{A\,(\theta_1 - \theta_2)}$$

By measuring the quantities appearing on the right hand side of the above equation, we can measure the coefficient of thermal conductivity of the material of the rod.

7.8 LESS METHOD FOR THE DETERMINATION OF THERMAL CONDUCTIVITY OR A BAD CONDUCTOR

Fig. 7.4 shows Lees and Chorlton's apparatus for the determination of thermal conductivity of a bad conductor. B is a thick nickel-polished cylindrical brass slab. D is a thin disc of the material whose thermal conductivity is to be measured. C is a flat circular hollow metal box. It has two openings to allow steam to flow through it. Two thermometers T_1 and T_2 are inserted in box C and slab B respectively. The apparatus is suspended from a report stand.

When steam is passed in the box, heat is transmitted to B through disc D. After a considerable time, steady state is reached. At this stage, the temperatures indicated by thermometers T_1 and T_2 become constant. Let θ_1 and θ_2 be the temperatures recorded by T_1 and T_2 respectively.

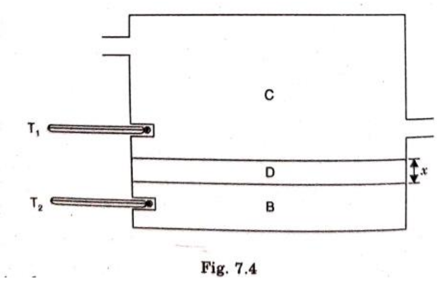


Fig. 7.4

In the steady state, the amount of heat conducted per unit time through D to B is equal to the rate of loss of heat due to radiation per unit time from B to the atmosphere. The rate of conduction of heat through the disc D is given by

$$q = \frac{KA\,(\theta_1 - \theta_2)}{x} \qquad \ldots\text{(i)}$$

Here, K is the coefficient of thermal conductivity of the material of D, A is cross-sectional area of disc D and x is the thickness of the disc D.

The rate of loss of heat due to radiation from the slab B is also equal to q. If m be the mass of B and s the specific heat, then

$$q = ms\left(\frac{d\theta}{dl}\right)_{\theta=\theta_2} \qquad \ldots\text{(ii)}$$

Here $\left(\frac{d\theta}{dl}\right)_{\theta=\theta_2}$ represents the rate of change of temperature of the slab B at the steady temperature θ_2.

Comparing (i) and (ii), $\frac{KA\,(\theta_1 - \theta_2)}{x} = ms\left(\frac{d\theta}{dl}\right)_{\theta=\theta_2}$

or $$K = \frac{ms\left(\frac{d\theta}{dl}\right)_{\theta=\theta_2} x}{A\,(\theta_1 - \theta_2)}$$

All the quantities, except $\left(\frac{d\theta}{dl}\right)_{\theta=\theta_2}$, appearing on the right hand side of the above equation are measured. For determination of $\left(\frac{d\theta}{dl}\right)_{\theta=\theta_2}$, the following method is used.

The disc D is removed. The slab B is kept in contact with C. The slab B is heated by a bunsen burner to a temperature which is nearly 10°C higher than θ_2. The slab is then allowed to cool. The temperature is noted at regular intervals of time until it falls to nearly 10°C below θ_2. On the basis of this data, a cooling curve is drawn. The slope of the cooling curve at temperature

θ_2 gives $\left(\frac{d\theta}{dl}\right)_{\theta=\theta_2}$

7.9 CONDUCTION OF HEAT THROUGH COMPOUND MEDIA

(i) Conductors in Series. Consider a bar of length, and thermal conductivity K_1 joined in series to another bar of length l_2 and thermal conductivity K_2 (Fig. 7.5). Let the faces A and B of the compound bar be maintained at temperatures θ_1 and θ_2 respectively such that $\theta_1 > \theta_2$. Since heat flows from higher temperature to lower temperature therefore the heat flows from face A to face B of the compound bar. When steady state is reached, rate of flow of heat in both the bars will be the same. If θ be the temperature of the common face of two bars, then

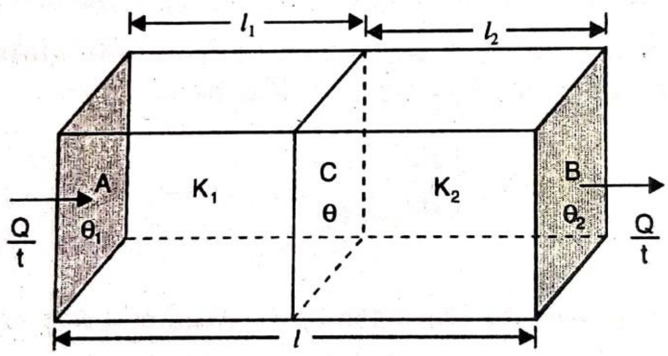


Fig. 7.5

and $$\frac{Q}{t} = \frac{K_2A\,(\theta_1 - \theta_2)}{l_2} \quad \text{or} \quad \theta_1 - \theta_2 = \frac{Q}{t}\left(\frac{l_2}{K_2A}\right)$$

Adding $$\theta_1 - \theta_2 = \frac{Q}{t}\left(\frac{l_1}{K_1A} + \frac{l_2}{K_2A}\right) \qquad \ldots\text{(i)}$$

If K be the equivalent thermal conductivity of the compound bar, then

$$\frac{Q}{t} = \frac{K_1A\,(\theta_1 - \theta_2)}{l_1 + l_2}$$

or $$\theta_1 - \theta_2 = \frac{Q}{t}\left(\frac{l_1 + l_2}{KA}\right) \qquad \ldots\text{(ii)}$$

Comparing (ii) and (i), we get

$$\frac{l_1 + l_2}{K} = \frac{l_1}{K_1} + \frac{l_2}{K_2}$$

If $l_1 + l_2 = l$, then

$$\frac{l}{K} = \frac{l_1}{K_1} + \frac{l_2}{K_2}$$

If n bars of lengths $l_1 + l_2 \ldots\ldots\ldots l_n$ and thermal conductivity $K_1 + K_2 \ldots\ldots\ldots K_n$ respectively, are connected in series, then equivalent thermal conductivity K is given by

$$\frac{l_1 + l_2 + \ldots\ldots\ldots + l_n}{K} = \frac{l_1}{K_1} + \frac{l_2}{K_2} + \ldots\ldots\ldots \frac{l_n}{K_n}$$

(ii) Conductors in Parallel. Fig. 7.6 shows two bars of same length l joined in parallel. The two bars have different cross-sectional areas A_1 and A_2 and different thermal conductivities K_1 and K_2. Let Q_1 and Q_2 be the amounts of heat flowing in the two bars at the opposite faces of the compound bar, in time *t*. If θ_1, and θ_2, are the steady temperatures at the opposite faces of the compound bar, then

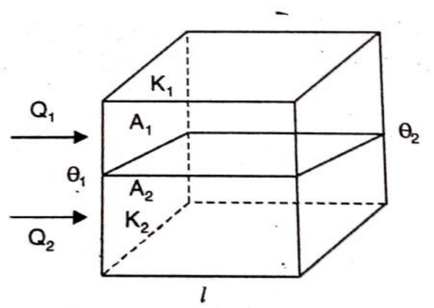


Fig. 7.6

$$Q_1 = \frac{K_1A_1\,(\theta_1-\theta_2)t}{l}$$

and

$$Q_2 = \frac{K_2A_2\,(\theta_1-\theta_2)t}{l}$$

If Q represents the total heat which flows in time t, then

$$Q = Q_1 + Q_2$$

$$\frac{K_1A_1\,(\theta_1-\theta_2)t}{l} + \frac{K_2A_2\,(\theta_1-\theta_2)t}{l}$$

or

$$\frac{Q}{t} = \left(\frac{K_1+A_1}{l} + \frac{K_2+A_2}{l}\right)(\theta_1 - \theta_2) \quad \ldots\text{(i)}$$

If K be the equivalent thermal conductivity, then

$$\frac{Q}{t} = \left(\frac{K_1(A_1+A_2)(\theta_1-\theta_2)}{l}\right) \quad \ldots\text{(ii)}$$

From (2) and (1), $K(A_1 + A_2) = K_1A_1 + K_2A_2$.

For n parallel conductors, the equivalent thermal conductivity is given by

$K(A_1 + A_2 + \ldots\ldots.+ A_n) = KA_1 + KA_2 + \ldots\ldots\ldots.+ KA_n$.

Sample Problem 7.1. Two rods, one semi-circular and the other straight, of the same material and of same cross-sectional area are joined as shown in Fig. 7.7. The ends A and B are maintained at a constant temperature difference. Calculate the ratio of the heat

conducted through a cross-section of a semi-circular rod to the heat conducted through a cross-section of the straight rod in a given time.

Solution. We know that $Q = \frac{KA(\theta_1 - \theta_2)t}{d}$

In the given problem, $Q \propto \frac{1}{d}$

$$\frac{Q_1}{Q_2} = \frac{2\pi}{\pi r} = \frac{2}{\pi}$$

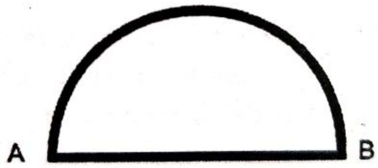


Fig. 7.7

Sample Problem 7.2. One end of a 0.25 m long metal bar is in steam and the other in contact with ice. If 12×10^{-3} kg of ice melts per minute, what is the thermal conductivity of the metal ? Given : cross-section of the bar = 5×10^{-4} m^2 and lateral heat of ice is 80 kcal/kg.

Solution $d = 0.25$ m, A = 5×10^{-4} m^2,

$\theta = (100 - 0)$ K = 100 K

t = 1 minute = 60 s

If Q is the amount of heat required to melt 12×10^{-3} kg of ice then

$$Q = 12 \times 10^{-3} \times 80 \times 1000 \text{ cal}$$

$$Q = K\frac{A\theta}{d}t \quad \text{or} \quad K = \frac{Qd}{A\theta t}$$

Or $$K = \frac{12 \times 10{-}3 \times 80 \times 1000 \times 0.25}{5 \times 10^{-4} \times 100 \times 60} \text{ cal s}^{-1}\text{ m}^{-1}\text{ K}^{-1}$$

$= 80$ **cal s^{-1} m^{-1} K^{-1}**

EXERCISES

1. Consider a hot water copper pipe that delivers a flow of water at 0.2 kg s^{-1}. If the pipe has a length of 20 m and the inlet temperature is 60°C, calculate the outlet temperature if the exterior of the pipe is at a temperature of 50°C. Given : Thermal conductivity of copper = 385 W m^{-1} K^{-1}, specific heat capacity of water = 4200 J kg^{-1} K^{-1} internal diameter of pipe = 2.0 cm and exterior diameter of pipe = 3.5 cm.

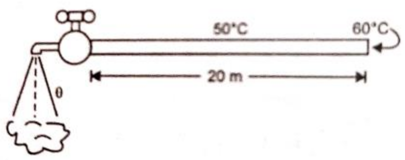


Fig. 7.8

[**Ans.** 39.4°C]

2. Find the rate of heat flow through a square iron plate of each sides equal to 4 cm and thickness 5 mm. Its opposite faces are kept at 90°C and 40°C respectively. K for iron is 0.15 cgs units.

   [Ans, 240 cal s^{-1}]

3. Calculate the thermal resistance of an aluminium rod of length 0.20 m and diameter 0.04 m. The thermal conductivity is 0.50 cal (cm s deg)$^{-1}$. The temperature difference along the length of the rod is 50°C. Also calculate the rate of heat transfer along the length of the rod.

   Ans. 3.18 s deg cal^{-1}, 15.7 cal s^{-1}]

4. Calculate the difference of temperature between two sides of an iron plate 2 cm thick when heat is conducted at the rate of 6×10^{-5} cal/min per square metre. The coefficient of thermal conductivity of metal is 0.2 cgs units.

   [**Ans.** 10°C]

5. Calculate the rate of loss of heat through a glass window of area 1000 sq. cm and thickness 4 mm when temperature inside is 37°C and outside is -5°C. Coefficient of thermal conductivity of glass is 0.0022 cal s^{-1} cm^{-1} K^{-1}. [**Ans.** 231 cal s^{-1}]

6. The air in a room is at 25°C and outside temperature is 0°C. The window of a room has an area of 2 m^2 and thickness 2 mm. Calculate the rate of loss of heat by conduction through window. Thermal conductivity for glass is 1.0 W m^{-1} degree^{-1}.

   [**Ans.** 25 kW]

7. There is a layer of ice 6 cm thick over the surface of a pond. The temperature of air is 283 kelvin. If the rate of loss of heat energy from the water be 200 joule per square metre per second, find the value of thermal conductivity of ice.

   [**Ans.** 1.2 W m^{-1} K^{-1}]

7.10 CONVECTION

Convection is the process in which heat is transferred from one point to another by the actual movement of the material particles from a region of high temperature to a region of lower temperature.

In this mode of heat transfer, heat is absorbed from the source by a portion of fluid that is being heated. The heated particles bodily move to the colder regions of the fluid. The heated particles share their heat with the colder particles. The space vacated by heated particles is taken up by colder particles. The space vacate by heated particles is taken up by colder particles. These 'colder particles' absorb heat from the source and move to the colder region of the fluid. In this way, the cycle is repeated and some sort of fluid current is established. The current is known as convection current. These currents are sometimes artificially generated with the help of pumps. Air-conditioning or central heating systems make use of artificially produced convection currents.

The gravity plays its own role in the 'natural formation of convection currents. Let us consider a glass vessel containing some water. When the vessel is heated, the portion of water near the bottom of the vessel suffers thermal expansion and its density decreases. Due to this decrease of density, the water becomes lighter than the water above. Considerations of buoyancy and gravity convince us that the lighter water has to move up and the heavier water has to move down. Thus, a current stops only when the whole mass of water is at the same temperature.

If the material is forced to move by a blower or pump, the process is called forced convection. If the material flows due to difference in density (caused by thermal expansion), the process is called natural or free convection.

There is no simple equation for convective heat transfer (as there is for conduction). For practical calculations, a convection coefficient h is defined by means of the equation :

$$H = hA\,\Delta T$$

where H is the heat current due to convection, A is the surface area and ΔT is the temperature difference.

7.11 PHENOMENA BASED ON CONVECTION

(i) Land and Sea Breezes

The heat from the Sun is absorbed more rapidly by land than by sea-water. Moreover, the specific heat of land is low as compared to that of sea-water. Consequently, the rise in temperature of land is higher as compared to that of sea-water. To sum-up, land is hotter than the sea during day time. As a result of this, the colder air over the sea blows towards the land. This is called sea-breeze.

At night, air blows from land towards sea. This is called land breeze.

(ii) Formation of Trade Winds

The surface of Earth near the equator gets heated strongly. So, the air in contact with the surface of Earth at the equator expands and rises upwards. As a result of this, a low pressure is created at the equator.

At the poles, the air in the upper atmosphere gets cooled and comes down, So, a high pressure is created at the poles.

Due to difference of pressures at the poles and equator, the air at the poles moves towards the equator, rises up, moves towards the poles and so on, In this way, a wind is formed in the atmosphere.

The rotation of the Earth also affects the motion of the wind. Due to anti- clockwise rotation of Earth, the warm wind blowing from equator to north drifts towards east.

The steady wind blowing from north-east to equator, near the surface of Earth, is called trade wind.

(iii) Monsoons

In summer, the peninsular mass of central Asia becomes more strongly heated than the waters of the Indian Ocean. This is due to the fact that the specific heat of water is much higher than that of the soil and rocks.

Hot air from the heated land mass rises up and moves towards the Indian ocean. Air filled with moisture flows over the Indian ocean on the south towards heated land mass. When obstructed by mountains, the moist air rushes upwards to great height. In the process, it gets cooled. Consequently, the moisture condenses and falls as rain.

(iv) Ventilation

Ventilator or exhaust fan in a room help to remove impure and warm air from a room. The fresh air from outside blows into the room. This is all due to the convection currents set up in the room.

(v) To Regulate Temperature in the Human Body

Heat transfer in the human body involves a combination of mechanisms. These together maintain a remarkably uniform temperature in the human body inspite of large changes in environmental conditions.

The chief internal mechanism is forced convection. The heart serves as the pump and the blood as the circulating fluid.

7.12 HEAT RADIATION

The energy emitted by a body in the form of radiation by virtue of its temperature is called thermal radiation or heat radiation. This energy is emitted by all bodies above absolute zero and is also called radiant energy. The most powerful source of radiant energy is the Sun.

7.13 CHARACTERISTICS OF HEAT RADIATIONS

Heat radiation belongs to the electromagnetic family i.e., it resembles γ-rays, X-rays, ultraviolet light, visible light and radiowaves. It can travel through vacuum and other transparent media. Its speed is the same as that of light. Like other electromagnetic radiations, it exhibits the phenomena of reflection, refraction, interference, diffraction and polarization.

The wavelength of heat radiation is longer than that of visible light. The wavelength of heat radiation ranges from 8×10^{-7} m to 3×10^{-4} m whereas the wavelength of visible light ranges from 4×10^{-7} m to 8×10^{-7} m.

7.14 PREVOST'S THEORY OF HEAT EXCHANGES

In the year 1902, the Swiss Physicist Pierre Prevost advanced this theory. The salient features of this theory are as under:

(i) All bodies above 0 K emit heat to the surroundings and gain hat from the surroundings at all times.

(ii) The amount of radiation emitted increases with temperature.

(iii) The rate of emission of thermal radiation depends only upon the temperature of the body and the nature of the surface.

(iv) The rate of emission of thermal radiation is in no way affected by the surroundings.

(v) There is a continuous exchange of heat between the body and the surroundings.

(vi) The fall or rise in temperature of a body is the net result of the exchange of heat between the body and the surroundings.

When we sit before a fire, we feel warm. This is because our body is receiving more thermal energy per unit area from the fire than it is losing by its own radiation. On the other hand, when we sit near a block of ice, we feel cold. This is because our body loses more heat energy by radiation than what it receives from ice. Thus, when a body absorbs more radiant energy than what it emits, there is rise in the temperature of the body, When a body absorbs less energy than what it emits, there is a fall in the temperature. However, if the quantity of thermal energy absorbed is equal to the quantity of thermal energy emitted, then there is no change in the temperature of the body.

When a body has the same temperature as that of its surroundings, it is a case of dynamic equilibrium and not that of a static equilibrium.

Prevost's theory of heat exchanges leads to the fact that good absorbers are good radiators and vice versa as proved below.

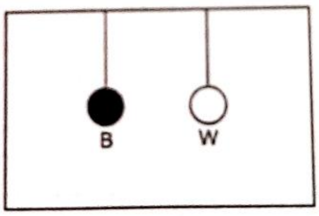


Fig. 7.9

Consider two bodies, one black B and the other white W, suspended by insulating threads inside a constant temperature enclosure (Fig. 7.9). The bodies will be in thermal equilibrium with the enclosure.

Since B is a black body therefore it should absorb most of the radiation incident on it. But its temperature remains constant. So, it must also be emitting at the same rate at which it absorbs. Thus, good absorbers are good radiators. Applying the same type of argument in the case of white body, we can conclude that poor absorbers are poor emitters.

7.15 ABSORBTANCE, REFLECTANCE AND TRANSMITTANCE

Let Q units of radiant energy be incident on a surface. Out of this, Q_1 is absorbed, Q_2 is reflected or scattered and the remaining Q_3 units transmitted.

Now, $\frac{Q_1}{Q}$ = absorbing power or absorptance 'a', $\frac{Q_2}{Q}$ = reflecting power or reflectance 'r' and $\frac{Q_3}{Q}$ = transmitting power or transmittance 't'

But the sum total of the absorbed, reflected and transmitted energies must be equal to the incident energy.

$$\therefore \quad Q_1 + Q_2 + Q_3 = Q$$

Or $$\frac{Q_1}{Q} + \frac{Q_2}{Q} + \frac{Q_3}{Q} = \frac{Q}{Q} = 1 \qquad \text{or} \qquad a + r + t = 1$$

If the surface does not transmit radiation, then $t = 0$,

In that case, $a + r = 1$

If r is less, then a is more and vice versa. So, poor reflectors are good absorbers and vice versa.

Absorbtance of a body is the ratio of the amount of thermal radiation absorbed by the body in a certain time to the total amount of thermal radiation incident on it in the same time.

No real body can absorb all the radiant energy incident on it. So, 'a' is always less than unity. However, a *perfect black body absorbs all the thermal radiation incident on it. So, its absorbtance is 100%.

Reflectance of a body is the ratio of the amount of thermal radiation reflected from it in a certain time to the total amount of thermal radiation incident on it in the same time.

Polished surfaces possess large reflectance.

Transmittance of a body is the ratio of the amount of thermal radiation transmitted by it in a certain time to the total amount of thermal radiation incident on it in the same time.

Note. The values of absorbtance, reflectance and transmittance are independent of the nature of material of the body. These merely depend on the wavelength of the incident radiation and the nature of the surface.

7.16 EMISSIVITY AND ABSORBTIVITY

Different sources of heat radiation emit different amounts of energy per second. The strength of a source of heat radiation is determined by "emissive power" or by "emissivity".

Emissive power is defined as the amount of energy emitted per unit time per unit area of a radiating force. The SI unit of emissive power is $Jm^{-2}\ s^{-1}$.

Def. 1. Emissivity is the ratio of the emissive power of a body to the emissive power of a black body at the same temperature. It is represented by ε.

$$\varepsilon = \frac{e}{E} \quad \text{or} \quad e = \varepsilon E$$

Def. 2. Emissivity of a body is the ratio of the energy radiated per unit time per unit area of the body to the energy radiated per unit time per unit area of a black body at the same temperature. It is denoted by ε. Emissivity depends on the nature of the surface and its temperature. The value of £ varies between 0 and 1.

A bright polished surface reflects most of the radiation falling on it. A rough black surface absorbs most of the radiation falling on it. Different surfaces absorb radiant energy differently. The absorbing power of a surface is determined by the "absorbtivity" of the surface.

Absorbtivity or the absorbing power of a surface for a given temperature and wavelength is the ratio of the amount of radiation absorbed by the surface in a given time to the total amount of radiation absorbed by the surface in a given time to the total amount of radiation incident upon it at the same time.

The absorbtive powers of a few substances are given below.	
Dull copper	...13%
Shellac	...72%
Indian ink	...85%

Lamp black	...98%

7.17 BLACK BODY

A perfectly black body is one which absorbs completely the radiations of all wavelengths falling on it. Since a perfectly black body neither reflects nor transmits any radiation therefore its absorbtance is unity. It is for the same reason that it appears black irrespective of the wavelength of incident radiation.

When a perfectly black body is heated to a suitable high temperature, it emits radiations of all possible wavelengths. This radiation is called black body radiation or full (total) radiation.

A perfectly black body cannot be realised in practice. The nearest approach to a perfect black body is a surface coated with lamp black or platinum black. Such a surface absorbs 96% to 98% of the incident radiation.

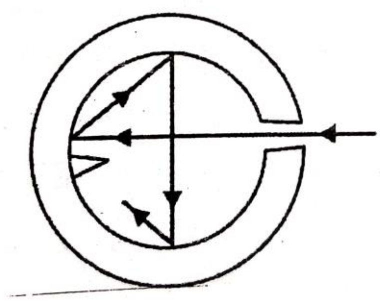


Fig. 7.10

For accurate experimental work, the black body designed by Fery is generally used. **Fery black body** is a closed double-walled hollow sphere having small opening O and a conical projection P opposite to the opening (Fig. 7.10). The projection will protect direct reflection of any radiation in the opening from the surface opposite it. It is painted black from inside. Radiation entering the opening O suffers multiple reflections at the inner walls. After a few reflections, almost the entire radiation gets absorbed. As an example, let 80% of energy be reflected at each reflection, the remaining 20% being absorbed. Then, at two reflections, 64% will be reflected and 36% will be absorbed. Thus, nearly 99% of the energy will be absorbed in 10 reflections.

When the body is heated, it becomes a source of thermal radiation. The radiation from a constant temperature enclosure depends only on the temperature of the enclosure. It does not depend on the nature of the substance of which the enclosure is made.

Note. Radiation falling on the hole is completely absorbed. When the sphere is heated, black body radiation emerges from the hole. So, it is the hole which is to be regarded as a black body and not the total enclosure.

7.18 KIRCHHOFF'S LAW

Statement. At any given temperature, the ratio of the emissive power to the absorbtive power corresponding to a certain wavelength

is constant for all bodies and this constant is equal to the emissive power of a perfectly black body at the same temperature and corresponding to the same wavelength.

This law can be proved as follows :

Consider a body placed in a uniformly heated enclosure maintained at a constant temperature T. Let an amount dQ of heat having wavelengths between λ and $\lambda + d\lambda$ be incident on a unit surface area of the body per unit time. If a_λ is the absorbtive power of the body corresponding to wavelength λ, then heat energy absorbed by the body per unit area per unit time = $a\lambda dQ$.

The remaining energy is reflected or transmitted.

∴ Heat energy reflected or transmitted per unit area per unit time

$$= dQ - a\lambda dQ = (1 - a_\lambda)\, dQ$$

If e_λ be the emissive power of the body corresponding to wavelength λ, then heat energy emitted by the body per unit area per unit time between wavelengths λ and $\lambda + d\lambda = e\lambda dQ$.

∴ Total heat energy coming from the body per unit area per unit time between wavelengths λ and $\lambda + d\lambda = (1 - a_\lambda)\, dQ + e\lambda\, d\lambda$.

The body is in thermal equilibrium with the enclosure. So, the heat energy coming from the body must be equal to the heat energy incident on the body.

∴ $$(1 - a_\lambda)\, dQ + e\lambda\, d\lambda = dQ$$

or $$e\lambda\, d\lambda = a\lambda dQ \qquad \ldots(1)$$

For a perfectly black body,

$$a\lambda = 1 \quad \text{and} \quad e\lambda = E\lambda \qquad \ldots(2)$$

∴ $$E\lambda\, d\lambda = dQ$$

Dividing (1) by (2), $$\frac{e_\lambda}{E_\lambda} = a\lambda$$

or $$\frac{e_\lambda}{a_\lambda} = E\lambda \qquad \ldots(3)$$

This relation shows that at any given temperature T and for radiations of the same wavelength λ, the ratio of the emissive power to the absorbtive power of a body is constant and is equal to the emissive power of a perfectly black body. This is Kirchhoff's law.

Note. In the above treatment, we have proved Kirchhoff law for a body placed inside a uniform temperature enclosure. But the Kirchhoff law is general and holds good under all condition of pure temperature radiation. This is due to the fact that the emissive and abortive power depend only on the nature of the body and not on its surroundings.

7.19 APPLICATIONS OF KIRCHHOFF'S LAW

It is clear from equation (3) that if e_λ is large, a_λ must also be large. So, if a body emits strongly the radiation of a particular wavelength, it must also absorb the same radiation strongly. Thus, good emitters are good absorbers.

(1) Let a piece of china with some dark painting on it be first heated to nearly 1300 K and then examined in dark room. It will be observed that the dark paintings appear much brighter than the white portion. This is because the paintings being better absorbers emit also much greater light.

(2) The silvered surface of a thermos flask does not absorb much heat from outside. This stops ice from melting quickly. Also, the silvered surface does not radiate much heat from inside. This prevents hot liquids from becoming cold quickly.

(3) A red glass appears red at room temperature. This is because it absorbs green light strongly. However, if it is heated in a furnace, it glows with green light. This is because it emits green light strongly at a higher temperature.

(4) Fraunhoffer, in 1817, discovered a large number of dark lines in the solar spectrum. These lines were named as Fraunhoffer lines. The satisfactory explanation of these lines was given by Kirchhoff on the basis of this law.

The central mass of the Sun is at a very high temperature and is called the photosphere. The light from the photosphere is such that it emits a continuous spectrum without dark lines. The photosphere is surrounded by a comparatively cooler gaseous envelope called chromosphere. It contains a number of elements such as H_2, N_2, O_2, Na, Cu etc. in the gaseous form. These elements absorb effectively the lines that they themselves emit in the photosphere. This gives rise to Fraunhoffer's lines.

7.20 STEFAN'S LAW OF BLACK BODY REDIATION

Statement. The total amount of heat energy radiated per second per unit area of a perfect black body is directly proportional to the fourth Power of the absolute temperature of the surface of the body.

This law is also known as Stefan's fourth power law..

If E be the energy radiated by a unit area of the surface of black body per second at absolute temperature T, then

$$E \propto T^4 \text{ or } E\ \sigma = \sigma T^4$$

where σ is a constant known as Stefan's constant. Its value in SI units is

$$5.67 \times 10^{-8}\ J\ m^{-2}\ S^{-1}\ K^{-4} \text{ or } W\ m^{-2}\ K^{-4}$$

Stefan derived this law experimentally in 1879. In 1894. Boltzmann gave a theoretical proof of this law bared on thermo dynamical considerations. So, this law is also known as **Stefan – Boltzmann law.**

It may be pointed out that the above law is not a law of cooling. It does not refer to the net laws of heat by a body. It merely deals with the amount of heat energy radiated by the body by virtue of its temperature, irrespective of what it gains from the surroundings. Moreover, Stefan's law applies to the whole range of wavelengths, without being limited to any particular wavelengths.

Stefan's law can be extended to represent the net loss of heat by a body.

Consider a black body at absolute temperature T surrounded by another black body at absolute temperature T_0. A unit area of the 'inner' black body Tones heat energy σT^4 per second. But it also gains heat energy σT^4 per second.

∴ Net loss of heat energy per unit area per unit time,

$$E = \sigma T^4 - \sigma T_0^4 = \sigma (T^4 - T_0^4)$$

If the body is not a perfect black body, then $E = e\sigma (T^4 - T_0^4)$

where e is called the radiation emissivity or emissivity of the surface.

The value of e depends upon the nature of the surface.

7.21 DEDUCTION OF NEWTON'S LAW OF COOLING FROM STEFAN'S LAW

$$E = \sigma (T^4 - T_0^4)$$

$$= \sigma (T^2 - T_0^2)(T^2 + T_0^2) = \sigma (T - T_0)(T + T_0)(T^2 + T_0^2)$$

If T is nearly equal to T_0, then

$$E = \sigma (T - T)(2T_0)(2T_0^2) = \sigma (T - T_0)(4T_0^3) = 4\sigma T_0^3 (T - T_0)$$

If A be the total surface area, then loss of heat energy per unit time or rate of loss of heat

$$= 4\, A\sigma\, T_0^3\, (T - T_0)$$

$\therefore$ Rate of loss of heat $\propto (T - T_0)$

So, the rate of loss of heat is proportional to the difference of temperature between the body and surroundings. This is Newton's law of cooling.

7.22 DISTRIBUTION OF ENERGY IN THE SPECTRUM OF BLACK BODY RADIATION

In 1899, Lummer and Pringsheim experimentally studied the distribution of energy in the spectrum of black body radiation at different temperatures.

An electrically heated enclosure with a hole served as a source of black body radiation. The temperature was measured by means of a thermocouple. The radiation was passed through a slit S and made incident on a concave reflector C_1 (Fig. 7.11). The radiation was then made to fall on a rock salt or flourite or *flourspar prism. The prism was placed on the turn table of a spectrometer. Like ordinary light, the thermal radiation disperses into its component wavelengths when passed through flourspar prism. The dispersed radiation in brought to a sensitive thermometer or **bolometer by a concave reflector C_2. The bolometer converts thermal energy into electrical energy. A sensitive galvanometer G is connected in series with the bolometer. The deflection of the galvanometer measures the intensity of radiation incident on the bolometer.

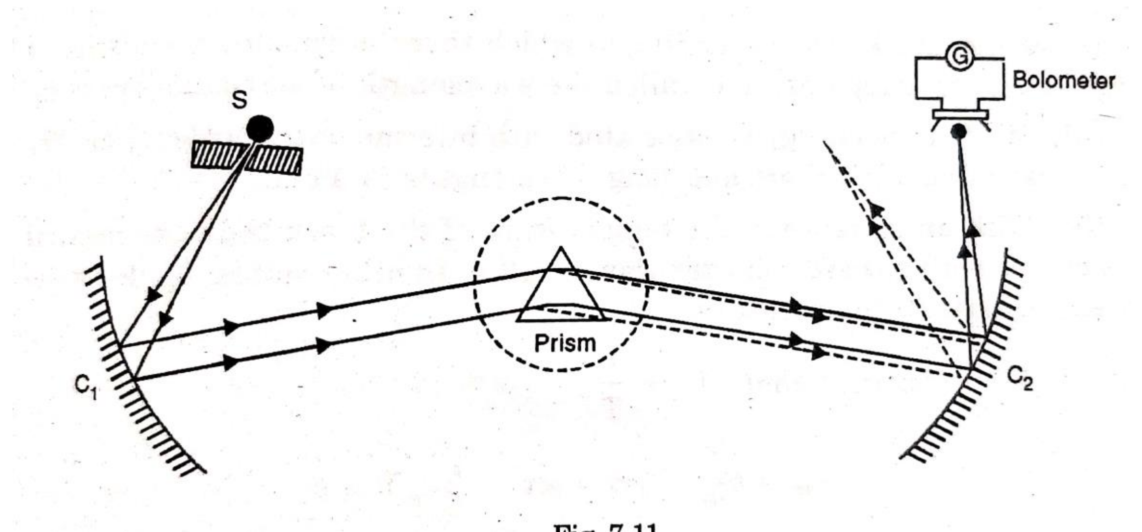


Fig. 7.11

By slow rotation of the prism, any narrow band of wavelengths can be made incident on the bolometer. The wavelength is measured with the help of diffraction grating. The intensity of radiation corresponding to different wavelengths is measured. A number of observations are taken by heating the body to different temperatures.

The results of the experiment are plotted graphically as shown in Fig. 7.12. Each curve represents the variation of monochromatic emittance E_λ with wavelength λ for a certain temperature.

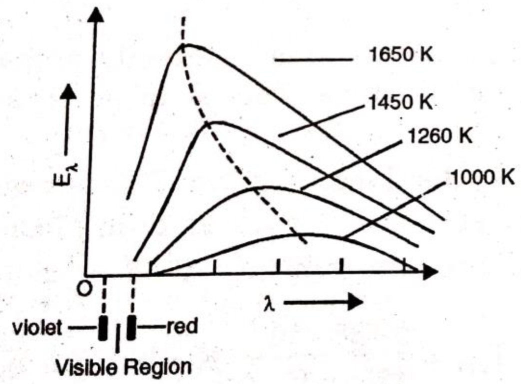


Fig. 7.12

The following conclusions may be drawn from the graphs.

(i) No curve enters the visible region. This is due to the fact that the radiation corresponding to the wavelength of the visible region is absorbed by the flour par prism.

(ii) The distribution of energy among the different wavelengths of the spectrum of black body radiation is not uniform. At a given temperature, the energy initially increases with wavelength, becomes maximum corresponding to a particular wavelength and then it begins to decrease.

The wavelength λ_m, corresponding to which there is maximum emission of energy at a given temperature is called the wavelength of maximum emission.

(iii) Most of the energy, is associated with intermediate wavelengths. The energy associated with short and long wavelengths is small.

(iv) With an increase in the temperature of the black body, the maxima of the curves shift towards shorter wavelengths. In other words, λ_m decreases with an increase in temperature.

It has been observed that $\lambda_m \propto \frac{1}{T}$

or $\lambda_m = b\frac{1}{T}$ or $\boxed{\lambda_m T = b}$ ...(i)

where b is called Wien's constant. Its value in SI units for a perfect black body is 2.888×10^{-3} m K.

Equation (1) is the mathematical statement of "Wien's displacement law. In words, the product of the wavelength λ_m corresponding to maximum radiant energy and the corresponding absolute temperature T is constant.

(v) Corresponding to each wavelength, there is an increase in energy emission with an increase in temperature.

(vi) The area under each curve represents the total energy $\int_0^\lambda e\lambda$ at the temperature for which the curve is drawn. The area increases with an increase in temperature.

It has been observed that the area is directly proportional to the fourth power of the absolute temperature. This is in accordance with the Stefan-Boltzmann law.

Importance of Wien's displacement law. This law can be used to determine the temperature of heavenly bodies such as Sun, Moon and the stars.

The value of λ_m for the Sun is nearly 4753 Å i.e., 4753×10^{-10} m.

$\therefore$ $T_{sun} = \frac{2.888 \times 10^{-3}}{4753 \times 10^{-10}}$ K = 6076.2 K

For Moon, λ_m = 14 micron, i.e., 14×10^{-6} m

$\therefore$ $T_{moon} = \frac{2.888 \times 10^{-3}}{14 \times 10^{-6}}$ K = 206.3 K

$\lambda_m T = b$
or $T = \frac{b}{\lambda_m}$

Sample Problem 7.3. A small hole is made in a hollow sphere whose walls are at 723°C. Find the total energy radiated per second per cm^2.

Given: Stefan's constant = 5.7×10^{-5} erg cm^{-2} s^{-1} K^{-4}

Solution. Temperature, T = 723 + 273) K = 996 K

Total energy radiated per second per cm^2 = σT^4

$= 5.7 \times 10^{-5} \times (996)^4$ erg

$= 5.61 \times 10^{-7}$ erg = **5.61 J**

Sample Problem 7.4. An indirectly heated filament is radiating maximum energy wavelength 2.16 × 10^{-5} cm. Find the net amount of heat energy lost per second per unit area. The temperature of surrounding air is 13°C.

Given: b = 0.288 cm K.

$\sigma = \mathbf{5.77 \times 10^{-5}\ erg\ s^{-1}\ cm^{-2}\ K^{-4}}$

Solution. Applying Wien's displacement law, λ_m T = b

or $$T = \frac{b}{\lambda_m} = \frac{0.288}{2.16 \times 10^{-5}} \text{ K} = 13333.3 \text{ K}$$

Temperature of surroundings,

T_0 = (13 + 273) K = 286K

$E = (T^4 - T_0^4)$

$= 5.77 \times 10^{-5}\ [(13333.3)^4 - (286)^4)\text{erg s}^{-1}\ \text{cm}^{-2}$

$= \mathbf{18.24 \times 10^{-8}\ J\ s^{-1}\ m^{-2}}$

EXERCISES

1. Calculate the radiant emittance of a black body at a temperature of 4000 K.
Given: $\sigma = 5.672 \times 10^{-8}$ J s^{-1} m^{-2} K^{-4}
[**Ans.** 1.452×10^{7} J s^{-1} m^{-2}]

2. A black body at temperature 400 K radiates at the rate of 1.452×10^{10} s^{-1} m^{-2}. Calculate the value of Stefan's constant.
[**Ans.** 5.672×10^{-5} erg s^{-1} cm^{-2} K^{-4}]

3. The energy emitted per second by a black body at 1227°C is E. If the temperature of the black body is increased to 2727°C, calculate the energy emitted per second in terms of E in second case.
[**Ans.** 16 E)

4. At what temperature will the filament of a 100 watt lamp operate, if it is supposed to be a perfectly black body of area 1 cm^2 ? Given : $\sigma = 5.672 \times 10^{-8}$ J s^{-1} m^{-2} K^{-4}.
[**Ans**. 2049 K]

5. A copper ball 2 cm in radius is heated in a furnace to 400°C. If its emissivity is 0.3, at what rate does it radiate energy? Given: $\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4}.
[**Ans.** 17.54 J s^{-1}]

6. To what temperature must a black body be raised in order to double total radiation if original temperature is 727°C ?
[**Ans.** 916.2°C]

7. Calculate the temperature at which a perfect black body radiation energy at the rate of I W cm^{-2}. Given : $\sigma = 5.67 \times 10^{-12}$ W cm^{-2} K^{-4}.
[**Ans.** 648 K]

8. Luminosity of Rigel star in Onion constellation is 17000 times that of our Sun. If the surface temperature of Sun is 6000 K, calculate the temperature of the star.
[**Ans.** 6.85×10^{4} K]

9. The boss A and B are kept in evacuated vessel maintained at temperature of 27°C. The temperature of A is 527°C and that of B is 127°C. Compare the rates at which heat is lost from A and B.
[**Ans.** 22.9]

10. Radiation from gives two maxima at wavelength of 4700 *A* and at 14 × 10^{-4} cm. which conclusion can you draw from this? Given: b = 0.2898 cm K.

SUMMARY

- Conduction, convection and radiation are the three modes of transfer of heat.
- Conduction is due to temperature difference and without the actual movement of the particles from their equilibrium positions.
- Thermal conductivity of a good conductor is determined by Searle's method.
- Thermal conductivity of a bad conductor is determined by Lee's method.
- In convection, the transfer of heat is by the actual movement of the particles.
- According to Prevost's theory, all bodies above 0 K emit heat to the surroundings and gain heat from the surroundings.
- A body which completely absorbs all the radiations falling on it is called perfectly black body.
- According to Kirchhoff's law, the ratio of the emissive power to the absorptive power is constant.
- According to Stefan's law, the total amount of heat energy radiated per second per unit of a perfect black body is directly proportional to the fourth power of the absolute temperature of the surface of the body.

TEST YOURSELF

1. What do you understand by the following modes of transfer of heat ?
(a) Conduction (b) Convection (e) Radiation

2. Define coefficient of thermal conductivity. Give its unit and dimensional formula

3. How can you determine the thermal conductivity of a good conductor ?

4. Give a method for the determination of thermal conductivity of a bad conductor.

5. Discuss conduction of heat in a compound bar made from n conductors in series.

6. What are the characteristics of heat radiation ?

7. Give the salient features of Prevost's theory of heat exchanges.

8. What do you understand by the term "black body* ? Write a note on Fery's black body.

9. Discuss the distribution of energy in the spectrum of black body radiation.

10. What is Wien's displacement law ? How can we use this law to determine the temperature of Moon?

11. What do you understand by the following terms ?

(i) Absorbtance (ii) Reflectance (iii) Tranmittance

(iv) Emissivity (v) Absorbtivity

12. State and explain Kirchhoff's law.

13. Give two important applications of Kirchhoff's law.

14. What is Stefan's law of black body radiation ?