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1. QUADRATIC EQUATIONS

LEARNING OBJECTIVES

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INTRODUCTION

In our earlier classes, we have already learnt about expressions of the type :

$$2x^2 + 5x - 6,\ \ 7x - 3,\ \ x^4 - x^3 + 2x + 9 \text{ etc.}$$

These are called *polynomials* in the variable 'x'.

We say that the *solution* of the *polynomial equation* $2x - 5 = 0$ is $x = 5/2$, because this value of x satisfies the equation $2x - 5 = 0$.

In the present chapter, we shall study the methods of finding real and complex roots of the polynomial equations of the type $ax^2 + bc + c = 0$ where a, b, c , are arbitrary complex numbers

IDENTITY AND EQUATION

An **identity** is a statement of equality between two expressions which is free for all value of the variable involved.

For example, $(x-1)^2+4(x-1)+10-x^2-2x-7=0$ is an identity, because the above statement is true for all values of x.

An **equation** is a statement of equality between two expressions which is not true for all values of the variable involved.

For example:

(i) $\sin x = 0$ is true for $x = 0, \pm 2\pi$ ....,

(ii) $x^2-5x+6=0$ is true for $x=2$, 3.

A polynomial equated to zero is called a **polynomial equation.** The **degree** of a polynomial equation is same as the *degree* of the corresponding polynomial.

For example, $2x^2-7x+6=0$ is a polynomial equation of degree 2.

A polynomial equation of degree 2 is called a **quadratic equation.**

ROOT OF AN EQUATION

A value of the variable for which an equation is satisfied is called a **root** of the equation, under consideration.

For example, $\frac{5}{2}$ is root of $4x^2-16x+15=0$, because

$$4\left(\frac{5}{2}\right)^2-16\left(\frac{5}{2}\right)+15=25-40+15=0.$$

Also, 1 is not a root of $4x^2-16x+15=0$, because

$$4(1)^2-16(1)+15=4-16+15=3\neq 0.$$

FACTORIZATION METHOD OF SOLVING A QUADRATIC EQUATION

The principal underlying this method is that if $xy=0$ then either $x=0$ or $y=0$.

Let $b=0$, then $ax^2+c=0$ *i.e.,* $x^2=-c/a$ or $x=\pm\sqrt{-c/a}$. So, let us assume that $b\neq 0$.

The **method of factorization** is applicable only if we can write *b* as the sum of two numbers whose product is *ac*. The value of *b* is changed in the given equation and the factorization is carried.

For example, consider the quadratic equation

$$x^2-x-6=0. \qquad .....(1)$$

Here *b* = - 1, We write $b=-1=(-3)+2,$ because (-3)(2) = -6 = (1)(-6).

$\therefore$ (1) $\Rightarrow$ $x^2-3x+2x-6=0$ $\Rightarrow$ $x(x-3)+2(x-3)=0$

$\Rightarrow$ $(x-3)(x+2)=0$ $\Rightarrow$ $x=3,\ -2$.

Example 1. *Solve the equation :* $\dfrac{x+2}{x+3}=\dfrac{x+4}{2x+3}$.

Sol. We have $\dfrac{x+2}{x+3}=\dfrac{x+4}{2x+3}$.

$\Rightarrow$ $(x+2)(2x+3)=(x+3)(x+4)$

$\Rightarrow$ $2x^2+3x+4x+6=x^2+4x+3x+12$

$\Rightarrow$ $x^2=6$ $\Rightarrow$ $x=\pm\sqrt{6}$

$\therefore$ The roots are $-\sqrt{6}$ and $\sqrt{6}$.

WORKING RULES FOR SOLVING PROBLEMS

Rule I. *If the equation to solve is* $ax^2+c=0$, *then write* $x^2=-c/a$ and then $x=\underline{+}\sqrt{-c/a}$. *The roots are* $\sqrt{-c/a}$ *and* $\sqrt{-c/a}$.

Rule II. *If the equation to solve is* $ax^2+bc+c=0$, *then find two number* l and m Such that $l+m=b$ and $lm=ac$. *Put* $b=l+m$ *in the equation and Factorize the L.H.S*

EXERCISE 1.1

SHORT ANSWER TYPE QUESTIONS

Solve the following equations by the method of factorization:

1. $7x^2+49=0$

2. $2x^2+1=0$

LONG ANSWER TYPE QUESTIONS

Solve the following equations by the method of factorization :

3. $2z^2-10=z$

4. $2x^2+3i\,x+2=0$

5. $abx^2-(a+b)x+1=0$

6. $\frac{x-p}{q}+\frac{x-q}{p}=\frac{q}{x-p}+\frac{p}{x-q}$

Answers

1. $\pm\sqrt{7}i$

2. $+\sqrt{2i}/2$

3. $-2,\ 5/2$

4. $i/2,-2i$

5. $\frac{1}{a},\frac{1}{b}$

6. $0,\ p+q,\ \frac{p^2+q^2}{p+q}$.

FORMULA METHOD OF SOLVING A QUADRATIC EQUATION

The 'formula method' of solving a quadratic equation is used when the 'factorization method' is not easily application.

Let $ax^2+bx+c=0,\qquad a\neq 0$ ....(1)

be the given quadratic equation where *a, b, c* are *complex numbers*.

$$(1)\quad\Rightarrow\quad ax^2+bc=-c\quad\Rightarrow\quad x^2+\frac{b}{a}x=-\frac{c}{a}\qquad(\because\ a\neq 0)$$

$$\Rightarrow\quad x^2+2\left(\frac{b}{2a}\right)x=-\frac{c}{a}\quad\Rightarrow\quad x^2+2\left(\frac{b}{2a}\right)x+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}$$

$$\Rightarrow\quad\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}\quad\Rightarrow\quad x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$\Rightarrow \qquad x = -\frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{2a}}$$

$$\Rightarrow \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

These are the required roots of the given equation.

Example 2. *Solve* : $x^2 + 2|x| - 8 = 0$

Sol. We have $x^2 + 2|x| - 8 = 0 \quad \therefore \quad |x|^2 + 2|x| - 8 = 0 \qquad (\because |x|^2 = x^2)$

$$\therefore \qquad |x| = \frac{-2 \pm \sqrt{4 + 32}}{2} = \frac{-2 \pm 6}{2} = -4, 2$$

$|x|$ = - 4 is impossible and $|x|$ = 2 $\Rightarrow$ $x = \pm\ 2$.

$\therefore$ Roots are **- 2, 2**.

WORKING RULES FOR SOLVING PROBLEMS

Step I. *Simplify the given equation and express it in the form* $ax^2 + bx + c = 0$.

Step II. *Identify the values of a , b and c.*

Step III. *Use the formula :* $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ *and simplify it.*

Step IV. *The values of* x *are the roots of the given equation.*

EXERCISE 1.2

SHORT ANSWER TYPE QUESTIONS

Solve the following equations by the formula method :

1. $x^2 - 9x + 20 = 0$

2. $x^2 - x - 12 = 0$

3. $x^2 + x + 1 = 0$

4. $x^2 + 2x + 2 = 0$.

5. $3x^2 - 7x + 5 = 0$

6. $9x^2 + 10x + 3 = 0$

7. $21x^2 - 29x + 11 = 0$

8. $x^2 + 4ix - 4 = 0$.

Answers

1. 4, 5 **2.** -3, 4 **3.** $-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ **4.** $-1 \pm i$

5. $\frac{7}{6} \pm \frac{\sqrt{11}}{6} i$ **6.** $-\frac{5}{9} \pm \frac{\sqrt{2}}{9} i$ **7.** $\frac{29}{42} \pm \frac{\sqrt{83}}{42} i$ **8.** $-2i, -2i$

EQUATIONS REDUCIBLE TO QUARATIC FORM

In this section, we shall solve equations which are not quadratic, but could be reduced to quadratic form by certain substitutions.

Type I. Equation of the form $aX^2 + bX^n + c = 0$**, where** X **is more function of x.**

WORKING RULES FOR SOLVING $aX^{2n} + bX^n + c = 0$**,**

WHERE X IS SOME FUNCTION OF X

Step I. *Put* $X^n = y$ *and get the quadratic equation* $ay^2 + by + c = 0$.

Step II. *Solve this equation and get two values of* y.

Step III. *Find the values of* x *by putting* $y = X^n$.

Example 3. *Solve the equation* : $(x^2 - 5x + 7)^2 - (x-2)(x-3) = 1$.

Sol. We have $\left(x^2 - 5x + 7\right)^2 - (x-2)(x-3) = 1$.

$\Rightarrow \quad (x^2 - 5x)^2 + 49 + 14(x^2 - 5x) - (x^2 - 5x + 6) = 1$

$\Rightarrow \quad (x^2 - 5x)^2 + 13(x^2 - 5x) + 42 = 0 \qquad \text{....(1)}$

Let y = x² – 5x $\quad \therefore \quad (1) \quad \Rightarrow \quad y^2 + 13y + 42 = 0 \Rightarrow (y+7)(y+6) = 0$

$\therefore$ y = - 7, - 6

$\therefore$ Either y = - 7	or y = - 6
$\therefore \quad x^2 - 5x = -7$ or $x^2 - 5x + 7 = 0$	$\therefore \quad x^2 - 5x = -6$ or $x^2 - 5x + 6 = 0$
$\therefore \quad x = \frac{5 \pm \sqrt{25 - 28}}{2} = \frac{5 \pm i\sqrt{3}}{2}$	$\therefore \quad x = \frac{5 \pm \sqrt{25 - 24}}{2} = \frac{5 \pm 1}{2} = 2,3$

∴ The roots are $\frac{5 \pm i\sqrt{3}}{2}$, 2 ,3 .

Type II. Equation of the form $a(p^X)^2 + b(p^X) + c = 0$.

WORKING RULES FOR SOLVING $a(p^X)^2 + b(p^X) + c = 0$
Step I. *Put* $p^x = y$ *and get the quadratic equation* $ay^2 + by + c = 0$
Step II. *Solve this equation and get two values of* y .
Step III. *Find the values of* x *by putting* $y = p^x$

Example 4. Solve the equation :

$$5^{2x} - 5^{x+3} + 125 = 5^x$$

Sol. We have $5^{2x} - 5^{x+3} + 125 = 5^x$.

$\Rightarrow \quad (5^x)^2 - 5^x.\ 5^3 + 125 - 5^x = 0 \quad \Rightarrow \quad (5^x)^2 - 5^x\,(5^3 + 1) + 125 = 0$

$\Rightarrow \quad (5^x)^2 - 126.5^x + 125 = 0 \qquad \ldots.(1)$

Let $y = 5^x$ $\quad \therefore \quad (1) \quad \Longrightarrow \quad y^2 - 126y + 125 = 0$

$$\therefore \quad y = \frac{126 \pm \sqrt{(-126)^2 - 4(1)(125)}}{2(1)} = 1,\ 125$$

∴ Either $y = 1$ $\Rightarrow \quad 5^x = 1 \ \Rightarrow \ 5^x = 5^0 \ \Rightarrow \ x = 0$	or $y = 125$ $\Rightarrow \quad 5^x = 125 \ \Rightarrow \ 5^x = 5^3 \Rightarrow \ x = 3$

∴ The roots are **0, 3**.

Type III. Equation of the form $aX + \frac{b}{X} + c = 0$, where X is some function of x.

WORKING RULES FOR SOLVING $aX + \frac{b}{X} + c = 0$, WHERE X IS SOME FUNCTION OF x
Step I. *Put* $X = y$ *and get the equation* $ay + \frac{b}{y} + c = 0$.

Step II. *Multiply both sides by* y *and get the quadratic equation*

$ay^2 + cy + b = 0$.

Step III. *Solve this equation and get two values of* y.

Step IV. *Find the values of* x *by putting* $y = X$

Example 5. *Solve the equation :*

$$8\sqrt{\frac{x}{x+3}} - \sqrt{\frac{x+3}{x}} = 2$$

Sol. We have $8\sqrt{\frac{x}{x+3}} - \sqrt{\frac{x+3}{x}} = 2$ . .....(1)

Let $y = \sqrt{\frac{x}{x+3}}$ $\therefore$ (1) $\Rightarrow$ $8y - \frac{1}{y} = 2$ $\Rightarrow 8y^2 - 1 = 2y \Rightarrow 8y^2 - 2y - 1 = 0$

$$\therefore \quad y = \frac{2 \pm \sqrt{4+32}}{16} = \frac{2 \pm 6}{16} = \frac{1}{2}, -\frac{1}{4}$$

$\therefore$ Either $y = 1/2$	or $y = -1/4$.
$\Rightarrow \sqrt{\frac{x}{x+3}} = \frac{1}{2}$	$\Rightarrow \sqrt{\frac{x}{x+3}} = -\frac{1}{4}$
$\Rightarrow \frac{x}{x+3} = \frac{1}{4} \Rightarrow 4x = x+3 \Rightarrow x = \frac{3}{3} = 1$.	This is impossible, because L.H.S. is non-negative.

$\therefore$ the root is **1.**

Type IV. Equation of the form λ(x+ a) (x+ b) (x + c) (x+ d) = k.

WORKING RULES FOR SOLVING λ(x+ a) (x+ b) (x + c) (x+ d) = k

Step I. *Express the given equation in the form λ(x+ a) (x+ b) (x + c) (x+ d) = k, if already not so. Note that the coefficient of x in each factor should be '1'.*

Step II. *Group a, b, c, d into two pairs having equal sums. Let a + b = c + d.*

Step III. *Write the equation in the form: λ(x+ a) (x+ b) (x + c) (x+ d) = k.*

Multiply the factors and get $\lambda\left[x^2+(a+b)x+ab\right]\left[x^2+(c+d)x+cd\right]=k$

i.e., $\lambda\left[x^2+(a+b)x+ab\right]\left[x^2+(c+d)x+cd\right]=k$, *because a + b = c + d.*

Step IV. *Put* $x^2+(a+b)x=y$ *and get the equation* $\lambda(y+ab)(y+cd)=k$ *.*

Simplify and get the quadratic equation $\lambda y^2+\lambda(ab+cd)y+\lambda abcd-k=0$ *.*

Step V. *Solve the equation and get two values of* y*.*

Step VI. *Find the value of x by putting* $y=x^2+(a+b)x$ *.*

Example 6. *Solve the equation :*

$$x(x+1)^2(x+2)=72$$

Sol. We have $x(x+1)^2(x+2)=72$

$\Rightarrow$ $(x+0)(x+1)(x+1)(x+2)=72$ .....(1)

Now $0+2=1+1$ (each = 2)

$\therefore$ (1) implies $\left[(x+0)(x+2)\right]\left[(x+1)(x+1)\right]=72$

$\Rightarrow$ $(x^2+2x)(x^2+2x+1)=72$ .....(2)

Let y = x² + 2x $\therefore$ (2) $\Rightarrow$ $y(y+1)=72$ $\Rightarrow$ $y^2+y-72=0$

$\Rightarrow$ $y=\dfrac{-1\pm\sqrt{1+288}}{2}=\dfrac{-1\pm 17}{2}=8,-9$

$\therefore$ Either $y = 8$ $\Rightarrow$ $x^2+2x=8$ $\Rightarrow$ $x^2+2x-8=0$ $\Rightarrow$ $x=\dfrac{-2\pm\sqrt{4+32}}{2}=\dfrac{-2\pm 6}{2}=2,-4$	or $y = -9$ $\Rightarrow$ $x^2+2x=-9$ $\Rightarrow$ $x^2+2x+9=0$ $\Rightarrow$ $x=\dfrac{-2\pm\sqrt{4-36}}{2}=\dfrac{-2\pm 4\sqrt{2}i}{2}$ $=-1\pm 2\sqrt{2}i$.

$\therefore$ The roots are $\mathbf{2,-4,-1\pm 2\sqrt{2}i}$.

Type V. Equation of the form $ax^4 \pm bx^3 + cx^2 \pm bx + a = 0$. In this equation, coefficients of terms equidistant from beginning and end are numerically equal.

WORKING RULES FOR SOLVING $ax^4 \pm bx^3 + cx^2 \pm bx + a = 0$

Step I. *Divide both sides of the equation by x^2 and get*

$$ax^2 \pm bx + c \pm \frac{b}{x} + \frac{a}{x^2} = 0$$

Step II. *Collect terms equidistant from beginning and end.*

Step III. *Put* $x + \frac{1}{x} = y$ or $x - \frac{1}{x} = y$, *as per requirement, and get a quadratic equation.*

Step IV. *Solve this equation and get two values of* y.

Step V. *Find the values of x by putting* $y = x + \frac{1}{x}\left(or\ x - \frac{1}{x}\right)$.

Example 7. *Solve the equation :*

$$6x^4 - 25x^3 + 12x^2 + 25x + 6 = 0$$

Sol. We have $6x^4 - 25x^3 + 12x^2 + 25x + 6 = 0$.

This is a *reciprocal equation.* Dividing throughout by x^2, we get

$$\frac{6x^4}{x^2} - 25\frac{x^3}{x^2} + 12\frac{x^2}{x^2} + 25\frac{x}{x^2} + \frac{6}{x^2} = \frac{0}{x^2}$$

$$\Rightarrow \qquad 6x^2 - 25x + 12 + \frac{25}{x} + \frac{6}{x^2} = 0$$

Grouping terms equidistant from beginning and end, we get

$$6\left(x^2 + \frac{1}{x^2}\right) - 25\left(x - \frac{1}{x}\right) + 12 = 0$$

Let $y = \mathrm{X} - \frac{1}{\mathrm{X}}$ $\quad \therefore \quad x^2 + \frac{1}{x^2} = \left(x^2 + \frac{1}{x^2} - 2\right) + 2 = \left(x - \frac{1}{x}\right)^2 + 2 = y^2 + 2$.

$\therefore$ (1) $\Rightarrow 6(y^2+2)-25y+12=0 \Rightarrow 6y^2-25y-24=0 \Rightarrow y=3/2, 8/3$

$\therefore$ Either $y=3/2$	or $y=8/3$
$\Rightarrow x-\frac{1}{x}=\frac{3}{2} \Rightarrow \frac{x^2-1}{x}=\frac{3}{2}$	$\Rightarrow x-\frac{1}{x}=\frac{8}{3} \Rightarrow \frac{x^2-1}{x}=\frac{8}{3}$
$\Rightarrow 2x^2-3x-2=0$	$\Rightarrow 3x^2-8x-3=0$
$\therefore x=\frac{3\pm\sqrt{9+16}}{4}=\frac{3\pm 5}{4}=-\frac{1}{2},2$	$\therefore x=\frac{8\pm\sqrt{64+36}}{6}=\frac{8\pm 10}{6}=-\frac{1}{3},3$

$\therefore$ The roots are **-1/2, 2, -1/3, 3.**

Type VI. Equation of the form $\sqrt{ax+b}\pm\sqrt{cx+d}=k$ **or** $\sqrt{ex+f}$

WORKING RULES FOR SOLVING $\sqrt{ax+b}\pm\sqrt{cx+d}=k$ **or** $\sqrt{ex+f}$

Step I. *Square both sides of the equation.*

Step II. *Transpose the terms, so that the expression under radical sign in on one side.*

Step III. *Square both sides again and solve it and get the values of x.*

Step IV. *Test all values of x so obtained and reject those values which do not Satisfy the given equation.*

Example 8. *Solve the equation :*

$$\sqrt{1-5x}+\sqrt{1-3x}=2$$

Sol. We have $\sqrt{1-5x}+\sqrt{1-3x}=2$ . ....(1)

Squaring, we get $(1-5x)+(1-3x)+2\sqrt{1-5x}\sqrt{1-3x}=4$

$\Rightarrow 2\sqrt{(1-5x)(1-3x)}=2+8x \Rightarrow \sqrt{(1-5x)(1-3x)}=1+4x$

Squaring again, we get $1-5x-3x+15x^2=1+16x^2+8x$

$\Rightarrow x^2+16x=0 \Rightarrow x(x+16)=0$

$\therefore \quad x = 0, -16$

$x = 0$ is a root of (1) if $\sqrt{1-5(0)} + \sqrt{1-3(-16)} = 2$

or if $\sqrt{1}+\sqrt{1} = 2$ if 2 = 2, which is true. $\therefore$ $x = 0$ is a root.

$x = -16$ is a root of (1) if $\sqrt{1-5(-16)} + \sqrt{1-3(-16)} = 2$

or if $\sqrt{81}+\sqrt{49} = 2$ if 9 + 7 = 2 if 16 = 2, which is not true.

$\therefore$ $x = -16$ is an extraneous root. $\therefore$ The only root is **0**.

Type VII. Equation of the form $p(ax^2 + bx + c) + q\sqrt{ax^2 + bx + c} = r$ **.**

WORKING RULES FOR SOLVING $p(ax^2 + bx + c) + q\sqrt{ax^2 + bx + c} = r$ **.**

Step I. *Put* $\sqrt{ax^2+bx+c} = y$ *and get the quadratic equation* $py^2 + qy - r = 0$.

Step II. *Solve this equation and get two values of* y. *If any or both values of* y *are negative, then we reject those values, because* $y = \sqrt{ax^2+bx+c}$ *is always non-negative.*

Step III. *Find the values of x by using* $y = \sqrt{ax^2+bx+c}$

Example 9. *Solve the equation :*

$$8 + 9\sqrt{(3x-1)(x-2)} = 3x^2 - 7x .$$

Sol. We have $\quad 8 + 9\sqrt{(3x-1)(x-2)} = 3x^2 - 7x .$

$$\Rightarrow \quad 9\sqrt{3x^2 - 6x - x + 2} = 3x^2 - 7x - 8$$

$$\Rightarrow \quad 9\sqrt{3x^2 - 7x + 2} = (3x^2 - 7x + 2) - 10 \quad \ldots(1)$$

Let $y = \sqrt{3x^2 - 7x + 2}$ $\quad\therefore$ (1) $\Rightarrow$ $9y = y^2 - 10$ $\Rightarrow$ $y^2 - 9y - 10 = 0$

$$\Rightarrow \quad y = \frac{9 \pm \sqrt{81 + 40}}{2} = \frac{9 \pm 11}{2} = -1, 10$$

$\therefore$ Either $y = -1$	or $y = 10$.
$\Rightarrow \sqrt{3x^2 - 7x + 2} = -1$	$\Rightarrow \sqrt{3x^2 - 7x + 2} = 10$
This is impossible because L.H.S. is non – negative.	$\Rightarrow \sqrt{3x^2 - 7x + 2} = 100$
	$\Rightarrow 3x^2 - 7x - 98 = 0$
	$\therefore x = \dfrac{7 \pm \sqrt{49 + 1176}}{6} = 7, -\dfrac{14}{3}$.

$\therefore$ The roots are **7, -14/3.**

Type VIII. Equation of the form $\sqrt{ax^2 + bx + c} \pm \sqrt{dx^2 + ex + f} = k$ **or** $gx + h$.

WORKING RULES FOR SOLVING

$$\sqrt{ax^2 + bx + c} \pm \sqrt{dx^2 + ex + f} = k \textbf{ or } gx + h$$

Step I. *Let the given equation be* $\sqrt{ax^2 + bx + c} \pm \sqrt{dx^2 + ex + f} = k$

Step II. *Put* $\sqrt{ax^2 + bx + c} = A$ *and* $\sqrt{dx^2 + ex + f} = B$ *and get the equation*

$$A - B = k \qquad \ldots(1)$$

Step III. *Simplify* $A^2 - B^2$ *and let it be* $p(x)$

$$\therefore \quad A^2 - B^2 = p(x), \qquad \ldots.(2)$$

where p(x) is either a quadratic polynomial or a linear polynomial or a constant

Step IV. *Divide (2) by (1) and get* $\dfrac{A^2 - B^2}{A - B} = \dfrac{p(x)}{k}$ *i.e.,* $A + B = \dfrac{p(x)}{k}$ $\qquad \ldots.(3)$

Step V. *Solve (1) and (3) and get the value of A (or of B).*

Step VI. *Find the value of x by putting* $A\sqrt{ax^2 + bx + c}$. *The value of B will also give the same value of x.*

Other forms of the given equation are also solved by following the same method.

Example 10. *Solve the equations :*

$$\sqrt{5x^2 - 6x + 8} - \sqrt{5x^2 - 6x - 7} = 1$$

Sol. We have $\sqrt{5x^2-6x+8}-\sqrt{5x^2-6x-7}=1$ ....(1)

Let A = $\sqrt{5x^2-6x+8}$ *and B* = $\sqrt{5x^2-6x-7}$

$\therefore$ (1) $\Rightarrow$ $A - B = 1$ ...(2)

Now, $A^2-B^2=(5x^2-6x+8)-(5x^2-6x-7)=15$

$\therefore$ $A^2-B^2=15$ ...(3)

Dividing (3) by (2), we get $\dfrac{A^2-B^2}{A-B}=\dfrac{15}{1}$.

$\Rightarrow$ $A+B=15$ ....(4)

(1) + (4) $\Rightarrow$ $2A=16$ *i.e.,* $A = 8$ $\therefore$ $\sqrt{5x^2-6x+8}=8$

or $5x^2-6x+8=64$ *i.e.,* $5x^2-6x-56=0$

$\therefore$ $=\dfrac{6\pm\sqrt{36+1120}}{10}=\dfrac{6\pm 34}{10}=4,\ -\dfrac{14}{5}$.

$\therefore$ The roots are **4, -14/5.**

Remark. In the above example, (2) – (4) implies – 2B = - 14 *i.e.,* B = 7

$\therefore$ $\sqrt{5x^2-6x-7}=7$ *i.e.,* $5x^2-6x-56=0$

Solving this equation, we shall get the same value of x as we got by using A = 8.

Type IX. Equation of the form $\dfrac{\sqrt{x+k}+\sqrt{x-k}}{\sqrt{x+k}-\sqrt{x-k}}=\lambda$

WORKING RULES FOR SOLVING $\dfrac{\sqrt{x+k}+\sqrt{x-k}}{\sqrt{x+k}-\sqrt{x-k}}=\lambda$

Step I. *Apply Componendo and Dividendo property* $\left(i.e., \dfrac{a}{b}=\dfrac{c}{d}\Rightarrow\dfrac{a+b}{a-b}=\dfrac{c+d}{c-d}\right)$ *on both sides of the given equation.*

Step II. *Square both sides of this equation and simplify to get the values of x.*

Step III. *Test all values of x so obtained and reject those values which do Not satisfy the given equation.*

Example 11. *Solve the equation :* $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=\dfrac{a}{x}$.

Sol. We have $$\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=\dfrac{a}{x}.$$

Applying componendo and dividend, we get

$$\frac{\left(\sqrt{a+x}+\sqrt{a-x}\right)+\left(\sqrt{a+x}-\sqrt{a-x}\right)}{\left(\sqrt{a+x}+\sqrt{a-x}\right)-\left(\sqrt{a+x}-\sqrt{a-x}\right)}=\frac{a+x}{a-x}$$

$\Rightarrow$ $$\frac{2\sqrt{a+x}}{2\sqrt{a-x}}=\frac{a+x}{a-x} \quad \Rightarrow \quad \sqrt{a+x}\left(a-x\right)=\sqrt{a-x}\left(a+x\right)$$

$\Rightarrow$ $\sqrt{a+x}\sqrt{a-x}\sqrt{a-x}-\sqrt{a+x}=0$

$\therefore$ Either $\sqrt{a+x}=0$ ...(1) or $\sqrt{a-x}=0$ ....(2)

or $\sqrt{a-x}-\sqrt{a+x}=0$ ....(3)

(1) $\Rightarrow$ $a+x=0$ *i.e.,* $x=-a$ (2) $\Rightarrow$ $a-x=0$ *i.e.,* $x=a$

(3) $\Rightarrow$ $\sqrt{a-x}=\sqrt{a+x}$ $\Rightarrow$ $a-x=a+x$ $\Rightarrow$ $2x=0$ *i.e.,* $x=0$

$x=0$ does not satisfy the given equation. $\therefore$ The root are $\pm$**a.**

Type X. Equation of the form $a(1+x)^{2/3}\pm b(1-x)^{2/3}=c(1-x^2)^{1/3}$

WORKING RULES FOR SOLVING $a(1+x)^{2/3}\pm b(1-x)^{2/3}=c(1-x^2)^{1/3}$

Step I. *Cube both sides of the equation by using the formula :*

$(a+b)^3=a^3+b^3+3ab(a+b)$ or $(a-b)^3=a^3-b^3-3ab(a-b)$.

Step II. *Replace* $a(1+x)^{2/3}\pm b(1-x)^{2/3}$ *by* $c(1-x^2)^{1/3}$ in this equation.

Step III. *Simplify the equation and get a quadratic equation in x.*

Step IV. *Solve this quadratic equation to get the required value of x.*

Example 12. *Solve the equation* $(1+x)^{2/3}+(1-x)^{2/3}=3(1-x^2)^{1/3}$.

Sol. We have $$(1+x)^{2/3}+(1-x)^{2/3}=3(1-x^2)^{1/3} \quad \text{....(1)}$$

Cubing, we get $$\left[(1+x)^{2/3}+(1-x)^{2/3}\right]^3=\left[3(1-x^2)^{1/3}\right]^3$$

$$\Rightarrow \quad (1+x)^2+(1-x)^2+3(1+x)^{2/3}(1-x)^{2/3}\left[(1+x)^{2/3}+(1-x)^{2/3}\right]=27(1-x^2)$$

$$\Rightarrow \quad (1+x^2+2x)+(1+x^2-2x)+3(1-x^2)^{2/3}.\ 3(1-x^2)^{1/3}=27-27x^2$$

[Using (1)]

$$\Rightarrow \quad 29x^2-25+9(1-x^2)=0 \Rightarrow 29x^2-25+9-9x^2=0$$

$$\Rightarrow \quad 20x^2=16 \Rightarrow x^2=\frac{16}{20}=\frac{4}{5}$$

$\therefore$ $x=\pm 2\sqrt{5}$. $\therefore$ The roots are $\pm 2/\sqrt{5}$.

EXERCISE 1.3

LONG ANSWER TYPE QUESTIONS

Solve the following equations :

1. (i) $x^4-8x-9=0$ (ii) $(x^2-5x)^2-30(x^2-5x)-216=0$

2. (i) $2^{x+1}+4^x=8$ (ii) $7^{1+x}+7^{1-x}=50$

3. (i) $\sqrt{3x+1}-\sqrt{x-1}=2$ (ii) $\sqrt{x+2}+\sqrt{x+7}=\sqrt{6x+13}$

4. (i) $3x^2+15x-2=2\sqrt{x^2+5x+1}$ (ii) $12+9\sqrt{(x-1)(3x+2)}=3x^2-x$

5. (i) $\sqrt{x^2+3x+32}+\sqrt{x^2+3x+5}=9$ (ii) $\sqrt{x^2-3x+36}-\sqrt{x^2-3x+9}=3$

Answers

1. (i) $\pm 3, \pm i$ (ii) -4, 2, 3, 9 **2.** (i) 1 (ii) ± 1

3. (i) 1, 5 (ii) 2

4. (i) $-\frac{16}{3}, \frac{1}{3}$ (ii) $-\frac{17}{3}$ **5.** (i) -4, 1 (ii) 0, 3

SIMULTANEOUS EQUATIONS IN TWO VARIABLES

The different methods of solving simultaneous equations are illustrated below:

Example 13. *Solve the equations :*

(i) $x^2 + y^2 = 185$, $x + y = 19$ (ii) $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = \frac{10}{3}$, $x + y = 10$

Sol. (i) We have $x^2 + y^2 = 185$ ...(1) and $x + y = 19$ .....(2)

(2) implies $y = 19 - x$.

Putting this value of y in (1), we get

$$x^2 + (19 - x^2) = 185 \quad i.e., \quad x^2 + 361 + x^2 - 38x = 185$$

$\Rightarrow$ $2x^2 - 38x + 176 = 0$ $\Rightarrow$ $x^2 - 19x + 88 = 0$

$\Rightarrow$ $(x-8)(x-11) = 0$ $\Rightarrow$ $x = 8, 11$

$\therefore$ Either $x = 8$ $\therefore$ $y = 19 - x = 19 - 8 = 11$	or $x = 11$ $\therefore$ $y = 19 - x = 19 - 11 = 8$

$\therefore$ The solution is x = **8**, y = **11** ; x =**11**, y =**8**.

(ii) We have $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = \frac{10}{3}$ ...(1) and $x + y = 10$ ....(2)

(1) $\Rightarrow$ $\frac{\sqrt{x}}{\sqrt{y}} + \frac{\sqrt{y}}{\sqrt{x}} = \frac{10}{3}$ $\Rightarrow$ $\frac{x+y}{\sqrt{xy}} = \frac{10}{3}$ $\Rightarrow$ $\frac{10}{\sqrt{xy}} = \frac{10}{3}$ [using (2)]

$\Rightarrow$ $\sqrt{xy} = 3$ $\Rightarrow$ $xy = 9$

$\Rightarrow \quad x(10-x)=9$ [using (2)]

$\Rightarrow \quad x^2-10x+9=0 \quad \Rightarrow x = 1, 9$

$\therefore$ Either $x=1$ $\therefore \quad y=10-x=10-1=9$	or $x=9$ $\therefore \quad y=10-x=10-9=1$

$\therefore$ The solution is x = **1**, y = **9**; x = **9**, y = **1**.

EXERCISE 1.4

LONG ANSWER TYPE QUESTIONS

Solve the following simultaneous equations :

1. $x+2y=1, x^2+y^2=10$

2. $x+y=20, xy=64$

3. $x+y=10, \sqrt{\dfrac{x}{y}}+\sqrt{\dfrac{y}{x}}=\dfrac{5}{2}$

4. $4x-3y=1, 12xy+13x^2=25$

Answers

1. $x=3, y=-1; x=-13/5, y=9/5$

2. $x=16, y=4; x=4, y=16$

3. $x=8, y=2; x=2, y=8$

4. $x=1, y=1; x=-25/29, y=-43/29$

SUMMARY

1. (i) An expression of the form $a_0x^n + a_1x^{n-1} + ... + a_n$, where n is a non-negative integer and $a_0, a_1, ..., a_n$ belong to some number system **F**, is called **a polynomial** in the variable x over **F.**

(ii) The **degree** of polynomial is defined as the highest index of the variable X occurring in the polynomial.

2. (i) An **identity** is a statement of equality between two expressions which is true for all values of the variable involved.

3. If *f(x)* = 0 is a polynomial equation and *f*(*a*) = 0, then *a* is called **a root** of the polynomial equation.

4. If $ax^2 + bx + c = 0,\ a \neq 0,$ then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where *a, b, c* be any complex numbers.

5. A quadratic equation has exactly two roots.

TEST YOURSELF

1. If the constant term in a quadratic equation is zero, then prove that one root is zero.

2. If x be a real number, find the least value of $3x^2 - 24x + 64$.

3. Solve :

(i) $x^2 - 7|x| + 12 = 0$

(ii) $(x^2 + 4x)^2 - 2x^2 - 8x + 1 = 0$

4. Solve :

(i) $\left(\sqrt{3} + \sqrt{2}\right)^x + \left(\sqrt{3} - \sqrt{2}\right)^x = 10$

(ii) $\left(\sqrt{5 + 2\sqrt{6}}\right)^x + \left(\sqrt{5 - 2\sqrt{6}}\right)^x = 10$.

(iii) $x^{(2/3)(\log_2 x - 1)} = \sqrt{2}$.

5. Show that the equation $e^{\sin x} - e^{-\sin x} = 4$ has no solution.

Answer

2. 16 **3.** (i) $\pm 3, \pm 4$ (ii) $-2 \pm \sqrt{5}, -2 \pm \sqrt{5}$

4. (i) -2, 2 (ii) -2, 2 (iii) $\frac{\sqrt{2}}{2}, 2\sqrt{2}$

SECTION – A

2. ARITHMETIC PROGRESSIONS

LEARNING OBJECTIVES

- Sequence
- Progression
- Series
- Definition of an Arithmetic progression (A.P.)
- Standard A.P.
- General Term of an A.P.
- Theorem
- Sum of First *n* Terms of an A.P.
- Arithmetic Means
- Single A.M. Between Any Two Given Numbers
- *n* A.M.s Between Any Two Given Numbers
- Use of A.P. in Solving Practical Problems

SEQUENCE

A succession of numbers formed according to a certain rule and arranged in a definite is called a **sequence.**

For example, $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots\ldots\ldots \frac{1}{2n}, \ldots\ldots\ldots\ldots$ is a sequence.

In a sequence, the numbers occurring at its first place, second place, third place, ……*n*th place are respectively called its first term, second term, third term, …………*n*th term.

The *n*th term of sequence is denoted by $T_n, t_n, a_n, a(n), u_n$, etc. For example, the succession of numbers 3, 7, 11, 15, ... form a sequence, given by the fule $T_n = 4n - 1, n \in N$. For this sequence, the 10th term, T_{10} is equal to 39 (=4(10)-1).

A sequence containing finite number of terms is called a **finite sequence** and a sequence having infinitely many terms is called an **infinite sequence**. For simplicity, an infinite sequence is generally referred as a 'sequence' only. For example, the sequence 2, ,4, 8, 16 is a finite sequence and the sequence 1, 2, 4, 7, 11, 16, ... is an infinite sequence.

Remark 1. A sequence can be thought of as a function defined on the set of natural numbers.

Remark 2. The sequence T_1, T_2, T_3, ... is generally written as (T_n).

Illustrations. (i) 1, 3, 7, 15, ... is a sequence and $T_n = 2^n - 1, n \in N$.

(ii) 5, 7, 9, 11, ... is a sequence, because each term (except first) is obtained by adding 2 to the previous term,

i..e., $T_{n+1} = T_n + 2, n \geq 1$.

(iii) 1, 4, 5, 9, 14,.... Is a sequence, because each term (except first two) is obtained by taking the sum of preceding two terms,

i.e., $T_{n+2} = T_n + T_{n+1}, n \geq 1$.

(iv) To define a sequence, we need not have an algebraic formula for its *n*th term. For example, the arrangement :

2, 3, 5, 7, 11, 13, 17, 19, ... of price numbers

is a sequence and there is no specific formula to evaluate the *n*th prime number.

Thus, a sequence can be described by any of the following ways:

I. A sequence may be described by writing first few terms of the sequence till the rule for writing down the other terms of the sequence become evident. For example, 1, 4, 9,..... is the sequence whose *n*th term is n^2.

II. A sequence may be described by giving a formula for its *n*th term. For example, the sequence 1, 4, 9, ... can be written as (n^2).

III. A sequence may be described by specifying its first few terms and a formula to determine the other terms of the sequence in terms of its preceding terms. Such a formula is called a **recursive formula**. For example, if $T_1 = 1$ and $T_{n+1} = 5T_n$ for $n \in N$.

PROGRESSION

A sequence is said to be a **progression** if its terms increase (respectively decrease) numerically.

For example, the following sequences are progressions:

(i) $2,4,6,8,\ldots\ldots\ldots\ \ldots$

(ii) $\frac{1}{3},\frac{1}{6},\frac{1}{9},\frac{1}{12},\ldots\ldots\ldots\ldots$

(iii) $1,-\frac{1}{2},\frac{1}{4},-\frac{1}{8},\ldots\ldots\ldots$

(iv) $1,4,9,16,\ldots\ldots\ldots\ \ldots.$

The sequence (iii) is a progression, because $|1| > \left|-\frac{1}{2}\right| > \left|\frac{1}{4}\right| > \left|-\frac{1}{8}\right| > \ldots\ldots\ldots$

SERIES

If $T_1, T_2, T_3 \ldots\ldots\ldots\ldots$ is a sequence, then the expression $T_1 + T_2 + T_3 \ldots\ldots\ldots\ldots$ is called the **series** corresponding to the given sequence.

A series is called **finite** or **infinite** according as the corresponding sequence is finite or infinite.

For example, 1 + 3 + 7 + 15 + ….. is a series and correspond to the sequence 1, 3, 7, 15, ………… .

Example 1. *Write the first three terms of the sequence whose nth term* T_n *is given by :*

(i) $\frac{2^n + 1}{2n + 1}$

(ii) $\frac{1-(-1)^n}{4}$.

Sol. (i) We have $T_n = \frac{2^n + 1}{2n + 1}$.

$\therefore \quad T_1 = \frac{2^1+1}{2(1)+1} = \frac{3}{3} = 1, \quad T_2 = \frac{2^2+1}{2(2)+1} = \frac{5}{5} = 1,$

and $\quad T_3 = \frac{2^3+1}{2(3)+1} = \frac{9}{7}$

(ii) We have $\quad T_n = \frac{1-(-1)^n}{4}$.

$\therefore \quad T_1 = \frac{1-(-1)^1}{4} = \frac{2}{4} = \frac{1}{2}, \quad T_2 = \frac{1-(-1)^2}{4} = \frac{0}{4} = 0$

and $\quad T_3 = \frac{1-(-1)^3}{4} = \frac{2}{4} = \frac{1}{2}$.

WORKING RULES FOR SOLVING PROBLEMS

Rule I. *A sequence is a succession of terms which are formed according to some definite rule.*

Rule II. *A sequence is either finite or infinite.*

Rule III. *A progression is a sequence if its terms increase (respectively decrease) numerically.*

Rule IV. *If $T_1, T_2, T_3, \ldots$ is a sequence, then $T_1 + T_2 + T_3 + \ldots$ is the series corresponding to the sequence $T_1, T_2, T_3, \ldots$*

EXERCISE 2.1

SHORT ANSWER TYPE QUESTIONS

1. For the sequence $\left\{\frac{4n+1}{n+7}\right\}$, find T_1, T_4.
2. For the sequence (T_n), where $T_n = (n-1)(2-n)(3+n)$, find three terms.
3. Find the first six terms of the sequence (a_n), where:

   (i) $a_1 = 2,\ a_2 = 4,$
   $a_n = 2a_{n-1} + 3a_{n-2},\ n \geq 3$

   (ii) $a_1 = 2$
   $a_n = 2(a_{n-1} + 1),\ n \geq 2.$
4. The **Fibonacci sequence** is defined by $a_1 = 1 = a_2,\ a_n = a_{n-1} + a_{n-2}\ (n > 2)$.

   Find $\frac{a_{n+1}}{a_n}$, for $n = 1, 2, 3, 4, 5$.

Answers

1. $\frac{5}{8}, \frac{17}{11}$ 2. 0, 0, -12

3. (i) 2, 4, 14, 40, 122, 364   (ii) 2, 6, 14, 30, 62, 126   4. $1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}$.

DEFINITION OF AN ARITHMETIC PROGRE SSION (A.P.)

A sequence is said to be an **arithmetic progression** (abbreviated as **A.P.**) If the difference of each term, expect the first its preceding term is always same.

For example, 2, 5, 8, 11, ………… is an A.P., because

5 – 2 = 3, 8 – 5 = 3, 11 – 8 =3, ……..

Thus, the sequence (T_n) is an arithmetic progression, if there exists a number, say, d such that $T_{n+1} - T_n = d$ for $n \geq 1$.

The constant number 'd' mentioned above is called the **common difference** of the corresponding A.P. The common difference of an A.P. is denoted by 'd'.

The first term of an A.P. is generally denoted by 'a'.

Remark. An arithmetic progression is a particular type of a 'progression'.

Illustrations:

(i) 1, 3, 5, 7, 9, ….. is an A.P. with common difference, 2 because

3 – 1 = 5 – 3 = 7 – 5 = 9 – 7 = ……. = 2.

(ii) 16, 13, 10, 7, 4, ….. is an A.P. with common difference, -3 because

13 – 16 = 10 – 13 = 7 – 10 = 4 – 7 = ……. = -3

Remark 1. An A.P. is characterized by its 'a' and 'd'.

Remark 2. If in a sequence, the terms are alternatively positive and negative, then it cannot be an A.P.

STANDARD A.P.

The **standard A.P.** is defined as $a, a + d, a + 2d,$ ….. This is an A.P. with 'a' as the first term and 'a' as the common difference.

GENERAL TERM OF AN A.P.

Theorem. If 'a' and 'd' be the first term and common difference of the A.P. (T_n), then prove that

$$T_n = a + (n - 1)d, \quad n \in \mathbf{N}. \qquad \ldots(1)$$

Proof. First term of A.P. = a

Common difference of A.P. = d

$\therefore$ The A.P. is $a, a + d, a + 2d,$ ……..

We have $T_1 = a = a + 0 = a + (1 - 1)d$

$T_2 = a + d = a + (2 - 1)d$

$T_3 = a + 2d = a + (3 - 1)d$

….. ….. …..

….. ….. …..

$\therefore \quad T_n = a + (n - 1)d, \quad n \in \mathbf{N}.$

Example 2. *Find the 20th and nth term of the sequence 4, 9, 14, 19, ……. .*

Sol. Given sequence is 4, 9, 14, 19, …….. ….(1)

Here $T_2 - T_1 = 9 - 4 = 5, \quad T_3 - T_2 = 14 - 9 = 5,$ ……..

$\therefore \quad T_2 - T_1 = T_3 - T_2 = \ldots\ldots = 5$

$\therefore$ (1) is an A.P. with $a = 4$ and $d = 5$.

Now $T_{20} = a + (20 - 1)\, d = 4 + 19(5) = \mathbf{99}$

and $T_n = a + (n - 1)\, d = 4 + (n - 1)\, 5 = \mathbf{5n - 1}$.

Example 3. *If $\log_{10} 2$, $\log_{10}(2^x - 1)$* and *$\log_{10}(2^x + 3)$ are in A.P. then find the value of x.*

Sol. $\log_{10} 2$, $\log_{10}(2^x - 1)$, $\log_{10}(2^x + 3)$ are in A.P.

$\Rightarrow \quad \log_{10}(2^x - 1) - \log_{10} 2 = \log_{10}(2^x + 3) - \log_{10}(2^x - 1)$.

$$\Rightarrow \quad \log_{10}\frac{2^x - 1}{2} = \log_{10}\frac{2^x + 3}{2^x - 1} \quad \Rightarrow \quad \frac{2^x - 1}{2} = \frac{2^x + 3}{2^x - 1}$$

$$\Rightarrow \quad \frac{y-1}{2} = \frac{y+3}{y-1}, \text{ where } y \text{ is } 2^x$$

$$\Rightarrow \quad y^2 - 2y + 1 = 2y + 6 \quad \Rightarrow \quad y^2 - 4y - 5 = 0 \quad \Rightarrow \quad y = -1, 5.$$

$y = -1 \quad \Rightarrow \quad 2^x = -1$. This is impossible.

$y = 5 \quad \Rightarrow \quad 2^x = 5 \quad \Rightarrow \quad x = \log_2 5$.

THEOREM

If a, b, c are in A.P., then prove that :

(i) a + k, b + k, c + k are in A.P. **(ii) a – k, b – k, c – k are in A.P.**

(iii) ka, kb, kc are in A.P. **(iv) a/k, b/k, c/k are in A.P. ($k \neq 0$).**

Proof. *a, b, c* are in A.P.

$\therefore \quad b - a = c - b \quad \ldots(1)$

(i) $a + k$, $b + k$, $c + k$ are in A.P. if $(b + k) - (a + k) = (c + k) - (b + k)$

if $b - a = c - b$, which is true. $\therefore$ **a + k, b + k, c +k** are in A.P.

(ii) $a - k$, $b - k$, $c - k$ are in A.P. if $(b - k) - (a - k) = (c - k) - (b - k)$

if $b - a = c - b$, which is true. $\therefore$ **a – k, b – k, c – k** are in A.P.

(iii) ka, kb, kc are in A.P. if $kb - ka = kc - kb$ if $k(b - a) = k(c - b)$

if $b - a = c - b$, which is true. $\therefore$ **ka, kb, kc,** are in A.P.

(iv) $\frac{a}{k}, \frac{b}{k}, \frac{c}{k}$ are in A.P. if $\frac{b}{k} - \frac{a}{k} = \frac{c}{k} - \frac{b}{k}$ if $\frac{b-a}{k} = \frac{c-b}{k}$

if $b - a = c - b$, which is true. $\therefore$ **a/k, b/k, c/k** are in A.P.

Example 4. *If a, b, c are in A.P., show that :*

(i) $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ *are in A.P.*

(ii) $a^2(b+c), b^2(c+a), c^2(c+b)$ *are in A.P.*

Sol. (i) $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ *are in A.P.* if $\frac{abc}{bc}, \frac{abc}{ca}, \frac{abc}{ab}$ *are in A.P.*

(Multiplying each term by abc)

if *a, b, c* are in A.P., which is given to be true.

$$\therefore \quad \frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab} \text{ are in A.P.}$$

(ii) $a^2(b+c), b^2(c+a), c^2(c+b)$ are in A.P.

if $a^2b + a^2c, b^2c + b^2a, c^2a + c^2b$ are in A.P.

if $a^2b + a^2c + abc, b^2c + b^2a + abc, c^2a + c^2b + abc$ are in A.P.

(Adding abc to each term)

if $a(ab + ac + bc), b(bc + ba + ac), c(ca + cb + ab)$ are in A.P.

if *a, b, c* are in A.P. which is given to be true.

(Dividing each term by ab +bc + ca)

$\therefore$ $a^2(b+c), b^2(c+a), c^2(c+b)$ *are in A.P.*

Example 5. *Find four numbers in A.P. whose sum is 20 and the sum of whose square is 120.*

Sol. Let the numbers be $a - 3d$, $a - d$, $a + d$, $a + 3d$.

$\therefore$ Sum $= (a - 3d) + (a - d) + (a + d) + (a + 3d) = 20$ (Given)

$\therefore$ $4a = 20$ *i.e.,* $a = 5$

$\therefore$ The numbers are $5 - 3d, 5 - d, 5 + d, 5 + 3d$.

Also, sum of squares $= (5 - 3d)^2 + (5 - d)^2 + (5 + d)^2 + (5 + 3d)^2 = 120$

(Given)

$\therefore \quad (25 + 9d^2 - 30d) + (25 + d^2 - 10d) + (25 + d^2 + 10d) + (25 + 9d^2 + 30d) = 120.$

$\Rightarrow \quad 20d^2 = 20$ i.e., $d^2 = 1$ or $d = \pm 1$.

Case I. d = 1. The numbers are 5 – 3(1), 5 – (1), 5 + (1), 5 + 3(1) or **2, 4, 6, 8.**

Case II. D = -1. The numbers are 5 – 3(-1), 5 – (-1), 5 + (-1), 5 +3(-1) or **8, 6, 4, 2.**

WORKING RULES FOR SOLVING PROBLEMS

Rule I. *The sequence $T_1, T_2, T_3, T_4, \ldots\ldots$ is an A.P. if $T_2 - T_1 = T_3 - T_2 = T_4 - T_3 = \ldots\ldots\ldots\ldots$.*

Rule II. *For the A.P. a, a + d, a + 2d, ……, we have $T_n = a + (n - 1)d$.*

Rule III. *The number k is a term in the A.P. a, a + d, a + 2d, ….. if there exists $n \in \boldsymbol{N}$ such that $k = a + (n - 1)d$ or equivalently $\frac{k-a}{d} + 1 \in \boldsymbol{N}$.*

Rule IV. *If a, b, c are in A.P., then :*

(i) *a + k, b + k, c + k are in A.P.* (ii) *a – k, b – k, c – k are in A.P.*

(iii) *ka, kb, kc are in A.P.* (iv) *a/k, b/k, c/k are in A.P. ($k \neq 0$).*

Rule V. *If the sum of n numbers in A.P. is given, then assume numbers to be :*

(i) *a – d, a, a + d for n = 3*

(ii) *a – 3d, a – d, a + d, a + 3d for n = 4*

(iii) *a – 2d, a – d, a, a + d, a + 2d for n = 5.*

EXERCISE 2.2

SHORT ANSWER TYPE QUESTIONS

1. Show that 4, 10, 16, 22, ……. is an A.P. Find its 7th and 9th terms.
2. Show that 6, $5\frac{1}{3}$, $4\frac{2}{3}$, 4, …… is an A.P. Find its 10th and *k*th terms.
3. Show that the linear function in *n* i.e., $f(n) = an + b$ determine an A.P., where *a* and *b* are constants.
4. Determine the number of terms in the sequence 17, $14\frac{1}{2}$, 12, ……., -38.
5. Determine *x* so that $2x + 1$, $x^2 + x + 1$ and $3x^2 - 3x + 3$ are consecutive terms of an A.P.
6. If 5 times the 5th term of an A.P. is equal to the 10 times the 10th term, final the 15th term of the A.P.
7. (i) Which term of the A.P. $8 - 6i$, $7 - 4i$, $6 - 2i$, …….. is (*a*) purely real (*b*) purely imaginary?

   (ii) Which term of the sequence 20, $19\frac{1}{4}$, $18\frac{1}{2}$, …….is the first negative term?

Answers

1. 40, 52	2. 0, $\frac{20-2k}{3}$	4. 23
5. 1, 2	6. 0	
7. (i) (a) 4th (b) 9th	(ii) 28th.	

SUM OF FIRST *n* TERMS OF AN A. P.

The sum of first *n* terms of an A.P. is denoted by S_n.

If (T_n) is an A.P., then we have $S_n = T_1 + T_2 + T_3 + \ldots\ldots + T_n$, $n \in \mathbf{N}$.

$\therefore$ In particular $S_1 = T_1$, $S_2 = T_1 + T_2$, $S_3 = T_1 + T_2 + T_3$ etc.

For example, 1, 4, 7, 10, …….is an A.P. and

$S_1 = 1$ $\qquad S_2 = 1 + 4 = 5$

$S_3 = 1 + 4 + 7 = 12 \qquad S_4 = 1 + 4 + 7 + 10 = 22$ etc.

In the next theorem, we shall establish a general formula for computing S_n for an A.P.

Theorem. If 'a' and 'd' be the first term and common difference of an A.P. then prove that the sum of first *n* terms of this A.P. is given by

$$S_n = \frac{n}{2}\{2n + (n-1)d\}, \quad n \in N.$$

Proof. By definition, $S_n = T_1 + T_2 + T_3 + \ldots\ldots.. + T_n$ .....(1)

Let l denotes then nth term i.e., then last term in the expression of S_n

$\therefore \qquad l = T_n = a + (n-1)d$ ....(2)

(1) implies $\qquad S_n = a + (a + d) + (a + 2d) + \ldots\ldots + (l - 2d) + (l - d) + l$ ...(3)

By reversing the order, we get

$$S_n = l + (l - d) + (l - 2d) + \ldots\ldots.. + (a + 2d) + (a + d) + a \quad \text{....(4)}$$

Adding (3) and (4), we get

$$2S_n = (a + l) + (a + l) + (a + l) + \ldots\ldots + (a + l) + (a + l) + (a + l)$$

$$= n(a + l) \qquad [\because (a + l) \text{ is added } n \text{ times}]$$

$$\therefore \qquad S_n = \frac{n}{2}(a+1). \qquad \text{[Form (1)]}$$

Substituting the value of l, we get

$$S_n = \frac{n}{2}\left[a + a + (n-1)d\right] \quad \text{or} \quad S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

[Form (2)]

Remark 1. The above theorem can also be proved by using *Principle of Mathematical Induction.*

Remark 2. *Form* I is used when the last term is know and the *Form* II is used when common difference is known.

Remark 3. We have $\quad S_1 = T_1$ and for $n > 1$, $S_n = (T_1 + T_2 + \ldots\ldots + T_{n-1}) + T_n$.

$\therefore$ $S_n = S_{n-1} + T_n$ *i.e.,* $T_n = S_n - S_{n-1}$.

$\therefore$ $\mathbf{T_1 = S_1}$ **and for n > 1,** $\mathbf{T_n = S_n - S_{n-1}}$.

Example 6. *Evaluate:*

(i) $\frac{1}{9}+\frac{2}{9}+\frac{1}{3}+........25$ *terms* (ii) *5 + 13 + 21 + ... + 181.*

Sol. (i) The series is $\frac{1}{9}+\frac{2}{9}+\frac{1}{3}+........$, Here $T_2 - T_1 = T_3 - T_2 = ... = \frac{1}{9}$.

$\therefore$ Given series is an arithmetic series with $a = \frac{1}{9}$ and $d = \frac{1}{9}$.

$\therefore$ Required sum $= S_{25} = \frac{25}{2}\left[2a + (25-1)d\right]$

$$= \frac{25}{2}\left[2\left(\frac{1}{9}\right)+24\left(\frac{1}{9}\right)\right] = \frac{25}{2}\left[\frac{26}{9}\right] = \frac{325}{9}.$$

(ii) The series is 5 + 13 + 21 + ... + 181. Here $T_2 - T_1 = T_3 - T_2 = ... = 8$.

$\therefore$ Given series is an A.S. with $a = 5$ and $d = 8$.

Let 181 be the nth term. $\therefore$ $T_n = 181$ *i.e.,* $5 + (n-1)\,8 = 181$.

Solving, we get $n = 23$.

$\therefore$ Required sum $= S_{23} = \frac{23}{2}(5+181) = \frac{23}{2}(186) = 2139$.

$$\left(S_n = \frac{n}{2}(a+l)\right)$$

Example 7. *If the sum of first n, 2n, 3n terms of an A.P. are S_1, S_2, S_3 respectively, show that $S_3 = 3(S_2 - S_1)$.*

Sol. Let a be the first term and d, the common difference of the A.P.

$\therefore$ $S_1 = \frac{n}{2}\left[2a(n+1)d\right]$, $S_2 = \frac{2n}{2}\left[2a(2n-1)d\right]$

and $S_3 = \frac{3n}{2}\left[2a(3n-1)d\right]$

R.H.S = 3($S_2 - S_1$) = $3\left[\frac{2n}{2}\left[2a+(2n-1)d\right]-\frac{n}{2}\left[2a+(n-1)d\right]\right]$

$= \frac{3n}{2}\left[2\left[2a+(2n-1)d\right]-\left[2a+(n-1)d\right]\right]$

$= \frac{3n}{2}\left[4a+4nd-2d-2a-nd+d\right]=\frac{3n}{2}\left[2a+(3n-1)d\right]=S_3$

= L.H.S

WORKING RULES FOR SOLVING PROBLEMS

Rule I. *For the sequence $T_1, T_2, T_3, \ldots\ldots.$ We have $S_n = T_1 + T_2 + T_3 + \ldots\ldots.. + T_n$ and $T_n = S_n - S_{n-1}$.*

Rule II. *For the A.P. a, a + d, a + 2d, ……, we have* $S_n = \frac{n}{2}[2a + (n-1)d]$.

Rule III. *For the A.P. a, a + d, a + 2d, ….., we have* $S_n = \frac{n}{2}(a + l)$, *where* $l = T_n = a + (n - 1)d$.

Rule IV. *The number k is the sum of the first n terms of the A.P. a, a + d, a + 2d, ……. if the equation* $\frac{n}{2}[2a + (n-1)d] = k$ *is true for some* $n \in$ **N.**

EXERCISE 2.3

SHORT ANSWER TYPE QUESTIONS

1. (i) Find the sum of first 50 natural numbers.
   (ii) Find the sum of first 35 even natural numbers.
   (iii) Find the sum of first 65 odd natural numbers.

2. Find the sum of indicated number of terms of each of the following A.P.
   (i) 5, 2, -1, …….n terms
   (ii) 0.9, 0.91, 0.92, …..100 terms
   (iii) -0.5, -1.0, -1.5, …..10 terms
   (iv) $x + y, x - y, x - 3y, \ldots\ldots 22$ terms.

3. Find the following sums:
   (i) 2 + 5 + 8 +…… + 44
   (ii) $6 + 5\frac{2}{3} + 5\frac{1}{3} + \ldots\ldots + \frac{2}{3}$.

4. If S_n denotes the sum of n terms of an A.P. whose common difference is d, show that $d = S_n - 2S_{n-1} + S_{n-2}$, $n > 2$.

LONG ANSWER TYPE QUESTIONS

5. (i) How many terms of the A.P. 18, 16, 14,....... are needed to give sum 78? Explain the double answer.

   (ii) If the sum of a certain number of terms of the A.P. 25, 22, 19, ... is 116. Find the last terms.

   (iii) If the first term of an A.P. is 22, the common difference is -4 and the sum to n terms is 64, find n. Explain the double answer.

6. Solve the equation : $1 + 6 + 11 + \ldots\ldots + x = 148$.

7. If S_1, S_2, S_3 are the sums of n terms of three A.P.s, the first term of each being unity and the respective command differences being 1, 2, 3, show that $S_1 + S_3 = 2S_2$.

8. If the sum of n terms of an A.P. is $pn + qn^2$, where p and q are constants, find the common difference.

ANSWERS

1. (i) 1275 (ii) 1260 (iii) 4225 (iv) $22(x - 20y)$
2. (i) $n(13 - 3n)/2$ (ii) 139.5 (iii) -27.5
3. (i) 345 (ii) $56\frac{2}{3}$
5. (i) 6, 13 (ii) 4 (iii) 4, 8
6. 36
8. $2q$.

ARITHMETIC MEANS

If three or more than tree numbers are in A.P., then the numbers lying between the first and the last numbers are called the **arithmetic means (A.M.s)** between them.

Equivalently, if $a, A_1, A_2, \ldots\ldots A_n, b$ are in A.P., then $A_1, A_2, \ldots\ldots A_n$ are called the n arithmetic means between a and b.

For example,

(i) 5, 10, 15 are in A.P. $\therefore$ 10 is the single A.M. between 5 and 15.

(ii) 5, 10, 15, 20, 25, 30 are in A.P. $\therefore$ 10, 15, 20, 25 are the four A.M.s between 5 and 30.

SINGLE A.M. BETWEEN ANY TWO GIVEN NUMBERS

Let a, b be any two numbers. Let A be the single A.M. between a and b.

$\therefore$ By definition, a, A, b are in A.P.

$\therefore$ $A - a = b - A$ (each = common difference)

$\Rightarrow$ $2A = a + b$ or $A = \dfrac{a+b}{2}$.

Remark. The single A.M. between any two numbers is simply referred as the A.M. between the numbers. Thus, the A.M. between given two numbers is equal to half their sum.

For example, the A.M. between 7 and 29 is $\dfrac{7+29}{2} = \dfrac{36}{2} = 18$.

n A.M.s BETWEEN ANY TWO GIVEN NUMBERS

Let a, b be any two numbers. Let $A_1, A_2, \ldots\ldots A_n$ be the n A.M.s between a and b.

$\therefore$ By definition, $a, A_1, A_2, \ldots\ldots, A_n, b$ are in A.P. Let d be the common difference of the A.P.

Now, $b = T_{n+2} = a + (n+1)d$. $\therefore$ $d = \dfrac{b-a}{n+1}$

$\therefore$ $A_1 = a + d = a + \dfrac{b-a}{n+1}$

$$A_2 = a + 2d = a + 2\left(\frac{b-a}{n+1}\right)$$

....

$$A_n = a + nd = a + n\left(\frac{b-a}{n+1}\right)$$

$\therefore$ **Then n A.M.s between a and b are**

$$a + \frac{b-a}{n+1}, a + 2\left(\frac{b-a}{n+1}\right), \ldots\ldots\ldots \ldots.. a + n\left(\frac{b-a}{n+1}\right).$$

Theorem. Prove that the sum of n A.M.s between any two number is equal to *n* times the A.M. between them.

Proof. Let $A_1, A_2, \ldots\ldots\ldots, A_n$ be the n A.M.s between number a and b.

$\therefore$ Sum of n A.M.s between a and b

$$= A_1 + A_2 + \ldots\ldots + A_n = (a + A_1 + A_2 + \ldots\ldots\ldots + A_n + b) - (a + b)$$

$$= \frac{n+2}{2}(a+b) - (a+b)$$

($\because$ $a, A_1, A_2, \ldots..A_n, b$ is an A.P. of $n + 2$ terms)

$$= (a + b)\left[\frac{n+1}{2} - 1\right] = n\left(\frac{a+b}{2}\right) = n$$ times the A.M. between a and b.

$\therefore$ **Sum of n A.M.s between a and b = n(A.M. between a and b).**

Example 8. *Insert three A.M.s between 11 and 14.*

Sol. Let A_1, A_2, A_3 be the three A.M.s between 11 and 14.

$\therefore$ 11, A_1, A_2, A_3, 14 are in A.P. Let d be the common difference of this A.P.

Now, $14 = T_5 = 11 + 4d$. $\therefore$ $d = 3/4$

$\therefore$ $A_1 = 11 + d = 11 + \frac{3}{4} = \frac{47}{4}$, $A_2 = A_1 + d = \frac{47}{4} + \frac{3}{4} = \frac{25}{2}$

and $A_3 = A_2 + d = \frac{50}{4} + \frac{3}{4} = \frac{53}{4}$.

Remark. $A_3 + d = \frac{53}{4} + \frac{3}{4} = 14$ = last term of the A.P. This ensures that the calculation work is correct.

Example 9. *If the A.M. between pth and qth terms of an A.P. be equal to the A.M. between rth and sth terms of the A.P., show that p + q = r + s.*

Sol. Let the A.P. be $a, a + d, a + 2d, \ldots\ldots$

$\therefore \quad T_p = a + (p - 1)d,\ T_q = a + (q - 1)d,\ T_r = a + (r - 1)d,\ T_s = a + (s - 1)d.$

We are given that:

A.M. between T_p and T_q = A.M. between T_r and T_s

$$\Rightarrow \quad \frac{T_p + T_q}{2} = \frac{T_r + T_s}{2} \quad \Rightarrow \quad T_p + T_q = T_r + T_s$$

$$\Rightarrow \quad [a + (p - 1)d] + [a + (q - 1)d] = [a + (r - 1)d] + [a + (s - 1)d]$$

$$\Rightarrow \quad (p - 1 + q - 1)d = (r - 1 + s - 1)d$$

$$\Rightarrow \quad p + q - 2 = r + s - 2$$

$\Rightarrow$ $\quad$ **p + q = r + s.** $\quad \therefore$ The result holds.

WORKING RULES FOR SOLVING PROBLEMS

Rule I. *If three or more than three numbers are in A.P., then the numbers lying between the first and the last numbers are the A.M.'s between them.*

Rule II. *The A.M. between a and b is* $\frac{a+b}{2}$.

Rule III. *The n A.M.s between a and b are* $a + \frac{b-a}{n+1},\ a + 2\left(\frac{b-a}{n+1}\right), \ldots\ldots a + n\left(\frac{b-a}{n+1}\right)$

Rule IV. *Sum of n A.M.s between any two numbers is equal to n times the A.M. between them.*

Rule V. *If* $\frac{a}{b} + \frac{c}{d}$, *then by componendo and dividend rule,* $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

EXERCISE 2. 4

SHORT ANSWER TYPE QUESTIONS

1. Find the A.M. between 5 and 9.
2. Find the A.M. between $(x - y)^2$ and $(x + y)^2$.
3. Find the sum of 500 A.M.s between 2 and 3.
4. Find the ratio of the sum of m A.M.s between any two numbers to the sum of n A.M.s between the same numbers.

LONG ANSWER TYPE QUESTIONS

5. Find n such that $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ may be the A.M. between a and b.
6. n A.M.s are instead between 5 and 86 such that the ratio of the first and the last mean in 2 : 11. Find n.
7. Between 1 and 31, m arithmetic means have been inserted in such a way that the ratio of the 7th and $(m - 1)$th means is 5 : 9. Find the value of m.

Answers

1. 7	2. $x^2 + y^2$	3. 1250	4. $m : n$
5. 0	6. 8	7. 14.	

USE OF A.P. IN SOLVING PRACTICAL PROBLEMS

In this section, we shall see how the formulae relating to A.P. can be made use of in solving practical problems.

Example 10. *Hari buys a scooter for Rs. 22,000. He pays Rs. 4,000 cash and agree to pay the balance in annual instalments of Rs. 1,000 plus 10% interest on the unpaid amount. Find the total payment for the scooter.*

Sol. Cost of scooter = Rs. 22,000

Cash payments = Rs. 4,000

$\therefore$ Balance = Rs. 18,000

There will be 18 $\left(=\frac{18,000}{1,000}\right)$ annual instalments each of Rs. 1,000 plus interest on unpaid amount. The first instalments will be of Rs. 1,000 plus interest on unpaid amount (= 18000) for one year.

$\therefore$ First instalment $= 1,000 + \frac{18,000 \times 1 \times 10}{100} = 2,800$

Second instalment $= 1,000 + \frac{17,000 \times 1 \times 10}{100} = 2,700$

Third instalment $= 1,000 + \frac{16,000 \times 1 \times 10}{100} = 2,600$

The instalements 2,800, 2,700, 2,600, ..., 18 terms, form an A.P. with a = 2,800, d = -100.

$\therefore$ Total amount paid for the scooter = cash payment + sum of instalments

$= 4,000 + S_{18}$ of the A.P. $= 4,000 + \frac{18}{2}[2(2,800) + 17(-100)]$

= 4,000 + 35,100 = **Rs. 39,100.**

EXERCISE 2. 5

LONG ANSWER TYPE QUESTIONS

1. A man starts repaying a loan with first instalment of Rs. 100. If he increase the instalment by Rs. 5 every months, what amount will be paid by him in the 30th instalment?
2. The income of a person is Rs.3,00,000 in the first year and he receives an increase of Rs. 10,000 in his income per year for the next 19 years. Find the total amount, he received in 20 years.
3. The interior angle of a polygon are in A.P. The smallest angle is 120^0 and the common difference 5^0. Find the number of sides of the polygon.
4. The ages of the students of a class form an A.P. whose common difference is 4 months. If the youngest students is 8 years old and the sum of the ages of all the students of the class in 168 years, find the number of students in the class

Answers

1. Rs. 245 2. Rs. 79,00,000 3. 9 4. 16

SUMMARY

1. A sequence is said to be a **progression** if its terms numerically increases (respectively decreases).
2. A sequence (T_n) is said to be an **arithmetic progression (A.P.)** if there exists a number, say d such that $T_{n+1} - T_n = d$, $n \geq 1$.
   The constant number 'd' mentioned above is called the **common difference** of the corresponding A.P.
3. If 'a' and 'd' be the first term and common difference of the A.P. (T_n), then $T_n = a + (n - 1)\ d$, $n \in \mathbf{N}$.
4. If 'a' and 'd' be the first term and common difference of the A.P. (T_n), then the sum of first n terms, S_n is give by

   a) $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$, $n \in \mathbf{N}$.

   b) $S_n = \frac{n}{2}(a + l)$, where l is the last term in S_n *i.e.*, $l = T_n = a + (n-1)d$.

   The form (a) is used when common difference 'd' is known and the form (b) is used when the last term 'l' is known.
5. $T_n = S_1$ and for $n > 1$, we have $T_n = S_n - S_{n-1}$.
6. If the sequence a, A_1, A_2, ……., A_n, b is an A.P., then the numbers A_1, A_2, ……., A_n are called the n **arithmetic means** between a and b.
7. The A.M. between given numbers a and b is equal to $\frac{a+b}{2}$.
8. The sum of n A.M.s between give numbers a and b equal to n times the A.M. between a and b.

TEST YOURSELF

1. If the mth term of an A.P. be $1/n$ and the nth term be $1/m$, then show that (mn)th term is 1.
2. The sum of the 4th and 8th terms of an A.P. is 24 and the sum of the 6th and 10th terms is 34. Find the first four terms of the A.P.

3. Show that the linear function in n, *i.e.*, $f(n) = an + b$ determine an arithmetic progression. Where a and b are constants.

4. If $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P., prove that a, b, c are in A.P.

5. The first and the last terms of an A.P. are a and l respectively. Show that the sum of nth term from the beginning and the nth term the end is $(a + l)$.

6. If the first term of an A.P. is 2 and the sum of first five terms is equal to one-fourth of the sum of the next five terms, find the sum of first 30 terms.

7. The third term of an A.P. is 7 and the seventh term exceeds three times the third term by 2. Find the first term, common difference and the sum of first 20 terms.

8. If S_1, S_2, S_3, ……., S_m be the sums of the first n terms of m A.P.s whose first terms are 1, 2, 3, …., m respectively and common differences 1, 3, 5,…., $2m - 1$ respectively. Show that
$$S_1 + S_2 + S_3 + \ldots\ldots + S_m = \frac{1}{2}mn(mn+1).$$

9. Insert A.M.s between 7 and 71 in such a way that the 5th A.M. is 27. Find the number of the A.M.s.

10. m A.M.s have been inserted between 1 and 31 in such a way that the ratio of the 7th and the $(m - 1)$th means is 5 : 9. Find the value of m.

Answers

2. $-\frac{1}{2}, 2, \frac{9}{2}, 7$ 6. -2550 7. -1, 4, 740 9. 15

10. 14

SECTION – A

3. GEOMETRIC PROGRESSIONS

LEARNING OBJECTIVES

- Definition of a Geometric Progression (G.P.)
- Standard G.P.
- General Term of a G.P.
- Sum of First *n* Terms of a G.P.
- Sum of Infinity of a G.P.
- Geometric Means
- Single G.M. Between any Two Given Positive Numbers
- *n* G.M.s Between any Two Given Positive Numbers
- Use of G.P. in Solving Practical Problems

DEFINITION OF A GEOMETRIC PROGRESSION (G.P.)

A succession of *non-zero* numbers is said to be a geometric progression (abbreviated as **G.P.**) if the ratio of each term, except the first one, by its preceding term is always same.

For example, 3, 6, 12, 24, ... is a G.P., because $\frac{6}{3} = 2, \frac{12}{6} = 2, \frac{24}{12} = 2$,.....

Thus, the sequence (T_n) with $T_n \neq 0$ is a geometric progression if there exists a number, say, r such that $\frac{T_{n+1}}{T_n} = r$ for $n \geq 1$.

The constant number 'r' mentioned above is called the **common ratio** of the corresponding G.P. The common ratio of a G.P. is denoted by 'r'.

The first term of a G.P., is generally denoted by 'a'.

Remark 1. In case of a G.P., neither $a = 0$ nor $r = 0$.

Remark 2. In a G.P., no term can be equal to '0'.

Remark 3. A geometric progression is a particular type of a 'progression'.

Illustrations: (i) 1, 2, 4, 8, 16, ……. is a G.P. with common ratio 2, because

$$\frac{2}{1} = \frac{4}{2} = \frac{8}{4} = \frac{16}{8} = ... = 2.$$

(ii) 9, 3, 1, $\frac{1}{3}, \frac{1}{9}$, …….. is a G.P. with common ratio $\frac{1}{3}$, because

$$\frac{3}{9} = \frac{1}{3} = \frac{1/3}{1} = \frac{1/9}{1/3} = ... = \frac{1}{3}.$$

Remark 1. A G.P. is characterized by its 'a' and 'r'.

Remark 2. If in a G.P., the terms are alternatively positive and negative, then its common ratio is always negative.

STANDARD G.P.

The **standard G.P.** is defined as a, ar, ar^2, ……. This is a G.P. with 'a' as the first term and 'r' as the common ratio.

Remark. If we multiply the common ratio with any term of a G.P., we get the next following term and if we divide any term by the common ratio, we get the preceding term.

GENERAL TERM OF A G.P.

Theorem. If 'a' and 'r' be respectively the first term and common ratio of the G.P. (T_n), then prove that

$$\mathbf{T_n = ar^{n-1}, \ \ n \in N.} \qquad ...(1)$$

Proof. First term of G.P. = a

Common ratio of G.P. = r

∴ The G.P. is a, ar, ar^3, ……

We have $T_1 = a = a\ .1 = ar^{1-1}$

$T_2 = ar = ar^{2-1}$

$T_3 = ar^2 = ar^{3-1}$

…………………….

……………………

$\therefore$ $\mathbf{T_n = ar^{n-1}, \ n \in N.}$

Example 1. *Find the 9th and nth terms of the sequence 3, 6, 12, 24, ……*

Sol. Given sequence is 3, 6, 12, 24, …… …(1)

Here, $\frac{T_2}{T_1} = \frac{6}{3} = 2, \quad \frac{T_3}{T_2} = \frac{12}{6} = 2, ...$

$\therefore$ $\frac{T_2}{T_1} = \frac{T_3}{T_2} = ... = 2.$ $\therefore$ (1) is a G.P. with $a = 3$ and $r = 2$.

Now, $T_9 = ar^{9-1} = ar^8 = 3(2)^8 = 3(256) = 768$ $\quad [T_n = ar^{n-1}]$

and $T_n = ar^{n-1} = 3(2)^{n-1}$.

Example 2. *There are four numbers such that the first three of these form an A.P. and the last three form a G.P. The sum of the first and the third numbers is 2 and that of the second and fourth is 26, what are these numbers ?*

Sol. Let the numbers be *a, b, c, d*. By the given conditions:

$a + c = 2$ ….(1) $\quad b + d = 26$ ….(2)

(1) $\Rightarrow$ $c = 2 - a$ and (2) $\Rightarrow$ $d = 26 - b$.

$\therefore$ The numbers are $a, b, 2 - a, 26 - b$.

Also, $a, b, 2 - a$ are in A.P. $\therefore$ $b - a = (2 - a) - b$ *i.e,* $2b = 2$ or $b = 1$.

$\therefore$ The numbers become $a, 1, 2 - a, 26 - 1 = 25$.

Also, last three numbers are in G.P. $\therefore$ $\frac{2-a}{1} = \frac{25}{2-a}$.

$\Rightarrow$ $(2 - a)^2 = 25$ $\Rightarrow$ $a^2 - 4a - 21 = 0$ $\Rightarrow$ $a = -3, 7$

Case I. a= -3. In this case, the numbers are -3, 1 , 2 – (-3), 25, *i.e.,* **-3, 1, 5, 25.**

Case II. a= 7. In this case, the numbers are 7, 1, 2, - 7, 25, *i.e.,* **7, 1, - 5, 25.**

WORKING RULES FOR SOLVING PROBLEMS

Rule I. *A sequence is* T_1, T_2, T_3, T_4, ... *is a G.P., if* $\frac{T_2}{T_1} = \frac{T_2}{T_2} = \frac{T_4}{T_3} = ...$

Rule II. *For the G.P. a, ar, ar*^{2}*, ... , we have* $T_n = ar^{n-1}$.

Rule III. *The number k is a term in the G.P. a, ar, ar*^{2}*, ... if there exists* $n \in \boldsymbol{N}$ *such that* $k = ar^{n-1}$ *or equivalently* $\frac{\log(kla)}{\log r} + 1 \in N$.

Rule IV. *If a, b, c are in G.P., then :*
(i) *ka, kb, kc are in G.P.* (ii) *alk, blk, clk are in G.P.* ($k \neq 0$).

Rule V. *If the product of n numbers in G.P. is given, the assume numbers to be:*
(i) $\frac{a}{r}, a, ar$ *for n = 3* (ii) $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$ *for n = 4*
(iii) $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$ *for n = 5.*

EXERCISE 3. 1

SHORT ANSWER TYPE QUESTIONS

1. Find the common ratio of the following G.P.:

   (i) 0.01, 0.0001, 0.000001,...... (ii) 1, -a, a^2, ...

   (iii) $\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}$,....... (iii) a^{m-n}, a^m, a^{m+n},.....

2. Show that each of the following is a G.P. Also find nth term in each case:

   (i) 128, 64, 32,...... (ii) 5/2, 5/4, 5/8, ...

   (iii) $\sqrt{3}, 3, 3\sqrt{3}$,....... (iv) $2, 2\sqrt{2}, 4$,..... is 128 ?

3. Determine the number of terms in the sequence 5/2, 5, 10, ..., 640.

4. Which term of the G.P.

   (i) 2, 8, 32, ... is 131072 ? (ii) $\sqrt{3}, 3, 3\sqrt{3}$,........ is 729 ?

(iii) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots\ldots$ is $\frac{1}{19683}$? (iv) $2, 2\sqrt{2}, 4, \ldots\ldots$ is 128 ?

5. Is 512 a term of the sequence 1/256, 1/64, 1/16, ……. ?

LONG ANSWER TYPE QUESTIONS

6. (i) The 5th, 8^{th} and 11^{th} terms of a G.P. are *p*, *q* and *r* respectively. Show that $q^2 = pr$.

   (ii) In any G.P. prove that $T_{n-r} \times T_{n+r} = (T_n)^2$.

7. (i) The 3rd term of a G.P. in 24 and 6th term is 192. Find the 10th term.

   (ii) The 5th term of a G.P. is 16 and 10th term is 1/2 . Find the G.P. Also find the 15th term.

8. (i) Find the value of *x* if -2/7, *x*, -7/2 are in G.P.

   (ii) For what value(s) of *k*, the numbers 1 + *k*, $\frac{5}{6} + k, \frac{13}{18} + k$ are in G.P.?

   (iii) If $\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx} = \frac{c+dx}{c-dx} (x \neq 0)$, then show that *a, b, c, d* are in G.P. ?

Answers

1. (i) 0.01 (ii) $-a$ (iii) 1/2 (iv) a^n

2. (i) 2^{8-n} (ii) $5/2^n$ (iii) $3^{n/2}$ (iv) ak^{2n-1}

3. 9 4. (i) 9th (ii) 12th (iii) 9^{th}

   (iv) 13th 5. No

7. (i) 3072 (ii) 256, 128, 64, ……..; $\frac{1}{64}$

8. (i) $\pm$ 1 (ii) -1/2

SUM OF FIRST *n* TERMS OF A G.P.

The sum of first *n* terms of a G.P. is denoted by S_n.

If (T_n) is a G.P., then we have $S_n = T_1 + T_2 + T_3 + \ldots\ldots + T_n$, $n \in \mathbf{N}$.

For example, 2, 6, 18, 54, ….. is a G.P. and

S_1 = 2, S_2 = 2 + 6 = 8, S_3 = 2 + 6 + 8 = 26 etc.

In the next theorem, we shall establish a general formula for computing S_n for a G.P.

Theorem. If a and r be respectively the first term and common ratio of a G.P., then prove that the sum of first n terms of this G.P. is given by

$$S_n = \begin{cases} na & \text{if} \quad r = 1 \\ \dfrac{n(1-r^n)}{1-r} & \text{if} \quad r \neq 1 \end{cases}$$

Proof. Let T_n be the nth term of the given G.P.

$\therefore \qquad T_n = ar^{n-1}, n \in N$

By definition, $\qquad S_n = T_1 + T_2 + ... + T_{n-1} + T_n$

$\therefore \qquad S_n = a + ar + ... + ar^{n-2} + ar^{n-1} \qquad ....(1)$

Case I. r = 1. In this case, (1) $\Rightarrow \qquad S_n = a + a + ... + a + a \qquad$ (n times)

$\therefore \qquad$ $\boldsymbol{S_n = na}$.

Case II. r $\neq$ 1. In this case, (1) $\Rightarrow \qquad rS_n = ar + ar^2 + ... + ar^{n-1} + ar^n \qquad .....(2)$

(1) – (2) $\Rightarrow \qquad S_n - rS_n = a + 0 + ... + 0 - ar^n$

$\Rightarrow \qquad (1-r)S_n = a(1-r^n) \qquad \Rightarrow \qquad S_n = \dfrac{a(1-r^n)}{1-r} \qquad (\because r \neq 1)$

Remark 1. Multiplying numerator and denominator by '- 1', we get $S_n = \dfrac{a(1-r^n)}{1-r}$.

This form of S_n is useful when $r > 1$.

Remark 2. This above theorem can also be proved by using P.M.I.

Remark 3. Let $l = T_n$. $\therefore \quad l = ar^{n-1}$.

$\therefore$ For $r < 1$. $\qquad S_n = \dfrac{a(1-r^n)}{1-r} = \dfrac{a - ar^n}{1-r} = \dfrac{a - (ar^{n-1})r}{1-r} = \dfrac{a - lr}{1-r}$

and for $r > 1$. $S_n = \dfrac{a(r^n - 1)}{r-1} = \dfrac{ar^n - a}{r-1} = \dfrac{(ar^{n-1})r - a}{r-1} = \dfrac{lr - 1}{r-1}$

$\therefore$ $$S_n = \begin{cases} \dfrac{a-1r}{1-r}, & \text{if } r < 1 \\ \dfrac{1r-a}{r-1}, & \text{if } r > 1 \end{cases}$$

These formula are used when 'last term' is given

Example 3. *Evaluate* : $1 - \frac{2}{3} + \frac{4}{9} + \text{........}10$ *terms*

Sol. The series is $1 - \frac{2}{3} + \frac{4}{9} + \text{...........}$

Here, $\dfrac{T_2}{T_1} = \dfrac{-2/3}{1} = -\dfrac{2}{3}$, $\dfrac{T_3}{T_2} = \dfrac{4/9}{-2/3} = -\dfrac{2}{3}, \text{.......... .}$

$\therefore$ $\dfrac{T_2}{T_1} = \dfrac{T_3}{T_2} = \text{........} = -\dfrac{2}{3}$

$\therefore$ Given series is a G.S. with $a = 1$ and $r = -2/3$.

$\therefore$ Required sum = $$S_{10} = \frac{1\left(1-\left(-\frac{2}{3}\right)^{10}\right)}{1-\left(-\frac{2}{3}\right)} = \frac{1-\frac{1024}{59049}}{1+\frac{2}{3}} = \frac{58025}{59049} \times \frac{3}{5} = \frac{11605}{19683}.$$

Example 4. *If S_n denotes the sum of n terms of a G.P., prove that*

$$(S_{10} - S_{20})^2 = S_{10}(S_{30} - S_{20}).$$

Sol. Let a and r be the first term and common ratio of the G.P. respectively.

$\therefore$ $$S_n = \frac{a(1-r^n)}{1-r}, n \in N.$$

L.H.S. $= (S_{10} - S_{20})^2 = \left[\dfrac{a(1-r^{10})}{1-r} - \dfrac{a(1-r^{20})}{1-r}\right]^2$

$$= \frac{a^2}{(1-r)^2}(1-r^{10}-1+r^{20})^2$$

$$= \frac{a^2}{(1-r)^2}\left[r^{10}\left(r^{10}-1\right)\right]^2 = \frac{a^2 r^{20}\left(r^{10}-1\right)^2}{\left(1-r\right)^2}.$$

R.H.S. $= S_{10}(S_{30}-S_{20}) = \frac{a\left(1-r^{10}\right)}{1-r}\left[\frac{a\left(1-r^{30}\right)}{1-r}-\frac{a\left(1-r^{20}\right)}{1-r}\right]$

$$= \frac{a\left(1-r^{10}\right)}{1-r}\times\frac{a}{1-r}\left(1-r^{30}-1+r^{20}\right) = \frac{a^2\left(1-r^{10}\right)}{\left(1-r\right)^2}r^{20}\left(1-r^{10}\right)$$

$$= \frac{a^2 r^{20}\left(1-r^{10}\right)^2}{\left(1-r\right)^2} = \frac{a^2 r^{20}\left(r^{10}-1\right)^2}{\left(1-r\right)^2}$$

$\therefore$ L.H.S. = R.H.

WORKING RULES FOR SOLVING PROBLEMS

Rule I. *For the G.P. a, ar, ar², ..., we have*

(i) $S_n = \frac{a\left(1-r^n\right)}{1-r}$ *for r < 1 and* (ii) $S_n = \frac{a\left(r^n-1\right)}{r-1}$ *for r > 1.*

Rule II. *For the G.P. a, ar, ar², ..., we have*

(i) $S_n = \frac{a-lr}{1-r}$ *for r < 1 and*

(ii) $S_n = \frac{lr-a}{r-1}$ *for n > 1, where* $l = T_n = ar^{n-1}$.

Rule III. *The number k is the sum of the first n terms of G.P. a, ar, ar², ...*

if the equation $\frac{a(1-r^n)}{1-r} = k$ *is true for some* $n \in N$.

EXERCISE 3. 2

SHORT ANSWER TYPE QUESTIONS

1. Find the sum of indicated numbers of terms of each of the following G.P. :

(i) $1, \frac{2}{3}, \frac{4}{9}, \ldots$ 10 terms

(ii) $x^3, x^5, x^7, \ldots n$ terms $(x \neq \pm 1)$

(iii) $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, \ldots.. n$ terms (iv) $1, -a, a^2, \ldots\ldots n$ terms $(a \neq -1)$

2. Given a G.P. with first term = 729, T_7 = 64 ; determine S_7.
3. (i) How many terms of the sequence 3, 3^2, 3^3,...... are needed to give the sum 120 ?

   (ii) How many terms of the G.P. 3, 3/2, ¾, ... are needed to give the sum 3069/512?
4. Evaluate the following :

   (i) $\sum_{j=1}^{11} \left(2 + 3^j\right)$ (ii) $\sum_{j=1}^{8} \left(2^j + 3^{j-1}\right)$
5. If $a + b + \ldots\ldots + l$ is a geometric series, show that its sum is $\dfrac{bl - a^2}{b - a}$.

LONG ANSWER TYPE QUESTIONS

6. (i) The fourth and seventh terms of a G.P. are 1/27 and 1/729 respectively. Find the sum of n terms of the G.P.

   (ii) the first term of a G.P. is 27 and its 8th term is 1/81. Find the sum of its 10 terms.
7. If $v = \dfrac{1}{1+\lambda}$, show that $v + v^2 + v^3 \ldots\ldots + v^n = \dfrac{1 - v^n}{\lambda}$.
8. The sum of some terms of G.P. is 315 whose first term and common ratio are 5 and 2 respectively. Find the last term and number of terms.

Answers

1. (i) $3\left(1 - \left(\dfrac{2}{3}\right)^{10}\right)$ (ii) $\dfrac{x^3\left(1 - x^{2n}\right)}{1 - x^2}$ (iii) $\dfrac{\sqrt{7}}{2}\left(\sqrt{3}+1\right)\left(3^{n/2} - 1\right)$

   (iv) $\dfrac{1 - (-a)^n}{1 + a}$

2. 463, 2059

3. (i) 4 (ii) 10

4. (i) 265741 (ii) 3790

6. (i) $\dfrac{3}{2}\left(1 - \dfrac{1}{3^n}\right)$ (ii) $\dfrac{81}{2}\left(1 - \dfrac{1}{3^{10}}\right)$

8. 6,160.

SUM TO INFINITY OF A G.P.

We know that for the G.P. $a, ar, ar^2, \ldots\ldots,$ the sum of first n terms is given by

$$S_n = \begin{cases} \frac{a(1-r^n)}{1-r}, & \text{if } r \neq 1 \\ na, & \text{if } r = 1. \end{cases}$$

This sum is defined for any natural number n.

Now we shall explore the possibility of finding the sum to infinity of a G.P.

Consider the G.P. 1, 2, 4, 8, ...

For this G.P., $a = 1$ and $r = 2$.

$$\therefore \quad S_n = \frac{1(2^n - 1)}{2 - 1} = 2^n - 1.$$

As n increases, $2^n - 1$ increase very rapidly.

$\therefore$ S_n keep on increasing as n increases. In this case, we do not expect to have a number which may be equal to the sum to infinity of the G.P.

Result. If a and r be respectively to first term and common ratio of a G.P. such the |r| < 1, then the sum to infinity (S) of the G.P. is given by

$$S = \frac{a}{1-r}.$$

Remark. The sum up to infinity of a G.P. is also denoted by S_∞.

Example 5. *Find the sum to infinity of the G.P. :*

(i) *4, 4/3, 4/9,......* (ii) *7, -1, 1/7,......*

Sol. (i) The G.P. is 4, 4/3, 4/9, ...

Here $a = 4$ and $r = \frac{T_2}{T_1} = \frac{4/3}{3} = \frac{1}{3}$ and $\left|\frac{1}{3}\right| = \frac{1}{3} < 1$.

$\therefore$ Sum to infinity is defined.

$\therefore \qquad S = \dfrac{4}{1-\frac{1}{3}} = \dfrac{4}{\frac{2}{3}} = 6. \qquad \left(S = \dfrac{a}{1-r}\right)$

(ii) The G.P. is 7, -1, 1/7, ...

Here $a = 7$ and $r = \dfrac{T_2}{T_1} = \dfrac{-1}{7} = -\dfrac{1}{7}$ and $\left|-\dfrac{1}{7}\right| = \dfrac{1}{7} < 1$.

$\therefore$ Sum to infinity is defined.

$\therefore \qquad S = \dfrac{7}{1-(-1/7)} = \dfrac{7}{8/7} = \dfrac{49}{9}. \qquad \left(S = \dfrac{a}{1-r}\right)$

Example 6. *If S_1, S_2, S_3.............. S_p denote the sums of infinite G.S. whose first terms are 1, 2, 3, ..., p respectively and whose common ratios are 1/2, 1/3, 1/4,...... 1/(p + 1) respectively. Show that*

$$S_1 + S_2 + S_3 + \dots\dots\dots .,+S_p = \frac{p(p+3)}{2}$$

Sol. For $S_1 : a = 1, r = \dfrac{1}{2}$ $\qquad \therefore \quad S_1 = \dfrac{1}{1-\frac{1}{2}} = 2$

For $\quad S_2 : a = 2, r = \dfrac{1}{3}$ $\qquad \therefore \quad S_2 = \dfrac{2}{1-\frac{1}{3}} = 3$

For $\quad S_3 : a = 3, r = \dfrac{1}{4}$ $\qquad \therefore \quad S_3 = \dfrac{3}{1-\frac{1}{4}} = 4$

....

For $\quad S_p : a = p, r = \dfrac{1}{p+1}$ $\qquad \therefore \quad S_p = \dfrac{p}{1-\frac{1}{p+1}} = p+1.$

$\therefore \qquad S_1 + S_2 + S_3 + \dots\dots + S_p = 2 + 3 + 4 + \dots\dots + (p+1)$

$(\therefore\ a = 2,\ d = 1,\ n = p)$

$$= \frac{p}{2}\left[2(2) + (p-1)1\right] = \frac{p}{2}\left[p+3\right] = \frac{p(p+3)}{2}$$

WORKING RULES FOR SOLVING PROBLEMS

Rule I. *The sum up to infinity for the G.P. a, ar, ar², ... is defined only when - 1 < r < 1.*

Rule II. *If – 1 < r < 1, then the sum up to infinity of the G.P. a, ar, ar², ... is defined and is equal to* $\frac{a}{1-r}$

EXERCISE 3. 3

SHORT ANSWER TYPE QUESTIONS

1. Find the sum of the following series :

(i) $1 - \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + ...$ (ii) 0.3 + 0.18 + 0.108 + 0.0648 +....

2. Find the sum of the following sequences :

(i) 2, -1, 1/2, -1/4,...... (ii) $\sqrt{2} + 1, 1, \sqrt{2} - 1, ...$

3. If $|x| < 1$ and $y = 1 + x + x^2 + ...\infty$, show that $x = \frac{y-1}{y}$.

LONG ANSWER TYPE QUESTIONS

4. If $A = 1 + r^a + r^{2a} + ...\infty$, and $B = 1 + r^b + r^{2b} + ...\infty$, show that

$$r = \left(\frac{A-1}{A}\right)^{1/a} = \left(\frac{B-1}{B}\right)^{1/b}$$

5. If $|a| < 1$, $|b| < 1$ and $x = 1 + a + a^2 + ...\infty$, show that

$$1 + ab + a^2b^2 + ...\infty = \frac{xy}{x+y-1}$$

6. Find the value(s) of p if S_∞ for the G.P. p, 1, $1/p$, ... is 25/4.

Answers

1. (i) $\frac{3}{4}$ (ii) $\frac{3}{4}$ 2. (i) $\frac{4}{3}$ (ii) $\frac{4+3\sqrt{2}}{2}$

6. $5, \frac{5}{4}$.

GEOMETRIC MEANS

If three or more than three *positive* numbers are in G.P., then the numbers lying between the first and the last numbers are called the **geometric means (G.M.s)** between them.

Equivalently, if a, G_1, G_2,……, G_n, b be a sequence of positive numbers which is a G.P., then G_1, G_2,……, G_n, are called the n geometric means between a and b.

For example,

i. 2, 6, 18 are in G.P. $\therefore$ 6 is the single G.M. between 2 and 18.

ii. 3, 12, 48, 192, 768 are in G.P.

$\therefore$ 12, 48, 192 are the three G.M.s between 3 and 768.

SINGLE G.M. BETWEEN ANY TWO GIVEN POSITIVE NUMBERS

Let a, b be any two positive numbers. Let G be the single G.M. between a and b.

$\therefore$ By definition, a, G, b are in G.P.

$\therefore$ $$\frac{G}{a} = \frac{b}{G}$$ (each = common ratio.)

$\Rightarrow$ $$G^2 = ab \quad \text{or} \quad G = \sqrt{ab}$$ (G is to be positive.)

Remark. The single G.M. between any two positive numbers is simplify referred as the G.M. between the numbers. Thus, the G.M. between given two positive numbers is equal to the positive square root of their product.

For example, the G.M. between 5 and 125 is $\sqrt{5\times125} - \sqrt{625} = 25$.

n G.M.s BETWEEN ANY TWO GIVEN POSITIVE NUMBERS

Let *a, b* be any two positive numbers. Let $G_1, G_2, \ldots\ldots, G_n$ be the *n*.G.M.s between *a* and *b*.

$\therefore$ By definition, $a, G_1, G_2, \ldots\ldots, G_n, b$ are in G.P. Let *r* be the common ratio of this G.P.

Now, $$b = T_{n+2} = ar^{n+1} \qquad \therefore\ r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$\therefore \qquad G_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$G_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$$

....

$$G_n = ar^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

$\therefore$ **The *n* G.M.'s between a and b are** $a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}, \ldots\ldots, a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$.

Theorem. Prove that the product of n G.M.'s between any two positive numbers is equal to nth power of the G.M. between them.

Proof. Let $G_1, G_2, \ldots\ldots, G_n$ be the *n* G.M.s between positive numbers *a* and *b*.

$\therefore$ $a, G_1, G_2, \ldots\ldots, G_n, b$ are in G.P. Let *r* be the common ratio of this G.P.

Now, $$b = T_{n+2} = ar^{n+1} \qquad \therefore\ r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

Product of *n* G.M.s between *a* and *b*

$= G_1 . G_2 \ldots\ldots G_n = ar . ar^2 . \ldots\ldots ar^n = a^n r^{1+2+\ldots\ldots+n}$

$$= a^n . r^{\frac{n}{2}[2(1)+(n-1).1]} = a^n r^{\frac{n(n+1)}{2}} = a^n \left[\left(\frac{b}{a}\right)^{\frac{1}{n+1}}\right]^{\frac{n(n+1)}{2}} = a^n \left(\frac{b}{a}\right)^{\frac{n}{2}}$$

$= a^{n-n/2} . b^{n/2} = a^{n/2} b^{n/2} = \left(\sqrt{ab}\right)^n =$ (G.M. between a and b)^n.

$\therefore$ **Product of n G.M.s between a and b**

= nth power of the G.M. between a and b

Example 7. *Find the G.M. between the numbers:*

(i) *72 and 882* (ii) *0.027 and 7.5.*

Sol. (i) G.M. between 72 and 882

$$= \sqrt{72 \times 882} = \sqrt{63504} = 252 .$$

(ii) G.M. between 0.027 and 7.5

$$= \sqrt{0.027 \times 7.5} = \sqrt{\frac{27}{1000} \times \frac{75}{10}} = \sqrt{\frac{2025}{10000}} = \frac{45}{100} = 0.45 .$$

Example 8. *The sum of two numbers is 6 times their geometric means. Show that the numbers are in the ratio 3 +* $2\sqrt{2}$ *: 3 -* $2\sqrt{2}$ *.*

Sol. Let the number be a and b. We assume that $a > b$.

$\therefore$ By the given conditions, $a + b = 6\sqrt{ab}$

$\Rightarrow \quad \dfrac{a+b}{2\sqrt{ab}} = 3$ **(Note this step)**

$\Rightarrow \quad \dfrac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \dfrac{3+1}{3-1}$ (By C and D law)

$\Rightarrow \quad \dfrac{\left(\sqrt{a}+\sqrt{b}\right)^2}{\left(\sqrt{a}-\sqrt{b}\right)^2} = 2 \Rightarrow \dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \sqrt{2}$ ($\because a > b$)

$$\Rightarrow \qquad \frac{(\sqrt{a}+\sqrt{b})+(\sqrt{a}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})-(\sqrt{a}-\sqrt{b})}=\frac{\sqrt{2}+1}{\sqrt{2}-1} \qquad \text{(By } C \text{ and } D \text{ law)}$$

$$\Rightarrow \qquad \frac{2\sqrt{a}}{2\sqrt{b}}=\frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$\Rightarrow \qquad \frac{a}{b}=\frac{2+1+2\sqrt{2}}{2+1-2\sqrt{2}}=\frac{3+2\sqrt{2}}{3-2\sqrt{2}}$$

$$\therefore \qquad a : b = 3 + 2\sqrt{2} : 3 - 2\sqrt{2}.$$

WORKING RULES FOR SOLVING PROBLEMS

Rule I. *If three or more than three positive numbers are in G.P., then the numbers lying between the first and the last numbers are the G.M.s between them.*

Rule II. *The G.M. between a and b is* $\sqrt{ab}$.

Rule III. *The n G.M.s between a and b are* $a(b/a)^{\frac{1}{n+1}}, a(b/a)^{\frac{2}{n+1}}, \ldots\ldots, a(b/a)^{\frac{n}{n+1}}$.

Rule IV. *Product of n G.M.s between any two positive numbers is equal to the nth power the G.M. between them.*

EXERCISE 3. 4

SHORT ANSWER TYPE QUESTIONS

1. (i) Find the G.M. between $\frac{8}{9}$ and $\frac{49}{50}$.

   (ii) Find the G.M. between 0.008 and 0.2

2. The G.M. between two positive numbers is 16. If one number is 32, find the other number.

3. If $k - 1$ is the G.M. between $k - 2$ and $k + 1$, then find the value of k.

4. If A and G are the A.M. and G.M. between positive numbers a and b respectively, then show that $A \geq G$.

LONG ANSWER TYPE QUESTIONS

5. If A and G be the A.M. and G.M. between positive numbers a and b respectively, then show that a and b are the roots of the equation $x^2 - 2Ax + G^2 = 0$.
6. If a, b, c are in A.P., x is the G.M. between a and b, y is the G.M. between b and c, show that b^2 is the A.M. between x^2 and y^2.
7. Find two positive numbers whose difference is 2 and whose A.M. exceeds the G.M. by 1/2.
8. If the A.M. between two positive numbers exceeds their G.M. by 2 and the ratio of the numbers be 1 : 4, find the numbers.

Answers

1. (i) $\frac{14}{15}$ (ii) 0.04 2. 8 3. 3

7. $\frac{1}{4}, \frac{9}{4}$ 8. 4, 16.

USE OF G.P. IN SOLVING PRACTICAL PROBLEMS

In this section, we shall see how the formulae relating to G.P. can be made use of in solving practical problems.

Example 9. *Nitin writes letter to four of his friends. He asks each of them to copy the letter and mail to four different persons with the request that they continue the chain similarly. Assuming that the chain is not broken and that it costs 50 paise to mail one letter, find the total money spent on postage till the 8th set of letters is mailed.*

Sol. No. of letters in the Ist set = 4. (These are letters sent by Nitin).

No. of letters in the IInd set = 4 + 4 + 4 + 4 = 16

No. of letters in the IIIrd set = 4 + 4 + …… 16 terms = 64

………………………………………………………………………………

The number of letters sent in the Ist set, IInd set, IIIrd set, …….. are respectively 4, 16, 64, …….., which is a G.P. with $a = 4,\ r = \frac{16}{4} = 4$.

$\therefore$ Total number of letters written in all the first 8 sets = S_8 of the above G.P.

$$= \frac{4(4^8 - 1)}{4 - 1} = 87380$$

$\therefore$ Total money spent on letters = 87380 X $\frac{50}{100} =$ Rs. 43,690.

EXERCISE 3. 5

LONG ANSWER TYPE QUESTIONS

1. Nidhi Gupta writes letters to five of her friends. She asks each of them to copy the letter and mail to five different persons with the request that they continue the chain similarly. Assuming that the chain is not broken and that it cost 15 paise to mail one letter, find the total money spent on postage till the 6th set of letters is mailed.
2. A manufacture reckons that the value of a machine which costs him Rs. 15,625 will depreciate each year by 20%. Find the estimated value of the machine at the end of 5 years.
3. An insect starts from a point and travels in a straight line 1.5 mm in the first second and one – third of the distance covered in the previous second in the succeeding second. In how much time would it reach a point 2.5 mm away from its starting point?
4. One grain of rice is placed on the first square of chess board, 2 on the second, 4 on the third and so on, every time doubling the number of grains. Find the total number of grains required assuming that the number of squares on chess board is 64.
5. The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of 2nd hour? 4th hour? nth hour?
6. A machine depreciates in value at the rate of 30% every year on the reducing balance. If the original cost be Rs. 20,000 and ultimate scrap value be Rs. 4,802, find the effective life of the machine.

7. What will Rs. 500 amount to in 10 years after its deposit in a bank which pays an annual rate of 10% interest compounded annually?

Answers

1. Rs. 2,929.50 2. Rs. 5,120 3. Never

4. $2^{64} - 1$ 5. 120, 480,30(2^n) 6. 4 years

7. Rs. 500 $(1.1)^{10}$.

SUMMARY

1. A sequence (T_n) of non-zero terms is said to be a **geometric progression (G.P.)** if there exists a number, say, r such that

$$\frac{T_{n+1}}{T_n} = r, n \geq 1.$$

The constant number 'r' mentioned above is called the **common ratio** of the corresponding G.P.

2. If 'a' and 'r' be the first term and common ratio of the G.P. (T_n), then $T_n = ar^{n-1}$, $n \in \mathbf{N}$.

3. If 'a' *and* 'r' be the first term and common ratio of the G.P. (T_n), then the sum of first n terms, S_n is given by

$$S_n = \begin{cases} na & \text{if } r = 1 \\ \frac{a(1-r^n)}{1-r} & \text{if } r < 1 \\ \frac{a(r^n-1)}{r-1} & \text{if } r > 1 \end{cases}$$

4. Using $l = T_n = ar^{n-1}$, we get

$$S_n = \begin{cases} \frac{a-lr}{1-r} & \text{if } r < 1 \\ \frac{lr-a}{r-1} & \text{if } r > 1 \end{cases}$$

These formulae are used when 'last term' is given.

5. If $|r| < 1$, then for the G.P. $a, ar, ar^2, \ldots\ldots,$ we have

$$S = \text{sum upto infinity} = \frac{a}{1-r}$$

TEST YOURSELF

1. Find three numbers in G.P. whose sum is 52 and the sum of whose products in pairs is 624.
2. If a, b, c, d are in G.P. prove that $a^n + b^n, b^n + c^n, c^n + d^n$ are also in G.P.
3. If the mth, nth and pth terms of a G.P., are in G.P., then show that m, n, p are in A.P.

4. The sum of the first three terms of a G.P. is 16 and the sum of the next three terms is 128. Determine the first term, common ratio and the sum to n terms of the G.P.
5. The sum of two numbers is 6 times their geometric mean. Show that the numbers are in the ratio $3 + 2\sqrt{2} : 3 - 2\sqrt{2}$.
6. Three numbers are in G.P. and their sum is 70. If the extremes be multiplied by 4 and the mean by 5, these will be in A.P. Find the numbers.
7. If a, b, c are in G.P., show that $\frac{a^2+ab+b^2}{bc+ca+ab}=\frac{b+a}{c+b}$.

Answers

1. 4, 12, 36 or 36, 12, 4

4. $\frac{16}{7}, 2\frac{16}{7}\left(2^n-1\right)$

6. 10, 20, 40 or 40, 20, 10.

7. Let $b = ar$ and $c = ar^2$

SECTION – A

4. PARTIAL FRACTIONS

LEARNING OBJECTIVES

- Introduction
- Resolution of a Fraction into Partial Fractions
- Method of Resolution into Partial Fractions
- Practical Problems

INTRODUCTIONS

In this chapter, we shall learn the method of writing a fraction as the sum of other fractions (called *partial fractions*) whose denominators are of lower degree than the denominator of the give fraction.

RESOLUTION OF A FRACTION INTO PARTIAL FRACTIONS

We know the method of finding the sum of two or more algebraic fractions by reducing the denominators of fractions to a common denominator, which is their L.C.M.

For example:

$$\frac{2}{x-3}+\frac{5}{x+2}=\frac{2x+4+5x-15}{(x-3)(x+2)}=\frac{7x-11}{x^2-x-6}$$

The reverse process of breaking up a single fraction into simpler fractions whose denominators are the factors of the denominator of the give fraction is called the **resolution of a fraction into its partial fractions.**

For example, $\frac{2x}{x^2-1}=\frac{1}{x-1}+\frac{1}{x+1}$

Here $\frac{1}{x-1}$ and $\frac{1}{x+1}$ are the partial fractions of the fraction $\frac{2x}{x^2-1}$.

METHOD OF RESOLUTION INTO PARTIAL FRACTIONS

If $f(x)$ and $g(x)$ are polynomials, then $\frac{f(x)}{g(x)}$ is called a **rational fraction.** If deg. $f(x)<\deg.g(x)$, then the rational fraction $\frac{f(x)}{g(x)}$ is called a **proper rational fraction,** otherwise $\frac{f(x)}{g(x)}$ is called an **improper rational fraction.**

For example, the rational fraction $\frac{x_2+x+3}{(x+2)(x^2+7)}$ is *proper* and the rational fraction $\frac{x^3+x+9}{x^2-5x-6}$ is *improper*.

If is an improper rational fraction, then we can divide $f(x)$ by $g(x)$ so as to write $\frac{f(x)}{g(x)}$ as the sum of polynomial and a proper rational fraction.

$\therefore$ **Any improper fraction can be expressed as the sum of a polynomial and a proper fraction.**

If can be proved mathematically that any proper fraction may be resolved into partial fractions and:

i. If $ax + b$ is any linear non-repeated factor in the denominator, then there corresponds a partial fraction of the form $\frac{A}{ax+b}$.

ii. If $ax + b$ is any linear factor repeater $r(\in N)$times in the denominator, then there correspond partial fractions of the form $\frac{A}{ax+b},\frac{B}{(ax+b)^2},\frac{C}{(ax+b)^3}$,……, r terms

iii. If ax^2+bx+c is any irreducible quadratic non-repeated factor in the denominator, then there corresponds a partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$

iv. If ax^2+bx+c is any irreducible quadratic factor repeated $r \in N$ times in the denominator, then there correspond partial fractions of the form $\frac{Ax+B}{ax^2+bx+c}$, $\frac{Cx+D}{(ax^2+bx+c)^2}$,..........., r terms.

The quantities A, B, C, D, …… are all constants independent of *x*.

$\therefore$ The given proper fraction can be expressed as the sum of its partial fractions.

The constants A, B, C, D, …… occurring in the numerators of the partial fractions are determined by simplifying the sum of partial fractions and then giving various values to x, to obtain equations involving unknown constants or by comparing the coefficients of like power of x.

Illustrations. (i) The partial fractions of the paper fraction $\frac{x-1}{(x+1)(x-2)(x+3)^3}$ are of the types $\frac{A}{x+1}, \frac{B}{x-2}, \frac{C}{x+3}, \frac{D}{(x+3)^2}$ and $\frac{E}{(x+3)^3}$.

(ii) The partial fractions of the proper fraction $\frac{x^2+4}{(x-7)(x+2)^2(x^2+2x+3)}$ are of the types $\frac{A}{x-7}, \frac{B}{x+2}, \frac{C}{(x+2)^2}$ and $\frac{Dx+E}{x^2+2x+3}$.

(iii) The partial fractions of the proper fraction $\frac{x^3+x+7}{(x-1)(x+2)^2(x^2+5)^2}$ are of the types $\frac{A}{x-1}, \frac{B}{x+2}, \frac{C}{(x+2)^2}, \frac{Dx+E}{x^2+5}$ and $\frac{Fx+G}{(x^2+5)^2}$.

PRACTICAL PROBLEMS

Type I. Problems Based on Non-Repeated Linear Factors.

Example 1. *Resolve the following fractions into partial fractions:*

(i) $\frac{3x-1}{x^2-1}$ (ii) $\frac{6x^3+5x^2-7}{3x^2-2x-1}$

Sol. (i) $\frac{3x-1}{x^2-1}$ is a proper fraction and the denominator has linear non-repeated factors.

Let $$\frac{3x-1}{x^2-1} = \frac{3x-1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}.$$

$$\therefore \quad \frac{3x-1}{(x-1)(x+1)} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)} \qquad .....(1)$$

Multiplying by $(x-1)(x+1)$, we get $3x - 1 = A(x+1) + B(x-1)$. ......(2)

Now we find the values of A and B.

Method I. By comparing the coefficients of like powers of *x*.

(2) $\Rightarrow$ $3x - 1 = (A + B)x + (A - B)$

$\therefore$ A + B = 3 and A – B = - 1. Solving those equations, we get A = 1 and B = 2.

Method II. By giving specific values of *x*.

Let us put $x = 1$ and $x = 2$ in (2).

$x = 1 \Rightarrow$ $3(1) - 1 = A(1 + 1) + B(1 - 1) \Rightarrow 2A = 2$ $\Rightarrow A = 1$

$x = 2 \Rightarrow$ $3(2) - 1 = A(2 + 1) + B(2 - 1) \Rightarrow 3A + B = 5$

$\Rightarrow B = 5 - 3A = 5 - 3(1) = 2$

$\therefore$ A = 1 and B = 2 .

In this method, it is always convenient to use those values of x which make linear factors in the denominator zero.

Here $x - 1 = 0 \Rightarrow$ $x = 1$ and $x + 1 = 0$ $\Rightarrow$ $x = -1$.

We put $x = 1$ and $x = -1$ in (1).

$x = 1$ $\Rightarrow$ $3(1) = 1 = A(1 + 1) + B(0)$ $\Rightarrow A = 2/2 = 1$

$x = -1 \quad \Rightarrow \quad 3(-1) - 1 = A(0) + B(-1 -1) \Rightarrow B = -4/(-2) = 2$

$\therefore$ A = 1 and B = 2 .

$$\therefore \ (1) \Rightarrow \frac{3x-1}{(x-1)(x+1)} = \frac{1}{x-1} = \frac{2}{x+1}$$

Remark. In the second method, we observed that

$$A = \text{value of } \frac{3x-1}{x+1} \text{ when } x = 1 \quad \text{and} \quad B = \text{value of } \frac{3x-1}{x-1} \text{ when } x = 1$$

$\therefore$ The value of A can be found out by putting x = 1 in the given proper fraction, after omitting $x - 1$ from the denominator. Similarly, the value of B can be found out by putting x = - 1 in the given proper fraction, after omitting $x + 1$ from the denominator.

Thus, when the factors in the denominator are *linear* and *non-repeated,* we can decompose the given proper fraction into partial fractions as follows:

$$\frac{3x-1}{(x-1)(x+1)} = \frac{3(1)-1}{(x-1)(1+1)} + \frac{3(-1)-1}{(-1-1)(x+1)} = \frac{2}{(x-1)2} + \frac{-4}{(-2)(x+1)} = \frac{1}{x-1} + \frac{2}{x+1}$$

This is called the **short out method** of finding the values of A, B etc. It is very important to bear in mind that this short cut method is to be used only when the denominator has *only linear and non-repeated factors*.

(ii) Given fraction is not a proper fraction. Dividing $6x^3 + 5x^2 - 7$ by $3x^2 - 2x - 1$, we get

$$6x^3 + 5x^2 - 7 = (2x+3)(3x^2 - 2x - 1) + (8x - 4).$$

$$\therefore \ \frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} = 2x + 3 + \frac{8x - 4}{(x-1)(3x+1)} \qquad (\because \ 3x^2 - 2x - 1 = (x-1)(3x+1)$$

$$= 2x + 3 + \left(\frac{8(1) - 4}{(x-1)(3(1)+1)} + \frac{8\left(-\frac{1}{3}\right) - 4}{\left(-\frac{1}{3} - 1\right)(3x+1)} \right)$$

(Using **short cut method**)

$$= 2x+3+\frac{4}{(x-1)4}+\frac{-20/3}{\left(-\frac{4}{3}\right)(3x+1)} = 2x+3+\frac{1}{x-1}+\frac{5}{3x+1}.$$

EXERCISE 4. 1

LONG ANSWER TYPE QUESTIONS

Resolve the following fractions into partial fractions:

1. $\frac{3x+4}{(x-2)(x-3)}$

2. $\frac{1}{(x+1)(2x+1)}$

3. $\frac{2x+1}{(x+1)(x-2)}$

4. $\frac{x+2}{2x^2-7x-15}$

5. $\frac{2x-1}{(x-1)(x+2)(x-3)}$

6. $\frac{x^2+8x+4}{x^3-4x}$

7. $\frac{2x^2+10x-3}{(x+1)(x^2-9)}$

8. $\frac{10x^2+9x-7}{(x+2)(x^2-1)}$

Answers

1. $-\frac{10}{x-2}+\frac{13}{x-3}$

2. $-\frac{1}{x+1}+\frac{2}{2x+1}$

3. $\frac{1}{3(x+1)}+\frac{5}{3(x-2)}$

4. $-\frac{1}{13(2x+3)}+\frac{7}{13(x-5)}$

5. $-\frac{1}{6(x-1)}-\frac{1}{3(x+2)}+\frac{1}{2(x-3)}$

6. $-\frac{1}{x}-\frac{1}{x+2}+\frac{3}{x-2}$

7. $\frac{11}{8(x+1)}-\frac{5}{4(x+3)}+\frac{15}{8(x-3)}$

8. $\frac{3}{x+1}+\frac{2}{x-1}+\frac{5}{x+2}$

Type II. Problems Based on Repeated Linear Factors

Example 2. *Resolve the following fractions into partial fraction:*

$$\frac{2x+1}{(x+2)(x-3)^2}$$

Sol. $\dfrac{2x+1}{(x+2)(x-3)^2}$ is a proper fraction. Let the partial fraction corresponding to the factor $x + 2$ be $\dfrac{A}{x+2}$ and the partial fractions corresponding to the factor $(x-3)^2$ be $\dfrac{B}{x-3}$ and $\dfrac{C}{(x-3)^2}$.

$$\therefore \qquad \frac{2x+1}{(x+2)(x-3)^2} = \frac{A}{x+2} + \frac{B}{x-3} + \frac{C}{(x-3)^2} \qquad \text{....(1)}$$

Multiplying both sides by $(x+2)(x-3)^2$, we get

$$2x+1 = A(x-3)^2 + B(x+2)(x-3) + C(x+2) \qquad \text{.....(2)}$$

Now $x + 2 = 0 \Rightarrow x = -2$ and $x - 3 = 0 \Rightarrow x = 3$

$x = -2$ in (2) implies $-3 = A(-5)^2 + B(0) + C(0) \Rightarrow A = \dfrac{-3}{25}$

$x = 3$ in (2) implies $7 = A(0) + B(0) + C(5) \Rightarrow C = \dfrac{7}{5}$

Comparing the coefficients of x^2 in (2), we get 0 = A + B.

$$\therefore \qquad B = -A = -\left(\frac{-3}{25}\right) = \frac{3}{25}$$

$$\therefore \qquad (1) \Rightarrow \frac{2x+1}{(x+2)(x-3)^2} = \frac{\frac{-3}{25}}{x+2} + \frac{\frac{3}{25}}{x-3} + \frac{\frac{7}{5}}{(x-3)^2}$$

$$= -\frac{3}{25(x+2)} + \frac{3}{25(x-3)} + \frac{7}{5(x-3)^2}.$$

EXERCISE 4. 2

LONG ANSWER TYPE QUESTIONS

Resolve the following fractions into partial fractions:

1. $\dfrac{2x^2 + 7x + 23}{(x-1)(x+3)^2}$

2. $\dfrac{3x - 2}{(x+3)(x+1)^2}$

3. $\dfrac{5x^2+18x+17}{(x+1)^2(2x+3)}$

4. $\dfrac{x^2+x+1}{(x+1)^2(x+2)}$

Answers

1. $\dfrac{2}{x-1}-\dfrac{5}{(x+3)^2}$

2. $\dfrac{11}{4(x+1)}-\dfrac{11}{4(x+3)}-\dfrac{5}{2(x+1)^2}$

3. $\dfrac{5}{2x+3}+\dfrac{4}{(x+1)^2}$

4. $\dfrac{3}{x+2}-\dfrac{2}{x+1}+\dfrac{1}{(x+1)^2}$

Type III. Problems Based on Non-repeated Quadratic Factors

Example 3. *Resolve the following fractions into partial fraction:*

$$\frac{2x+3}{(x+1)(x^2+1)}$$

Sol. $\dfrac{2x+3}{(x+1)(x^2+1)}$ is a proper fraction. Let the partial fraction corresponding to the factor x + 1 be $\dfrac{A}{x+1}$ and the partial fraction corresponding to the factor x^2+1 be $\dfrac{Bx+C}{x^2+1}$.

$$\therefore \quad \frac{2x+3}{(x+1)(x^2+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1} \quad \ldots(1)$$

Multiplying both sides by $(x+1)(x^2+1)$, we get

$$2x+3=A(x^2+1)+(Bx+C)(x+1) \quad \ldots\ldots(2)$$

$x=-1$ in (2) $\Rightarrow$ $1=A(2)+(-B+C)(0)$ $\Rightarrow$ $A=\dfrac{1}{2}$

Comparing the coefficients of x^2 in (2), we get 0 = A + B. $\therefore$ $B=-A=-\dfrac{1}{2}$

Comparing the coefficient of x in (2), we get 2 = B + C. $\therefore$ $C=2-B=2+\dfrac{1}{2}=\dfrac{5}{2}$

$$\therefore \ (1) \Rightarrow \frac{2x+3}{(x+1)(x^2+1)}=\frac{1/2}{x+1}+\frac{(-1/2)x+5/2}{x^2+1}=\frac{1}{2(x+1)}+\frac{5-x}{2(x^2+1)}.$$

EXERCISE 4. 3

LONG ANSWER TYPE QUESTIONS

Resolve the following fractions into partial fractions:

1. $\dfrac{x}{1+x^3}$

2. $\dfrac{x}{x^3+x^2+x+1}$

3. $\dfrac{3}{(1-x)(1+x^2)}$

4. $\dfrac{1}{1+x+x^2+x^3}$.

Answers

1. $-\dfrac{1}{3(1+x)}+\dfrac{x+1}{3(x^2-x+1)}$

2. $-\dfrac{1}{2(x+1)}+\dfrac{x+1}{2(x^2+1)}$

3. $\dfrac{3}{2(1-x)}+\dfrac{3x+3}{2(1+x^2)}$

4. $\dfrac{1}{2(1+x)}+\dfrac{x+1}{2(1+x^2)}$

SUMMARY

1. The process of breaking up a single fraction into simpler fractions whose denominators are the factors of the denominator of the given fraction is called the **resolution of a fraction into its partial fractions.**

2. (i) If deg. $f(x)$ < deg. $g(x)$, then the fraction $\frac{f(x)}{g(x)}$ is called a **proper fraction.**

   (ii) If deg. $f(x) \geq$ deg. $g(x)$, then the fraction $\frac{f(x)}{g(x)}$ is called in **improper fraction.**

3. Any improper fraction can be expressed as the sum of a polynomial and a proper fraction.

4. Any proper fraction can be resolved into partial fractions.

TEST YOURSELF

Resolve the following fractions into partial fractions :

1. $\frac{x+2}{x^2-7x+12}$
2. $\frac{8-x}{2x^2+3x-2}$
3. $\frac{12x+11}{x^2+x-6}$
4. $\frac{x}{x^2-3x-18}$
5. $\frac{5x+4}{x^2+2x}$
6. $\frac{x^3}{x^2-4}$

Answers

1. $\frac{6}{x-4}-\frac{5}{x-3}$
2. $\frac{3}{2x-1}-\frac{2}{x+2}$
3. $\frac{7}{x-2}+\frac{5}{x+3}$
4. $\frac{2}{3(x-6)}+\frac{1}{3(x+3)}$
5. $\frac{2}{x}+\frac{3}{x+2}$
6. $x+\frac{2}{x-2}+\frac{2}{x+2}$

SECTION – A

5. PERMUTATIONS

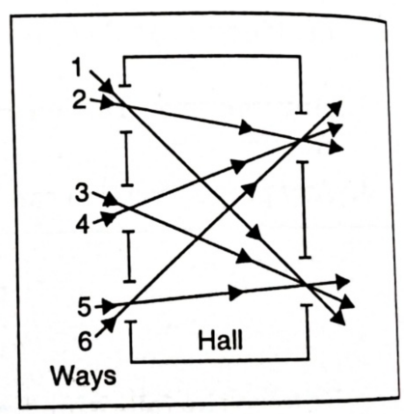
LEARNING OBJECTIVES

- Introduction
- Fundamental Principle of Counting
- Factorial Notation
- Definition of Permutations
- Practical Problems Involving Permutations
- Permutations of Things not all Different
- Circular Permutations
- Permutations with Repetitions

INTRODUCTIONS

In this chapter, we shall discuss the problem of arranging certain things in a definite order, taking particular number of things at a time. ab and ba are two arrangements of a and b in different orders.

Suppose there are three entrances to hall and two exists to come out of that hall. Therefore, there will be exactly six ways of going in and coming out of the hall. This is explained in the adjoining diagram.



In mathematical terminology, the acts like 'going in' and 'coming out' are called **operations.** Thus, we can say that there are three ways of performing first operation and two ways of performing second operations.

From the figure, we observe that there are in all 6 ways of going in and coming out of the hall. The total number of ways (6) is also equal to the product of 3 and 2.

FUNDAMENTAL PRINCIPLE OF COUNTING

This principle states that if an operating can be performed in 'm' different ways, following which another operation can be performed in 'n' different ways, the both operation, in succession can be performed in exactly 'mn' different ways.

In the illustration given in the above section, the number of ways of performing operations are 3 and 2 and so by **F.P.C.** the total number of ways of performing both operations in the specified order is equal to 3 X 2, i.e., 6.

The **F.P.C.** can also be generalized, for even more than two operations, as follows:

If an operation can be performed in 'm_1' different ways, following which another operation can be performed in 'm_2' different ways, following which another operations can be performed in 'm_3' different ways and so forth, then all operations, in succession can be performed in exactly m_1 X m_2 X m_3 X .... different ways.

WORKING RULES FOR SOLVING PROBLEMS

Step I. *Identify the independent operations involved in the given problem.*

Step II. *Find the number of ways of performing each operation.*

Step III. *Multiply these numbers to get the total number of ways of performing all the operations.*

Example 1. *In a class there are 25 boys and 15 girls. The teacher wants to select 1 boy and 1 girl to represent the class in a function. In how many ways can the teacher make this selection ?*

Sol. No of ways of selecting one boy out of 25 boys = 25

No. of ways of selecting one girl out of 15 girls = 15.

$\therefore$ By **F.P.C.,** total number of ways of selecting one boy and one girl

$$= 25 \times 15 = 375.$$

Example 2. *How many odd numbers less than 1000 can be formed using the digits 0, 1, 8, 9 (repetitions of digits is allowed) ?*

Sol. The numbers less than 1000 can be either one digited or two digited or three digited.

One digited numbers

The one digited odd numbers are 1 and 9. These are 2 in number.

Two digited numbers

No. of ways of filling unit's place = 2 (either 1 or 9)

No. of ways of filling ten's place = 3 (either 1 or 8 or 9)

∴ By **F.P.C.,** number of 2 digited odd numbers = 2 X 3 = 6

Three digited numbers

No. of ways of filling unit's place = 2 (either 1 or 9)

No. of ways of filling ten's place = 4 (either 0 or 1 or 8 or 9)

No. of ways of filling hundred's place = 3 (either 1 or 8 or 9)

∴ By **F.P.C.,** number of 3 digited odd numbers = 2 X 4 X 3 = 24

∴ Total numbers = 2 + 6 + 24 = 32.

EXERCISE 5. 1

SHORT ANSWER TYPE QUESTIONS

1. In how many ways can two friends sit in three vacant seats in a bus?
2. A hall has three entrances and four exists. In how many ways can a man enter and exit from the hall?
3. In a class there are 30 boys and 18 girls. The teacher wants to select 1 boy and 1 girl to represent to class in a function. In how ways can the teacher make this selection?
4. Given seven flags of different colours, how many different signals can be generated, if a signals requires the use of two flags, one below the other?
5. A lady wants ot select a cotton saree and one polyester saree from a textile shop. If there are 20 cotton varieties and 45 polyester varieties, in how many ways can she choose two sarees?

LONG ANSWER TYPE QUESTIONS

6. A puzzle has eight empty spaces, each to be filled by the word 'yes' or 'no'. in how many ways can it be solved?

7. There are 6 multiple choice questions in an examination. How many sequences of answer are possible, if the first three questions have four choices each and the next three have five choices each?
8. Five coins are tossed simultaneously. In how many ways can these fall?
9. In how many ways can 5 women draw water from 5 taps, if no tap remains unused?
10. A club consists of 100 numbers. In how many ways can the members select a president, a vice president, a secretary, if a member can hold only one position at a time?

Answers

1. 6	2. 12	3. 540
4. 42	5. 900	6. $2^8 = 256$
7. 4 X 4 X 4 X 5 X 5 X 5 = 8000	8. 8. 32	
9. 5 X 4 X 3 X 2 X 1 = 120	10. 100 X 99 X 98 = 970200.	

FACTORIAL NOTATION

Let $n \in N$. The continued product of first n natural numbers (beginning with 1 and with n) is called **factorial** n and is denoted by $n\,!$

Thus, $n\,! = 1.2.3. \ldots\ldots. (n-1).n$

In particular, $5\,! = 1.2.3.4.5 = 120$

$8\,! = 1.2.3.4.5.6.7.8 = 40320.$

Factorial zero is defined as equal to 1 and we write $0\,! = 1$.

It is easily seen that $n\,! = n\,.\,(n = 1)\,!$

$= n(n-1).\,(n-2)\,!$

$= n(n-1)\,(n-2)\,.\,(n-3)\,!$

……………………………

$\therefore$ We have $8\,! = 8 \text{ X } 7\,! = 8 \text{ X } 7 \text{ X } 6\,! = 56 \text{ X } 6\,!$ etc.

Example 3. *Evaluation :*

(i) $\frac{9\,!}{8\,!}$ (ii) $\frac{16.15.14.13.12\,!}{15\,!}$

(iii) $\frac{11\,!}{7!\,4\,!}$ (iv) $\frac{1\,!}{5\,!}+\frac{1\,!}{6\,!}+\frac{1\,!}{7\,!}$.

Sol. (i) $\frac{9\,!}{8\,!}=\frac{9\times 8\,!}{8\,!}=9.$

(ii) $\frac{16\times 15\times 14\times 13\times 12\,!}{15\,!}=\frac{16\,(15\times 14\times 13\times 12\,!)}{15\,!}=\frac{16(15\,!)}{15\,!}=16$

(iii) $\frac{11\,!}{7!\,4\,!}=\frac{11\times 10\times 9\times 8\times 7\,!}{7\,!\,(4\times 3\times 2\times 1)}=\frac{11\times 10\times 9\times 8}{4\times 3\times 2}=330$

(iv) $\frac{1\,!}{5\,!}+\frac{1\,!}{6\,!}+\frac{1\,!}{7\,!}=\frac{7\times 6}{7\times 6\times 5\,!}+\frac{7}{7\times 6\,!}+\frac{1}{7\,!}=\frac{42}{7\,!}+\frac{7}{7\,!}+\frac{1}{7\,!}=\frac{50}{7\,!}$.

Example 4. *Convert into factorials:*

(i) 4.5.6.7.8.9.10.11 (ii) 2.4.6.8.10.

Sol. (i) $4.5.6.7.8.9.10.11=\frac{1\times 2\times 3\times 4\times 5\times 6\times 7\times 8\times 9\times 10\times 11}{1\times 2\times 3}=\frac{11\,!}{3\,!}$

(ii) $2.4.6.8.10=\frac{1\times 2\times 3\times 4\times 5\times 6\times 7\times 8\times 9\times 10}{1\times 2\times 3\times 4\times 5}=\frac{10\,!}{5\,!}$

EXERCISE 5. 2

SHORT ANSWER TYPE QUESTIONS

1. Find the value of :

   (i) 7 ! (ii) 8 ! (iii) (3 !)(6 !) (iv) $\frac{9\,!}{4\,!\,5\,!}$

2. Show that $2n\,!\;n\,!=2n^2\,.(n-1)\,!\,(2n-1)\,!\;\; n\in N$.

3. Show that $\frac{12\,!}{6\,!}=1.3.5.7.9.11.2^6$.

4. Evaluate : $\dfrac{n\,!}{(n-r)\,!}$ when :

   (i) $n = 10,\ r = 4$ (ii) $n = 9,\ r = 5$

5. Show that $n\,!\,(n + 2) = n\,! + (n + 1)\,!$.

6. Find n if:

   (i) $(n + 1)\,! = 12\ .\ (n - 1)!$ (ii) $(n + 2)\,! = 20\ .\ n\,!$

LONG ANSWER TYPE QUESTIONS

7. Show that 55 ! + 1 is not divisible by any number from 2 to 55.

8. Find the L.C.M. and H.C.F. of :

   (i) 3 !, 6 !, 7 ! (ii) 10 !, 12 !, 17 !

9. Show that $\dfrac{6\,!}{2\,!\,4\,!} + \dfrac{6\,!}{3\,!\,3\,!} = \dfrac{7\,!}{3\,!\,4\,!}$

10. If $(n + 2)\,! = 60\ .\ (n - 1)\,!$, find the value of n.

11. Show that $\dfrac{(2n+1)\,!}{n!} = 1.3.5.\ \ldots\ldots\ (2n + 1)2^n,\ \ n \in N$.

12. If $\dfrac{n\,!}{2\ .\ (n-2)\,!} = \dfrac{2\,.n\,!}{4\,!\ (n-4)\,!}$, them find the value of n.

Answers

1. (i) 5040	(ii) 40320	(iii) 4320	(iv) 126
4. (i) 5040	(ii) 15120	6. (i) 3	(ii) 3
8. (i) 7 !, 3 !	(ii) 17 !, 10 !	10. 3	12. 5.

DEFINITION OF PERMUTATIONS

An arrangement in a definite order of a number of things taking some or all of them at a time is called a **permutation.** The total number of permutations of n distinct things taking r $(1 \leq r \leq n)$ at a time is denoted by ${}^nP_r$, or by $P(n,\ r)$. We define ${}^nP_0 = 1$.

For example, the permutations of 3 things *a*, *b*, *c* taking 2 at a time are:

ab	*bc*	*ca*
ba	*cb*	*ac*

$\therefore\ {}^3P_2 = 6$

The value of ${}^3P_2$ can also be found out by considering the problem of filling two places by using two out of *a, b, c*. Thus, the first place can be filled in 3 ways. After filling the first place, the second place can be filled in by using any of 3 – 1 = 2 things.

$\therefore$ By **F.P.C.,** the value of ${}^3P_2 = 3 \times 2 = 6$.

Theorem I. For $1 \leq r \leq n$, prove that ${}^nP_r = n(n-1)(n-2)$ …….r factors.

Proof. ${}^nP_r$ = number of permutations of n different things taking r at a time

= number of ways in which r places in a row can be filled with n different things

Now, no. of ways of filling 1st place $= n$

No. of ways of filling 2nd place $= n - 1$

($\because$ only $n - 1$ things are left after filling the 1st place)

No. of ways of filling 3rd place $= n - 2$

……………………………………

……………………………………

No. of ways of filling last *i.e., r*th place $= n - (r - 1)$

$\therefore$ By **F.P.C.,** the total no. of ways of filling all the r places

$= n\,(n-1)\,(n-2)$ ……. r factors

$\therefore$ **${}^nP_r = n(n-1)(n-2)$ ……r factors.**

Thus, the number of permutations of n things taking r at a time is given by

$$ {}^nP_r = n(n-1)(n-2) \text{ ……r factors}, \quad 1 \leq r \leq n. $$

Theorem II. For $0 \leq r \leq n$, prove that $^nP_r = \dfrac{n\,!}{(n-r)\,!}$

Proof. Let $0 < r \leq n.$

$$^nP_r = n\,(n-1)\,\ldots\ldots\, r \text{ factors}$$

$$= n\,(n-1)\,\ldots\ldots\,[n-(r-1)] = n\,(n-1)\,\ldots\ldots\,(n-r+1)$$

$$= \frac{n(n-1)\ldots\ldots\ldots\ldots(n-r+1).(n-r)(n-r-1)\ldots\ldots\ldots3.2.1}{(n-r)(n-r-1)\ldots\ldots\ldots3.2.1}$$

$$= \frac{n\,!}{(n-r)\,!}$$

$$\therefore \quad ^nP_r = \frac{n\,!}{(n-r)\,!},\ 0 < r \leq n.$$

Also $\quad ^nP_0 = 1$ and $\dfrac{n\,!}{(n-0)\,!}, \dfrac{n\,!}{n\,!} = 1$

$$\therefore \quad ^nP_r = \frac{n\,!}{(n-r)\,!},\ \mathbf{0 < r \leq n.}$$

Corollary. Show that $^nP_n = n\,!$.

Proof. $\quad ^nP_r = n(n-1)\,(n-2)\,\ldots\ldots\, n$ factors

$$= n(n-1)\,(n-2)\ldots\ldots\,(n-(n-1))$$

$$= n(n-1)\,(n-2)\,\ldots\ldots\,1 = \mathbf{n\,!}.$$

∴ **The number of permutations of n different things all taking all at a time is equal to n !.**

Remark 1. In evaluating ^nP_r , the formula :

(i) $^nP_r = n\,(n-1)(n-2)\,\ldots\ldots\, r$ factors is used when the value of r is known.

For example, $^8P_r = n(n-1)\,(n-2)\,(n-3)$

(ii) $\quad ^nP_r = \dfrac{n\,!}{(n-r)\,!}$ is used the value of r is not known.

For known, $\quad {}^8P_r = \dfrac{8\,!}{(8-r)\,!}$.

Remark 2. It may be noted carefully that in ${}^nP_r$, we count only those permutations in which repetition of things is not allowed.

Example 5. *Evaluate :*

(i) ${}^5P_3$ (ii) ${}^7P_2$ (iii) ${}^{18}P_3$ (iv) ${}^6P_6$

Sol. (i) $${}^5P_3 = \frac{5\,!}{(5-3)\,!} = \frac{5\times4\times3\times2\times1}{2\times1} = 5\times4\times3 = 60 .$$

$$\left({}^nP_r = \frac{n\,!}{(n-r)\,!} \right)$$

Alternatively, ${}^5P_3$ = 5 X 4 X 3 = 60 $\quad$ (${}^nP_r = n(n-1)$ …… r factors)

(ii) ${}^7P_2$ = 7 X 6 = 42

(iii) ${}^{18}P_3$ = 18 X 17 X 16 = 4896

(iv) $${}^6P_6 = \frac{6\,!}{(6-6)\,!} = \frac{6\,!}{(0)\,!} = \frac{6\times5\times4\times3\times2\times1}{1} = 720 .$$

Example 6. *Find n, if* ${}^{2n-1}P_n : {}^{2n+1}P_{n-1} = 22 : 7$.

Sol. We have ${}^{2n-1}P_n : {}^{2n+1}P_{n-1} = 22 : 7$, *i.e.,* $\dfrac{{}^{2n-1}P_n}{{}^{2n+1}P_{n-1}} = \dfrac{22}{7}$

$$\Rightarrow \quad \frac{\dfrac{(2n-1)\,!}{[(2n+1)-n]\,!}}{\dfrac{(2n+1)\,!}{[(2n+1)-(n-1)]\,!}} = \frac{22}{7} \Rightarrow \frac{(2n-1)(n+2)\,!}{(n-1)(2n+1)\,!} = \frac{22}{7}$$

$$\Rightarrow \quad \frac{(2n-1)\,!\,(n+2)(n+1)\,.\,n\,.\,(n-1)\,!}{(n-1)(2n+1)\,2n\,.(2n-1)\,!} = \frac{22}{7} \Rightarrow \frac{(n+2)(n+1)}{(2n+1)2} = \frac{22}{7}$$

$$\Rightarrow \quad 7(n^2+3n+2) = 44(2n+1)$$

$$\Rightarrow \quad 7n^2-67n-30 = 0 \Rightarrow n = 10, -3/7$$

$\therefore$ $\quad$ **n = 10** $\quad$ ($\because$ n = -3/7 is not possible)

WORKING RULES FOR SOLVING PROBLEMS

Rule I. *^nP_r (or P(n, r) denotes the number of permutations of n distinct things taking r at a time, $1 \leq r \leq n$.*

Rule II. *If value of r is given, then use : $^nP_r = n(n-1)(n-2)$ ……r factors.*

Rule III. *If value of r is not given, then use :* $^nP_r = \dfrac{n!}{(n-r)!}$

Rule IV. *$^nP_r = n! = n(n-1)(n-2)$ …….. 3. 2. 1.*

EXERCISE 5. 3

SHORT ANSWER TYPE QUESTIONS

1. Evaluate :

   (i) $^7P_3$, (ii) $^9P_5$ (iii) $^{20}P_4$ (iv) $^{10}P_8$.

2. Find n if $16\ ^nP_3 = 13\ ^{n+1}P_3$.
3. Find n if $^{2n}P_3 = 100\ ^nP_2$.
4. Find n if $^nP_6 = 3\ ^nP_5$.
5. Show that $^nP_n = 2\ ^nP_{n-2}$.
6. Show that $^nP_n = {^nP_{n-1}}$.
7. Show that $^nP_r = n\ ^{n-1}P_{r-1}$.

LONG ANSWER TYPE QUESTIONS

8. Find n if $^nP_5 = 42\ ^nP_3$, $n \geq 5$
9. Find n if $30\ ^nP_6 = {^{n+2}P_7}$, $n \geq 6$.
10. Find n if $^{10}P_n = 2\ ^9P_n$.
11. Find n if $^nP_4 : {^{n+1}P_5} = 1 : 9$
12. Find n if $^{10}P_{n+1} : {^{11}P_n} = 30 : 11$.
13. If $1 \leq r \leq s \leq n$, then show that ^nP_s is a multiple of ^nP_r .

Answers

1. (i) 210	(ii) 15120	(iii) 116280	(iv)1814400
2. 15	3. 13	4. 8	8. 10
9. 8, 19	10. 5	11. 8	12. 5.

PRACTICAL PROBLEMS INVOLVING PERMUTATIONS

In this section, we shall learn the use of the formulae regarding permutations, in solving practical problems.

Example 7. *In how many ways, the letters of the following words can be arranged :*

(i) **RAM** (ii) **POWER**

(iii) **COMBINE** (iv) **EQUATION ?**

Sol. (i) Total number of letters in the word **RAM** is 3 and all are different.

$\therefore$ Number of arrangements of these three letters = $P(3,3) = 3! =$ **6**

Remark. The different arrangements of the letters **R, A** and **M** are **RAM, RMA, ARM, AMR, MRA** and **MAR.**

(ii) Total number of letters in the word **POWER** is 5 and all are different.

$\therefore$ Number of arrangements of these five letters = $P(5, 5) = 5! =$ **120**

(iii) Total number of letters in the word **COMBINE** is 7 and all are different.

$\therefore$ Number of arrangements of these seven letters = $P(7, 7) = 7! =$ **5040**

(iv) Total number of letters in the word **EQUATION** is 8 and all are different.

$\therefore$ Number of arrangements of these eight letters = $P(8, 8) = 8! =$ **40320**.

Example 8. *In an examination hall, there are four rows of chairs. Each row has 8 chairs one behind the other. There are two classes sitting for the examination with 16 students in each class. It is desired that in each row, all students belong to the same class and that no two adjacent rows are allotted to the same class. In how many ways can these 32 students be seated ?*

Sol. No. of ways of choosing rows for classes = 2.

Ist way	**IInd way**
I, III First Class	I, III Second class
II, IV Second Class	II, IV First class

No. of ways of arranging 16 students of first class in 2 rows each having 8 chairs = $P(16, 16) = 16\,!$.

No. of ways of arranging 16 student of second class in 2 rows each having 8 chairs = $P(16, 16) = 16$.

$\therefore$ By **F.P.C.,** total number of arrangements = 2 X 16 ! X 16 ! = $\mathbf{2(16\,!)^2}$.

EXERCISE 5. 4

SHORT ANSWER TYPE QUESTIONS

1. In how many ways , the letters of the following words can be arranged :

   (i) **AND** (ii) **MOHAN**

   (iii) **DELHI** (iv) **PERSONAL ?**

2. In how many ways can five children stand in a queue ?
3. Determine the number of ways in which 4 books, one each on physics, chemistry, biology and mathematics can be arranged on shelf.
4. How many different five letters words (may or may not be meaningful) can be formed out of the letter of the word **'KNIFE'** if repetition of letters is not allowed?
5. In how many ways can six women draw water from six taps?
6. Determine the number of permutations of the letters of the words **HEXAGUN** taken all at a time.
7. How many three digit number of there, with distinct , with each digit odd?
8. Six students are contesting election for the president ship of the student union. In how many ways can their names be listed on the ballot paper?

LONG ANSWER TYPE QUESTIONS

9. Find the number of permutations of English letters **A, B, C, D, E** taking 2 at a time. Also verify your result.
10. Determine the number of different 5-letter words formed from the letters of the word **EQUATION.**
11. Twelve students complete in a race. In how many ways can the first three places be taken?

12. Determine the number of 5-letter words formed from the letters of the word **DAUGHTER.**

Answers

1. (i) 6	(ii) 120	(iii) 120	(iv) 40320
2. 120	3. 24	4. 120	5. 720
6. 5040	7. 5 X 4 X 3 = 60		8. 720
9. 5 X 4 = 20	10. 8 X 7 X 6 X 5 X 4 = 6720		
11. 12 X 11 X 10 = 1320		12. 6720.	

PERMUTATIONS OF THINGS NOT ALL DIFFERENT

In this section, we shall consider the method of counting the possible arrangements of things, which are not all different.

Theorem I. If p_1 objects are of first kind and p_2 objects be of second kind, then prove that the total number of permutations of all the $p_1 + p_2$ objects is given by $\frac{(p_1+p_2)!}{p_1!\,p_2!}$**.**

Proof. Let the required number of permutations be x. We fix one permutation among these x permutations.

Now we imagine that the p_1 alike objects are replaced by p_1 different objects. These p_1 different objects can be arranged among themselves in p_1 ! ways.

Similarly, we imagine that the p_2 alike objects are also replaced by p_2 different objects and these can be arranged in p_2 ! ways.

Therefore, if both the replacements are done simultaneously then each one of x permutations give rise to $p_1 ! \, p_2 !$ permutations.

$\therefore$ x permutations gives rise to $(p_1 ! \, p_2 !)\, x$ permutations.

Now each of these $p_1 ! \, p_2 ! \, x$ is a permutations of $p_1 + p_2$ different objects taken all at a time

$$\therefore \qquad P_1 ! \, p_2 ! \, x = (p_1 + p_2) !$$

$\therefore \qquad x = \dfrac{(p_1 + p_2)!}{p_1! \, p_2!}$.

Theorem II. If p_1 objects are of the ith kind and I = 1, 2, 3, ..., r then prove that the total number of permutations of all the $p_1 + p_2 + \dots\dots + p_r$ object is given by $\dfrac{(p_1 + p_2 + \dots\dots + p_r)!}{(p_1!)(p_2!)\dots\dots(p_r!)}$.

Proof. This result is a generalization of **Theorem I** and we accept it without proof.

Example 9. *In how many ways can the letters of the following words be arranged :*

(i) **TALL** (ii) **APPLE**

Sol. (i) In the word **TALL,** there are 4 letters.

T occurs once, A occurs once and **L** occurs twice.

$\therefore$ Total number of arrangements = $\dfrac{4\,!}{2\,!} = \dfrac{24}{3} = 12$

$$\begin{vmatrix} T \to 1 \\ A \to 1 \\ L \to 2 \end{vmatrix}$$

(In the denominator, we have avoided writing 1 ! two times, because 1 ! = 1.)

(ii) In the word, **APPLE**, there are 5 letters.

P occurs twice and the rest are all different.

$\therefore$ Total number of arrangements = $\dfrac{5\,!}{2\,!} = \dfrac{120}{2} = 60$.

$$\begin{vmatrix} A \to 1 \\ P \to 2 \\ L \to 1 \\ E \to 1 \end{vmatrix}$$

Example 10. *How many different words can be formed with the letters of the word* **HARYANA** *? How many of these :*

(i) *have* **H** *and* **N** *together ?* (ii) *begin with* **H** *and end with* **N** *?*

(iii) *have three vowels together ?*

Sol. In the word **HARYANA,** there are 7 letters.

A occurs thrice and the rest are all different.

$\therefore$ Total number of words = $\frac{5\,!}{2\,!}$ = $\frac{5040}{6} = 840$

$$\left| \begin{array}{lll} H & \rightarrow & 1 \\ A & \rightarrow & 3 \\ R & \rightarrow & 1 \\ Y & \rightarrow & 1 \\ N & \rightarrow & 3 \end{array} \right.$$

(ii) In this case, we consider the pair (**HN**) as one object.

$\therefore$ No. of objects to be arranged is 6 in which **A** is repeated thrice.

$\therefore$ No. of arrangements = $\frac{6\,!}{3\,!}$ = $\frac{720}{6} = 120$

$$\left| \begin{array}{lll} \overline{HN} & \rightarrow & 1 \\ A & \rightarrow & 3 \\ R & \rightarrow & 1 \\ Y & \rightarrow & 1 \end{array} \right.$$

(iii) The Three vowels in the word **HARYANA** are **A, A, A.**

$\therefore$ We are to arrange 5 objects **(AAA), H, R, Y, N** and this can be done in

5 ! = 120 ways.

$\therefore$ Total number of words = 120.

$$\left| \begin{array}{lll} \overline{AAA} & \rightarrow & 1 \\ H & \rightarrow & 1 \\ R & \rightarrow & 1 \\ Y & \rightarrow & 1 \\ N & \rightarrow & 1 \end{array} \right.$$

EXERCISE 5. 5

SHORT ANSWER TYPE QUESTIONS

1. In how many different ways, can the letters of the following words be arranged:

   (i) **ALL** (ii) **MOON**

   (iii) **NOON** (iv) **AGAIN ?**

2. In how many different ways, can the letters of the following words be arranged:

   (i) **COMMERCE** (ii) **ALLAHABAD**

   (iii) **CHANDIGARH** (iv) **EXAMINATION?**

3. In how many different ways, can the letters of the following words be arranged :

(i) **INDEPENDENCE** (ii) **ASSASSINATION**
(iii) **KURUKSHETRA** (iv) **YAMUNANAGAR?**

4. How many different arrangements can be made by using 6 red and 5 black identical balls?
5. The seven objects are *x, x, y, y, y, y, z.* Find the number of their permutations.
6. There are 5 red, 4 white and 3 blue marbles in a bag. These are drawn one by one and arranged in a row. Assuming that all the 12 marbles are drawn, determine the number of different arrangements.

LONG ANSWER TYPE QUESTIONS

7. In how many ways can the letter of the word **PERMUTATIONS** be arranged if the
   (i) words start with **P** (ii) words start with **PS**.
   (iii) vowel are all together
   (iv) there are always 4 letters between **P** and **S**.
8. How many 7-digit numbers can be formed using the digits 1, 2, 0, 2, 4, 2, 4?
9. How many numbers greater than a million can be formed with the digits :
   (i) 2, 3, 1, 3, 4, 2, 3 (ii) 2, 3, 0, 3, 4, 2, 3?

Answers

1. (i) 3 (ii) 12 (iii) 6 (iv) 60
2. (i) 5040 (ii) 7560 (iii) 907200 (iv) 4989600
3. (i) 1663200 (ii) 10810800 (iii) 4989600 (iv) 831600
4. $\frac{11\,!}{6\,!\,5\,!} = 462$ 5. 105 6. 27720
7. (i) $1 \times \frac{11\,!}{2\,!} = 19958400$ (ii) $1 \times \frac{10\,!}{2\,!} = 1814400$
   (iii) $5\,! \times \frac{(7+1)\,!}{2\,!} = 2419200$ (iv) $7 \times \frac{10\,!}{2\,!} = 12700800$
8. 360 9. (i) $\frac{7\,!}{2\,!\,3\,!} = 420$ (ii) $\frac{7\,!}{2\,!\,3\,!} - \frac{6\,!}{2\,!\,3\,!} = 360$
10. 60, 10.

CIRCULAR PERMUTATIONS

The **circular permutations** are permutations of certain things in the form of a circle.

For example, *ABCD, BCDA, CDAB and DABC* are four different linear permutations, but round a circle these four different arrangements gives only *one* circular permutation *ABCD* read in the anticlockwise direction. In circular permutations, there is neither a beginning nor and end.

Theorem. Prove that the number of circular permutations of n different things is given by (n – 1) !.

Proof. Let x be the required number of circular permutations. To each one of these x circular permutations, there corresponds n linear permutations starting from each one of n things in the circular permutations and read in the anticlockwise direction.

∴ All circular permutations give rise to $x.n$ linear permutations.

∴ $$x \cdot n = n\,!$$

∴ $$x = \frac{n\,!}{n\,!} = (n-1)\,!$$

Thus, the number of circular permutations of n different things is given by (n – 1) !.

In circular permutations, the permutations are always read in anticlockwise direction. The circular permutation *ABCDE* of the objects *A, B, C, D* and *E* is as given in Fig. (i).

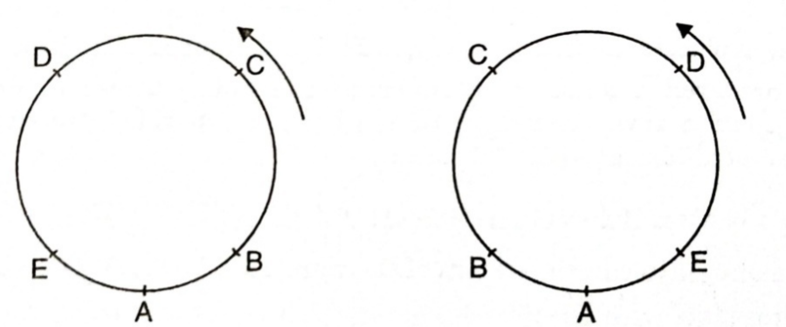


Fig. (*i*) Fig. (*ii*)

This is not to be read as *AEDCB* because the permutations *AEDCB* is as given in Fig. (ii).

∴ Anticlockwise wand clockwise permutations are different permutations.

In case, there is no difference between anticlockwise and clockwise permutations, then the circular permutations of n things is $\frac{(n-2)!}{2}$, because in this case if we turn around an anticlockwise circular permutations, we shall get the corresponding clockwise circular permutation. For example if we turn around the circular permutation in Fig (i), we shall get the circular permutation as in Fig. (ii) Problems relating to forming necklace with n different beads are solved by using the formula $\frac{(n-2)!}{2}$.

Remark. In an anticlockwise circular permutation and its corresponding clockwise circular permutation, each item has same neighbor on both sides with the only difference that a neighbor on left side would be the neighbor on the right side in the corresponding circular permutation.

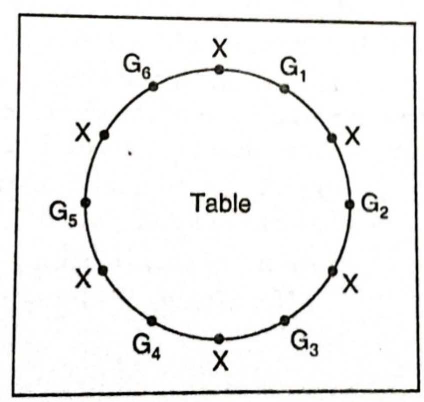
Example 11. *Find the number of ways in which 7 dissimilar things can be arranged in a (i) line (ii) circle.*

Sol. Number of things = 7.

(i) $\therefore$ Number of ways of arranging these things in a line = 7 ! = 5040

(ii) Number of ways of arranging these things in a circle = (7 – 1) ! = 6 ! = 720.

Example 12. *There are six gentlemen and four ladies to dine at a round table. In how many ways can they seat themselves so that no two ladies are together?*



Sol. No. of gentle men = 6

No. of ladies = 4

In order to have no two ladies together, we shall first arrange all the 6 gentlemen and then we shall arrange ladies to in between gentlemen.

No. of ways of arranging gentlemen

= (6 – 1) ! = 120

The ladies can occupy seats marked '*X*'.

$\therefore$ No. of ways of arranging ladies

$= {}^6P_4 = 360$

∴ By **F.B.C.,** the required number of arrangements

$= 120 \text{ X } 360 = 43200.$

WORKING RULES FOR SOLVING PROBLEMS

Rule I. *Number of ways of arranging in dissimilar things in a circle = (n – 1) !.*

Rule II. *Number of ways of formatting a necklace of n dissimilar beads*

$$= \frac{(n-1)!}{2}$$

EXERCISE 5. 6

SHORT ANSWER TYPE QUESTIONS

1. In how many ways 10 boys be arranged in a (i) line (ii) circle ?
2. In how many ways can 8 heads of different colours form a necklace ?
3. In how many ways can a garland of 15 different flowers be made ?

LONG ANSWER TYPE QUESTIONS

4. A round table conference is to be held between delegates of 15 countries. In how many ways, can they be seated, if two particular delegates may wish to sit together?
5. In how many ways can 5 gentlemen and 5 ladies be seated at a round table so that two particular gentlemen are always together?
6. The chef minister of 11 states of India meet to discuss the current issues. In how many ways can they seat themselves at a round table so that the chief ministers of states X and Y sit together?
7. In how many ways 11 members of a committee sit at a round table so that the secretary and the joint secretary are always the neighbours of the president?

Answers

1. (i) 3628800	(ii) 362880	2. 2520	3. (14 !)/2
4. 2 X (13 !)	5. 80640	6. 725760	7. 80640.

PERMUTATIONS WITH REPETITIONS

In this section, we shall study the method of counting various arrangements in which things are allowed to repeat in the same permutation.

Theorem. Prove that the number of permutations of n different things taken r at a time when each thing is allowed to repeat any number of times in any arrangement is given by n^r :

Proof. The number of permutations of n different things taking r at a time is same as the number of ways of fillings r places with n different things.

Now, no. of ways of filling 1st place = n

No. of ways of filling 2nd place = n

($\because$ The thing used in filling the 1st place can also be used in filling the 2nd place.)

No. of ways of filling 3rd place = n

....

No. of ways of filling rth place = n

$\therefore$ By **F.P.C.,** the total no. of ways of filling all the r places

$$= n \,.\, n \,.\, n \,........\, n = n^r \,.$$

Thus, the number of permutations of n different things taken r at a time when each thing is allowed to repeat any number of times in any arrangement is given by n^r.

Remark. In the above theorem, the value of r can also be greater than n.

Example 13. *Find the number of two-digit numbers by using the digits 2, 3, 5, 7. The repetitions of digits is allowed. Also, verify your answer.*

Sol. The digits are 2, 3, 5, 7.

No. of ways of filling unit's place = 4.

No. of ways of filling ten's place = 4.

$\therefore$ By **F.P.C.,** total number of numbers = 4 X 4 = 16.

Verification. The required numbers are 22, 23, 25, 27, 32, 33, 35, 37, 52, 53, 55, 57, 72, 73, 75, 77. These are 16 in number.

EXERCISE 5. 7

SHORT ANSWER TYPE QUESTIONS

1. Find the number of two-digit numbers by using the digits 4, 6, 9. The repetition of digits is allowed.
2. Find the number of two – digit numbers by using the digits 0, 3, 7. The repetition of digits is allowed.

LONG ANSWER TYPE QUESTIONS

3. How many three – digit numbers can be formed by using the digits :

   (i) 1, 4, 7, 9 (ii) 2, 3, 6, 8 (iii) 0, 2, 3, 6, 8, 7

   The repetition of digits is allowed.
4. How many four – digit numbers can be formed by using the digits 2, 3, 4, 5, 6, when repetition of digits is allowed?
5. How many four – digit numbers can be formed by using the digits 0, 3, 4, 5, 6, when repetition of digits is allowed?

Answers

1. 9 2. 6 3. (i) 64 (ii) 64

(iii) 100 4. 625 5. 500

SUMMARY

1. The **fundamental principle of counting** (F.P.C.) states that if an operation can be performed in m different ways and if for each such choice, another operation can be performed in n different ways, then both operations, in succession can be performed in exactly mn different ways. The principle can also be generalized, for even than two operations.
2. For $n \in N$, the **factorial** of n is defined as $n\,! = 1 \times 2 \times 3 \times \ldots\ldots \times n$.

   $0\,!$ is defined as 1.
3. The arrangements of a number of things taking some or all of them at a time are called **permutations.** The total number of permutations of n distinct things taking r $(1 \leq r \leq n)$ at a time is denoted by ${}^nP_r$ or $P(n, r)$.
4. For $1 \leq r \leq n$, ${}^nP_r = n(n-1)(n-2)\ldots\ldots r$ factors.

   In particular, ${}^nP_r = n(n-1)(n-2)\ldots\ldots n$ factors $= n(n-1)(n-2)\ldots 3.2.1. = n\,!$
5. We define, ${}^nP_0 = 1$.
6. For $0 \leq r < n$, ${}^nP_r = \dfrac{n\,!}{(n-r)\,!}$.

   In particular, ${}^nP_n = \dfrac{n\,!}{(n-n)\,!} = \dfrac{n\,!}{0\,!} = n\,!$

TEST YOURSELF

1. A sample of 3 bulbs is tested. A bulb is labeled as 'G' if it is good and 'D' if it is defective. Find the number of all the possible outcomes.
2. John wants to go abroad by ship and return by air. He has a choice of 6 different ships to go and 4 airlines to return. In how many ways can ho perform his journey?
3. There are 5 routes from places A to place B and 3 routes from place B to place C. Find how many different routes are there from A to C via B.
4. For a group photograph, 3 boys and 2 girls stand in a line in all possible ways. How photos could be taken if each photo corresponds to each such arrangement?

5. A coin is tossed three times and the outcomes are recorded. How many possible outcomes are there? How many possible outcomes if the coin is tossed four times? Five times? n times?

6. How many 3-digit numbers can be formed form the digits 1, 2, 3, 4 and 5 assuming:

   (i) repetition of digits allowed. (ii) repetition of digits not allowed ?

7. How many 5-digit telephone numbers can be constructed using the digits 0 to 9 if each number starts with 67, for example 67125 etc, and no digit appears more than once?

Answers

1. 8	2. 24	3. 15	4. 120
5. 8, 16, 32, 2^n	6. (i) 125	(ii) 60	7. 8 X 7 X 6 = 336

SECTION – A

6. COMBINATIONS

LEARNING OBJECTIVES

- Introduction
- Definition of Combinations
- Practical Problems Involving Combinations
- Division into Groups
- Combinations when all Things are not Alike

INTRODUCTIONS

In this chapter, we shall discuss the problem of arranging certain things taking particular number of things at a time. The selections are different from permutations in the sense that in a permutation, the order of things is taken into consideration whereas in case of selections, the order of things is immaterial and we consider only the things which are occurring in a selection. For example, *ab* and *ba* are two distinct permutations but same selection.

DEFINITION OF COMBINATIONS

A selection (group) of a number of things taking some or all of them at a time is called a **combination.** The total number of combinations of n distinct things taking $r(1 \leq r \leq n)$ at a time is denoted by ${}^nC_r$ or by $C(n, r)$. We define ${}^nC_0 = 1$.

For example, the combinations of 3 things a, b, c taking 2 at a time are:

ab   *bc*   *ca*

$\therefore \quad {}^3C_2 = 3$

Remark. In a combination, the order of objects is immaterial whereas in a permutation, the order of objects matters. For example, *abc, acb, bca* represent the same combination and three different permutations. Thus, we see that permutations are 'arrangements in definite order, whereas combinations are 'selections' in which order of objects does not matter.

Theorem I. For $0 \le r \le n$, prove that $^nC_r = \dfrac{n\,!}{r\,!(n-r)!}$.

Proof. Let $^nC_r = x$ and $r > 0$.

Each one of these x combinations contains r things and these r things can be arranged among themselves in $r\,!$ ways. Hence one combination give rise to $r\,!$ permutations.

$\therefore$ x combinations will give rise to $x\,.\,r\,!$ permutations.

But the number of permutations of n things taking r at a time is $\dfrac{n\,!}{(n-r)!}$.

$$\therefore \quad x\,.\,r\,! = \frac{n\,!}{(n-r)!} \qquad i.e., \quad x = \frac{n\,!}{r\,!(n-r)!}\,.$$

$$\therefore \quad {}^nC_r = \frac{n\,!}{r\,!(n-r)!}\,,\ 0 < r \le n$$

Also, $\quad {}^nC_0 = 1 \quad$ and $\quad \dfrac{n\,!}{0\,!(n-0)!} = \dfrac{n\,!}{1 \times n\,!} = 1$

$$\therefore \quad {}^nC_r = \frac{n\,!}{r\,!(n-r)!}\,,\ 0 \le r \le n.$$

Corollary I. $\quad {}^nC_r = \dfrac{n\,!}{r\,!(n-r)!} = \dfrac{n(n-1)......(n-r+1)(n-r)........3.2.1}{r\,!(n-r)!}$

$$= \frac{n(n-1)......(n-r+1)}{r\,!}$$

$$\therefore \quad {}^nC_r = \frac{n(n-1)(n-2).........(n-r+1)}{1,2,3,.....r}\,,\ \text{for } 1 \le r \le n.$$

This form of ^nC_r is generally used in practical problems, because it does not involve factorials.

From the above form of ^nC_r , we have

$$^{n}C_r = \frac{n(n-1)(n-2)\ldots\ldots r \text{ factors}}{1,2,3,\ldots.r}$$

For example, $^{8}C_4 = \frac{8\times7\times6\times5}{1\times2\times3\times4} = 70$.

Corollary II. We have $^{n}C_0 = 1$ and $^{n}C_n = \frac{n\,!}{n\,!(n-r)!} = \frac{1}{0\,!} = 1.$

$\therefore$ $\mathbf{^{n}C_0 = 1}$ and $\mathbf{^{n}C_n = 1.}$

Theorem II. For 0 ≤ r ≤ n, prove that $^{n}C_n = {^{n}C_r}$.

Proof. $^{n}C_{n-r} = \frac{n\,!}{(n-r)\,!(n-(n-r)!} = \frac{n\,!}{(n-r)!\,r\,!} = {^{n}C_r}$

Remark 1. If $r > \frac{n}{2}$, then we simplify the calculation of $^{n}C_r$ = by writing it equal to $^{n}C_{n-r}$.

For example, $^{15}C_{11} = {^{15}C_{15-11}} = {^{15}C_4} = \frac{15\times14\times13\times12}{1\times2\times3\times4} = 1365$.

Remark 2. We have seen that $^{n}C_r = {^{n}C_{n-r}}$ for $1 \le r \le n$.

$\therefore$ If $^{n}C_p = {^{n}C_q}$, then $^{n}C_p = {^{n}C_q} = {^{n}C_{n-q}}$ $(\because\ {^{n}C_q} = {^{n}C_{n-q}})$

$\therefore$ $p = q$ or $p = n - q$ *i.e.*, $n = p + q$.

Thus, if $^{n}C_p = {^{n}C_q}$, then either p = q or p + q = n.

Theorem III. (Pascal's rule). If n and r are natural numbers such that 1 ≤ r ≤ n, then prove that

$$^{n}C_r + {^{n}C_{r-1}} = {^{n+1}C_r}\ .$$

Proof. L.H.S. = $^{n}C_r + {^{n}C_{r-1}}$

$$= \frac{n\,!}{r\,!\,(n-r)!} + \frac{n\,!}{(r-1)!(n-r+1)!} = \frac{n\,!}{r\,.(r-1)\,!(n-r)!} + \frac{n\,!}{(r-1)!(n-r+1)(n-r)!}$$

$$= \frac{n\,!}{(r-1)\,!(n-r)\,!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right] = \frac{n\,!}{(r-1)\,!(n-r)!} + \left[\frac{n-r+1+r}{r(n-r+1)}\right]$$

$$= \frac{(n+1)\,.\,n\,!}{r\,.(r-1)!\,(n-r+1)\,.\,(n-r)!} = \frac{(n+1)\,!}{r\,!\,(n-r+1)!} = {^{n+1}C_r} = \text{R.H.S.}$$

$\therefore \qquad {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r, \qquad 1 \le r \le n.$

Example 1. *Evaluate the following :*

(i) ${}^9C_4$ (ii) ${}^{51}C_{49}$ (iii) ${}^{100}C_{96}$.

Sol. (i) ${}^9C_4 = \dfrac{9!}{4!(9-4)!} = \dfrac{9!}{4!5!} = \dfrac{9 \times 8 \times 7 \times 6 \times 5!}{4 \times 3 \times 2 \times 1 \times 5!} = \dfrac{9\times 8\times 7\times 6}{4\times 3\times 2\times 1} = 126$

Alternative method. ${}^9C_4 = \dfrac{9\times 8\times 7\times 6}{4\times 3\times 2\times 1} = 126$

$$\left(\because \quad {}^nC_r = \frac{n(n-1)\ldots\ldots r \text{ factors}}{1 \times 2 \times \ldots \times r}\right).$$

(ii) ${}^{51}C_{49} = {}^{51}C_{51-49} = {}^{51}C_2 = \dfrac{51\times 50}{1\times 2} = 1275$ $\qquad (\because \quad {}^nC_r = {}^nC_{n-r})$

(iii) ${}^{100}C_{96} = {}^{100}C_{100-96} = {}^{100}C_4 = \dfrac{100\times 99\times 98\times 97}{1\times 2\times 3\times 4} = 3921225.$

Example 2. *Show that the product of k consecutive natural numbers is divisible by k !.*

Sol. Let us consider k consecutive natural numbers $n+1, n+2, \ldots\ldots, n+k$.

Now, $\dfrac{(n+1)(n+2)\ldots\ldots(n+k)}{k!} = \dfrac{1.2.3\ldots\ldots n.(n+1)(n+2)\ldots\ldots(n+k)}{1.2.3\ldots\ldots n.k!}$

$$= \frac{(n+k)!}{n!\,k!} = {}^{n+k}C_k \text{, a natural number.}$$

$\therefore \quad (n+1)(n+2)\ldots\ldots(n+k)$ is divisible by $k!$.

WORKING RULES FOR SOLVING PROBLEMS

Rule I. *${}^nC_r$ (or $C(n, r)$) denotes the number of combinations of n distinct things r at a time, $1 \le r \le n$.*

Rule II. *If value of r is given, then use :* ${}^nC_r = \dfrac{n(n-1)(n-2)\ldots r \text{ factors}}{1.2.3\ldots\ldots r}$.

Rule III. *If value of r is not given, then use :* ${}^nC_r = \dfrac{n!}{r!(n-r)!}$.

Rule IV. (i) ${}^nC_r = {}^nC_{n-r}$

(ii) *If* ${}^nC_p = {}^nC_q$ *then either* $p = q$ *or* $p + 1 = n$.

Rule V. ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$, $1 \le r \le n$.

Rule VI. ${}^nC_0 = 1$ *and* ${}^nC_n = 1$.

EXERCISE 6. 1

SHORT ANSWER TYPE QUESTIONS

1. Evaluate :

   ${}^{11}C_2$ (ii) ${}^{20}C_3$ (iii) ${}^{20}C_{18}$ (iv) ${}^{21}C_{20}$.

2. If $n = 7$ and $r = 3$, then verify that ${}^nC_r = {}^nC_{n-r}$.
3. Show that:

   (i) $2 \,.\, C(7, 4) = C(8, 4)$ (ii) $2 \,.\, C(8, 4) \neq C(9, 4)$.

4. If ${}^nC_9 = {}^nC_8$, find ${}^nC_{17}$.
5. Using ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$, evaluate the following :

   (i) ${}^{10}C_4 + {}^{10}C_5$ (ii) ${}^{61}C_{57} - {}^{60}C_{56}$.

Answers

1. (i) 55 (ii) 1140 (iii) 190 (iv) 21

4. 1 5. (i) 462 (ii) 34220.

PRACTICAL PROBLEMS INVOLVING COMB INATIONS

In this section, we shall learn the use of formulae regarding combinations in solving practical problems.

Example 3. *16 players of cricket go to England for playing test matches. In how many ways can the team of 11 be selected ?*

Sol. Number of ways of selecting 11 players out of 16 players

$= {}^{16}C_{11} = {}^{16}C_5$ $(\because {}^nC_r = {}^nC_{n-r})$

$$= \frac{16\times15\times14\times13\times12}{5\times4\times3\times2\times1} = 4368.$$

Example 4. *Find the number of diagonals that can be drawn by joining the angular points of octagon.*

Sol. An octagon has 8 angular points. By joining any two angular points, we get a line, which is either a side of a diagonal.

$\therefore$ Number of lines $= {}^8C_2 = \frac{8\times7}{1\times2} = 28$

Number of sides = 8

$\therefore$ Number of diagonals = 28 – 8 = 20.

EXERCISE 6. 2

SHORT ANSWER TYPE QUESTIONS

1. How many chords can be drawn through 21 points on a circle?

2. Find the number of ways in which a team of 8 players can be formed out of 15 players.

3. From a class of 32 students, 4 are to be chosen for a competition. In how many ways can this be done?

4. If there are 10 persons in a party and each two of them shakes hands with each other, how many hand shakes happen in the party?

   (**Explanation.** When two persons shake ahnds, it is counted as one hand shake, not two. Therefore this is a problem on combinations, not permutations).

LONG ANSWER TYPE QUESTIONS

5. In how many ways can a students, choose 5 subjects out of 9 subjects, if 2 subjects are compulsory for every students?
6. In an examination, a student has to answer 4 questions out of 5 questions, Questions nos. 1 and 2 are however compulsory. Determine the number of ways in which the student can make the choice.

7. In how many ways can 5 members forming a committee out of 10 be selected, so that :

   (i) two particular members must be included ?

   (ii) two particular members must not be included ?

Answers

1. ${}^{21}C_2 = 210$	2. 6435	3. 35960	4. 45
5. 35	6. 3	7. (i) 56	(ii) 56.

DIVISION INTO GROUPS

To find the number of ways in which (m + n) things can be divided into two groups containing m and n things respectively.

The number of ways in which m things can be selected out of $(m + n)$ things is ${}^{m+n}C_m$. Also, whenever a group of m things is selected, a group of n things is automatically left out. Hence the number of ways in which $(m + n)$ things are divided into two groups containing m and n things respectively

$$= {}^{m+n}C_m = \frac{(m+n)!}{m\,!\,n\,!}.$$

Corollary 1. If we have to divide $2m$ things into two groups containing m things each, then by putting $n = m$, we have :

$$\text{Required number of ways} = \frac{(2m)!}{(m\,!)^2}.$$

Corollary 2. If no distinction is made between the groups, the groups can be interchanged in 2 ! ways without performing a new division.

$$\therefore \quad \text{Required number of ways} = \frac{(2m)!}{2\,!\,(m\,!)^2}.$$

Note. Similarly, the number of ways in which $(m + n + p)$ things can be divided into three groups containing m, n and p things respectively is $\frac{(m+n+p)}{m\,!\,n\,!\,p\,!}$

If $m = n = p$, then the number of groups $= \frac{(3m)!}{(m\,!)^3}$.

However, if no distinction is made between the groups, then the number of ways of division = $\frac{(3m)!}{3\,!\,(m\,!)^3}$, because each group is repeated 3 ! times.

COMBINATIONS WHEN ALL THINGS ARE NOT ALIKE

To find the total number of ways in which a selection can be made out of (p + q + r) things of which p are alike of one kind, q alike of another kind, r alike of third kind.

There are (p + 1) ways of making a selection out of p like things according as we make selection of 1 or 2 or 3, ..., or p or none of them. Hence p like things can be dealt with in (p + 1) ways.

Similarly, q like things can be dealt with in (q + 1) ways and r like things can be dealt with in (r + 1) ways.

Hence, the number of ways of dealing with all the things = (p + 1)(q + 1)(r + 1).

But this includes the case in which all are excluded. Rejecting this case, the required number of ways.

$$= \mathbf{(p + 1)(q + 1)(r + 1) - 1}$$

Example 5. *In how many ways can a selection be made out of 2 mangoes, 3 apples and 3 oranges ?*

Sol. 2 mangoes can be disposed of in 3 ways, for we can choose 1, 2 or none of these mangoes. Similarly, 3 apples and 3 oranges can be disposed of in 4 ways each. Therefore the number of ways disposing of these fruits = 3 X 4 X 4 = 48. But this also includes the case in which no fruit is selected. We reject this case.

∴ Required number of ways = 48 – 1 = 47.

WORKING RULES FOR SOLVING PROBLEMS

Rule I. (i) *Number of ways in which (m + n) dissimilar things can be divided into two groups containing m and n things respectively* = $\frac{(m+n)\,!}{m\,!\,n\,!}$.

(ii) *Number of ways in which (m + n + p) dissimilar things can be divided into three groups containing m, n and p things respectively*

Rule II. (i) *Number of ways in which 2m dissimilar things can be divided into two groups, without distinction, each containing m things* = $\frac{(2m)\,!}{2\,!\,(m\,!)^2}$.

(ii) *Number of ways in which 3m dissimilar things can be divided into three groups, without distinction, each containing m things* = $\frac{(3m)\,!}{3\,!\,(m\,!)^3}$.

Rule III. *Number of selections of some or all things out of (p + q +r) things of which p are alike of one kind, q alike of another kind, r alike of third kind.*

$$= (p + 1)(q + 1)(r + 1) - 1.$$

EXERCISE 6. 3

SHORT ANSWER TYPE QUESTIONS

1. In how many ways can 11 distinct things be divided into two groups containing respectively 5 and 6 things?
2. In how many ways can 11 things be divided into groups of 6, 3 and 2 ?
3. In how many ways can 18 books be divided equally among 3 students ?

Long ANSWER TYPE QUESTIONS

4. In how many ways can 15 different books be divided equally (i) among 5 boys (ii) into 5 heaps?
5. In how many ways can 52 playing cards be placed in 4 heaps of 13 cards each? In how many ways can these be dealt out to four players giving 13 cards each ?
6. In how many ways can 20 students be divided into four equal groups? In how many ways can these be sent to four different schools?

Answers

1. 462 2. 4620 3. $\frac{18\,!}{(6\,!)^3}$ 4. (i) $\frac{15\,!}{(3\,!)^5}$

(ii) $\frac{15\,!}{5\,!\,(3\,!)^5}$ 5. $\frac{52\,!}{4\,!\,(13\,!)^4}$, $\frac{52\,!}{(13\,!)^4}$ 6. $\frac{20\,!}{4\,!\,(5\,!)^4}$, $\frac{20\,!}{(5\,!)^4}$

SUMMARY

1. The selections (groups) of a number of things taking some or all of them at a time are called **combinations.** The total number of combinations of n distinct things taking r $(1 \leq r \leq n)$ at a time is denoted by ${}^{n}C_{r}$ or by $C(n, r)$.
2. By definition, ${}^{n}C_{0} = 1$.
3. For $0 \leq r \leq n$, ${}^{n}C_{r} = \dfrac{n!}{r!(n-r)!}$
4. For $1 \leq r \leq n$, ${}^{n}C_{r} = \dfrac{n(n-1)\ldots\ldots.(n-r+1)}{1.2\ldots\ldots\ldots.r}$
5. In particular, ${}^{n}C_{0} = {}^{n}C_{n} = 1$.
6. If $1 \leq r \leq n$, then ${}^{n}C_{r} = {}^{n}C_{n-r}$
7. If $1 \leq r \leq n$, then ${}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$

TEST YOURSELF

1. If $1 \leq r \leq n$, show that n X $C(n-1, r-1) = (n-r+1)$ X $C(n, r-1)$.
2. Show that ${}^{4n}C_{2n} : {}^{2n}C_{n} = [1.\ 3.\ 5.\ \ldots..(4n-1)] : [1\ .\ 3.\ 5.\ \ldots\ldots.\ (2n-1]^{2}$.
3. From a class of 12 boys and 10 girls, 10 students are to be chosen for a competition, including at least 4 boys and 4 girls. The 2 girls who won the prizes last year should be included. In how many ways can the selections be made?
4. A students has three library tickets and 8 books of his internet in the library. Of these 8, he does not want to borrow Chemistry part II, unless Chemistry part I is also borrowed. In how many ways can he choose the three books to be borrowed?
5. In a small village, there are 87 families, of which 52 families have at most 2 children. In a Rural Development Programme, 20 families are to be chosen for assistance, of which at least 18 families must have at most 2 children. In how many ways can the choice be made?

Answers

3. 104874 4. 41

5. ${}^{52}C_{18}$ X ${}^{35}C_{2}$ X ${}^{52}C_{19}$ X ${}^{35}C_{1}$ + ${}^{52}C_{20}$ X ${}^{35}C_{0}$.

SECTION – A

7. BINOMIAL THEOREM (FOR POSITIVE INTEGRAL INDEX)

LEARNING OBJECTIVES

- Introduction
- Binomial Theorem of Positive Integral Index
- Method of Writing Expansion for $(a + b)^n$
- Some Observations
- Pascal Triangle
- Some Particular Expansions
- General Term
- Middle Terms
- Particular Terms
- Some Applications of Binomial Theorem

INTRODUCTION

A **binomial** is an algebraic expression of two terms which are connected by the operations '+' or '-'. For example, $x - y$, $a + 3b$, $x^3 + 4y$ etc. are binomials. We know that:

$$(a + b)^1 = a + b$$

$$= {}^1C_0 a^1 b^0 + {}^1C_1 a^0 b^1$$

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$= {}^2C_0 a^2 b^0 + {}^2C_1 a^1 b^1 + {}^2C_2 a^0 b^2$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$= {}^3C_0a^3b^0 + {}^3C_1a^2b^1 + {}^3C_2a^1b^2 + {}^3C_3a^0b^3$$

$$(a + b)^4 = (a + b)(a + b)^3 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$= {}^4C_0a^4b^0 + {}^4C_1a^3b^1 + {}^4C_2a^2b^2 + {}^4C_3a^1b^3 + {}^4C_4a^0b^4.$$

For $n = 1, 2, 3, 4$, the expansion of $(a + b)^n$ has been expressed in a very *systematical manner* in terms of combinatorial coefficients. The above expressions suggest the conjecture that $(a + b)^n$ should be expressible in the form

$${}^nC_0a^nb^0 + {}^nC_1a^{n-1}b^1 + \dots\dots\dots + {}^nC_{n-1}a^1b^{n-1} + {}^nC_na^0b^n$$

for every natural number **n**.

In fact, this conjecture is valid and we can establish it by using *principle of mathematical induction.*

BINOMIAL THEOREM FOR POSITIVE INTEGRAL INDEX

For any natural number n,

$$(a + b)^n = {}^nC_0a^nb^0 + {}^nC_1a^{n-1}b^1 + \dots\dots\dots + {}^nC_{n-1}a^1b^{n-1} + {}^nC_na^0b^n.$$

Remark 1. If $n = 0$, then $(a + b)^n = (a + b)^0 = 1$

and $\quad {}^nC_0\, a^nb^0 + \dots\dots + {}^nC_na^0b^n = {}^0C_0a^0b^0 = {}^0C_0 \,.1.\, 1$

$$= {}^0C_0 = \frac{0\,!}{0\,!(0-0)\,!} = \frac{1}{1\times 1} = 1$$

$\therefore$ For $n = 0$, we have $(a + b)^0 = {}^0C_0$ and its both parts are equal to q.

$\therefore$ The binomial theorem is also true for $n = 0$.

2. In the summation notation, the binomial theorem can be written as:

$$(a + b)^n = \sum_{k=0}^{n} {}^nC_k\, a^{n-k}\, b^k, \quad \textbf{for} \quad \mathbf{n = 0, 1, 2, \dots\dots .}$$

METHOD OF WRITING EXPANSION FOR $(a + b)^n$

The first term in the expansion of $(a + b)^n$ is ${}^nC_0a^nb^0$. For the second term, the coefficient is taken as ${}^nC_1$, the power of a is decreased by one and the power of b is increased by one. So, the second term is ${}^nC_1a^{n-1}b^1$. For the third term, the coefficient is taken as ${}^nC_2$, the power of a is again decreased by one and the power of b is increased by one. This process goes on, till we get the last term as ${}^nC_na^0b^n$.

SOME OBSERVATIONS

For $n \in N$, in the expansion of $(a + b)^n$, we observe that:

i. The number of terms is $n + 1$.
ii. The exponent of a decreases from n to 0.
iii. The exponent of b increases from 0 to n.
iv. The sum of exponents of a and b in any terms in n.
v. The coefficient of any term is ${}^nC_k$, where k in the exponent of b.
vi. ${}^nC_0, {}^nC_1, {}^nC_2, \ldots\ldots, {}^nC_n$ are called the **binomial coefficients.**

Since ${}^nC_r = {}^nC_{n-1}$, we have ${}^nC_0 = {}^nC_n$, ${}^nC_1 = {}^nC_{n-1}$, ${}^nC_2 = {}^nC_{n-2}, \ldots\ldots.$

$\therefore$ The binomial coefficients in $(a + b)^n$, which are equidistant from beginning and end are equal.

PASCAL TRIANGLE

A French mathematician *Blaise Pascal* (1623 – 1662 A.D) used an arithmetic triangle to derive the coefficients of a binomial expansion. This triangle is called **Pascal triangle** and is given below:

Pascal Triangle

Index, n	Binomial coefficients
$n = 0$	1
$n = 1$	1 1
$n = 2$	1 2 1
$n = 3$	1 3 3 1
$n = 4$	1 4 6 4 1
$n = 5$	1 5 10 10 5 1
$n = 6$	1 6 15 20 15 6 1
$n = 7$	1 7 21 35 35 21 7 1
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-----------	----------------------------------

In the Pascal triangle each row start and end with 1 and each coefficient in a row is equal to the sum of the two coefficients one just before it and oter just after it in the preceding row.

SOME PARTICULAR EXPANSIONS

For $n \in N$, we have :

(i) $(a - b)^n = (a + (-b))^n$

$= {}^nC_0 a^n(-b)^0 + {}^nC_1 a^{n-1}(-b)^1 +, {}^nC_2 a^{n-2}(-b)^2 + \ldots\ldots\ldots + {}^nC_n a^0(-b)^n$

$\mathbf{= {}^nC_0 a^n b^0 - {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \ldots\ldots\ldots + (-1)^n\, {}^nC_n a^0 b^n}$.

(ii) $(1 + x)^n = {}^nC_0 1^n x^0 + {}^nC_1 1^{n-1} x^1 + {}^nC_2 1^{n-2} x^2 + \ldots\ldots\ldots + {}^nC_n 1^0 x^n$

$$= \mathbf{1 + nx + \frac{n(n-1)}{2\,!} x^2 + \ldots\ldots\ldots\ldots + x^n.}$$

(iii) $(1 - x)^n = (1 + (-x))^n = {}^nC_0 1^n (-x)^0 + {}^nC_1 1^{n-1}(-x)^1 + {}^nC_2 1^{n-2}(-x)^2 + \ldots . + {}^nC_n 1^0(-x)^n$

$$\mathbf{= 1 - nx + \frac{n(n-1)}{2\,!} x^2 + \ldots\ldots + (-1)^n x^n.}$$

Remark 1. It is available to remember the following values.

For $n \in N$, ${}^nC_0 = 1$, ${}^nC_1 = \frac{n}{1} = n$, ${}^nC_2 = \frac{n(n-1)}{12}$, ${}^nC_3 = \frac{n(n-1)(n-2)}{123}$,....., ${}^nC_{n-3} = {}^nC_3$, ${}^nC_{n-2} = {}^nC_2$, ${}^nC_{n-1} = {}^nC_1$, ${}^nC_n = {}^nC_0$.

Remark 2. In ${}^nC_r$, if $r > \frac{n}{2}$, then it is useful to find the value of ${}^nC_r$, by writing ${}^nC_r$ as ${}^nC_{n-r}$

For example, in ${}^{18}C_{12}$, we have $12 > \frac{15}{2}$.

$\therefore$ We write ${}^{15}C_{12} = {}^{15}C_{15-12} = {}^{15}C_3 = \frac{15.14.13}{1.2.3} = 455$.

WORKING RULES FOR EXPANDING (a + b)ⁿ, $n \in N$

Step I. *The value of index, n implies that there will be n + 1 terms in the expansion of $(a + b)^n$.*

Step II. *Write the first term : ${}^nC_0\, a^n b^0$.*

Step III. *For the second term, take coefficient as ${}^nC_1$, decrease the power of a by 1 and increase the power of b by 1. Thus, the second term in ${}^nC_1\, a^{n-1} b^1$.*

Step IV. *For the third term, take coefficient as ${}^nC_2$, power of a as n – 2 and power of b as 2. Continue this process repeatedly till the last term ${}^nC_n\, a^0 b^n$ is obtained*

.

Step V. *For evaluation ${}^nC_r$, it is useful to write ${}^nC_r$ as ${}^nC_{n-r}$, if $r > \frac{n}{2}$.*

Example 1. *Expand the following by using binomial theorem:*

(i) $\left(-3x - \frac{1}{3x}\right)^2$ (ii) $\left(x^2 + \frac{2}{x}\right)^4, x \neq 0$.

Sol. (i) $\left(-3x - \frac{1}{3x}\right)^2 = \left[(-3x) + \left(-\frac{1}{3x}\right)\right]^3$

$$= {}^3C_0(-3x)^3\left(-\frac{1}{3x}\right)^0 + {}^3C_1(-3x)^2\left(-\frac{1}{3x}\right)^1 + {}^3C_2(-3x)^1\left(-\frac{1}{3x}\right)^2 + {}^3C_3(-3x)^0\left(-\frac{1}{3x}\right)^3$$

$$= 1(-27x^3)(1) + 3(9x^2)\left(-\frac{1}{3x}\right) + 3(-3x)\frac{1}{9x^2} + 1(1)\left(-\frac{1}{27x^3}\right)$$

$$= 27x^3 - 9x - \frac{1}{x} - \frac{1}{27x^3}$$

(ii) $\left(x^2 + \frac{2}{x}\right)^4, {}^4C_0(x^2)^4\left(\frac{2}{x}\right)^0 + {}^4C_1(x^2)^3\left(\frac{2}{x}\right)^1 + {}^4C_2(x^2)^2\left(\frac{2}{x}\right)^2 + {}^4C_3(x^2)^1\left(\frac{2}{x}\right)^3 + {}^4C_4(x^2)^0\left(\frac{2}{x}\right)^4$

$$= 1 . x^8 . 1 + 4 . x^6 . \frac{2}{x} + 6 . x^4 \frac{4}{x^2} + 4 . x^2 . \frac{8}{x^3} + 1.1. \frac{16}{x^4}$$

$$= x^8 + 8x^5 + 24x^2 + \frac{32}{x} + \frac{16}{x^4}$$

Example 2. *Show that* $\sum_{r=0}^{n} 3^r\, {}^nC_r = 4^n$.

Sol. L.H.S. $= \sum_{r=0}^{n} 3^r\, {}^nC_r$

$$= 3^0\, {}^nC_0 + 3^1\, {}^nC_1 + 3^2\, {}^nC_2 + \ldots\ldots\ldots\ldots\ldots\ldots + 3^n\, {}^nC_n$$

$$= {}^nC_0 1^n 3^0 + {}^nC_1 1^{n-1} 3^1 + {}^nC_2 1^{n-2} + 3^2 \ldots\ldots\ldots\ldots + {}^nC_n 1^0 3^n$$

$$= (1 + 3)^n = 4^n = \text{R.H.S.}$$

EXERCISE 7.1

SHORT ANSWER TYPE QUESTIONS

How many terms are there in the binomial expansion of :

1. $(a + 3b)^4$
2. $\left(\frac{2}{p} + \frac{p}{2}\right)^8$
3. $\sqrt{(3x + 2y)^8}$
4. $\left[(x - 5y)^5\right]^3$
5. $\{[(x + y)^2]^3\}^6$?

LONG ANSWER TYPE QUESTIONS

Expand the following by using binomial theorem (Q. No. 6 – 12) :

6. $(2x-3x^2)^5$ 7. $(y^2+3x)^8$ 8. $\left(x-\frac{1}{y}\right)^{11}$

9. $(3x^2-2ax+3a^2)^3$ 10. $(1-x+x^2)^4$ 11. $(1+2x-3x^2)^5$.

12. Find the coefficient of a^4 in the expansion of the product $(1+2a)^4(2-a)^5$.

Answers

1. 5 2. 9 3. 5 4. 16

5. 37 6. $32x^5-240x^6+720x^7-1080x^8+810x^9-243x^{10}$

7. $y^{16}+24y^{14}x+252y^{12}x^2+1512y^{10}x^3+5670y^8x^4+13608y^6x^5+20412y^4x^6+17496y^2x^7+6561x^8$

8. $x^{11}-\frac{11x^{10}}{y}+\frac{55x^9}{y^2}-\frac{165x^8}{y^3}+\frac{330x^7}{y^4}-\frac{462x^6}{y^5}+\frac{462x^5}{y^6}-\frac{330x^4}{y^7}+\frac{165x^3}{y^8}-\frac{55x^2}{y^9}+\frac{11x^2}{y^{10}}-\frac{1}{y^{11}}$

9. $27x^6-54ax^5+117a^2x^4-116a^3x^3+117a^4x^2-54a^5x+27a^6$

10. $1-4x+10x^2-16x^3+19x^4-16x^5+10x^6-4x^7+x^8$

11. $1+10x+25x^2-40x^3-190x^4+92x^5+570x^6-360x^7-675x^8+810x^9-243x^{10}$

12. – 438 .

GENEARL TERM

For $n \in N$, we have $(a+b)^n = {}^nC_0a^2b^0 + {}^nC_1a^{n-1}b^1 + {}^nC_2a^{n-2} + \ldots\ldots. + {}^nC_na^0b^n$.

Let T_{r+1} $(0 \le r \le n)$ be the $(r+1)$th term in the expansion.

$T_{0+1} = T_1 = {}^nC_0 a^n b^0$

$T_{1+1} = T_2 = {}^nC_1 a^{n-1} b^1$

$T_{2+1} = T_3 = {}^nC_2 a^{n-2} b^2$

....

....

$T_{n+1} = {}^nC_n a^{n-n} b^n$.

$\therefore$ **For $0 \le r \le n$, we have $T_{r+1} = {}^nC_r a^{n-r} b^r$.**

Example 3. *Find the general terms in the expansions of :* $\left(2x + \frac{1}{x}\right)^5$.

Sol. We have T_{r+1} in $(a + b)^n = {}^nC_r\, a^{n-r} b^r$, $0 \le r \le n$.

$$\therefore \quad T_{r+1} \text{ in } \left(2x + \frac{1}{x}\right)^5 = {}^5C_r\,(2x)^{5-r}\left(\frac{1}{x}\right)^r = {}^5C_r\,(2)^{5-r}\,x^{5-r}\,\frac{1}{x^r}.$$

$$\therefore \quad T_{r+1} = {}^5C_r\,(2)^{5-r}\,x^{5-2r},\ 0 \le r \le 5.$$

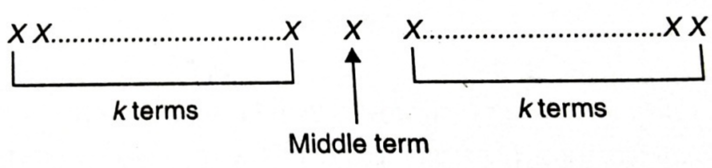
MIDDLE TERMS

The number of terms in the expansion of $(a + b)^n$ depend upon the index n. The index n is either even or odd.

Case I. n is even. Let $n = 2k$.

$\therefore$ The number of terms is $n + 1$ *i.e.,* $2k + 1$.

The middle term has k terms before it.

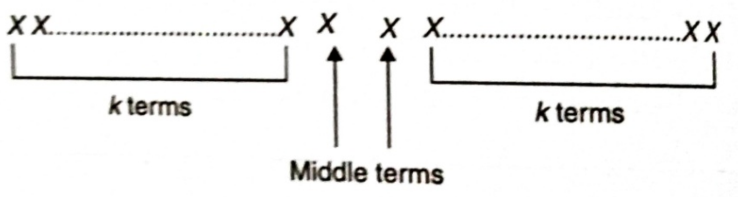


∴ The middle term $= T_{k+1} = T_{\frac{n}{2}+1} = T_{\frac{n+2}{2}}$.

Case II. n is odd. Let $n = 2k + 1$.

∴ The number terms is $n + 1$ *i.e.,* $(2k + 1) + 1 = 2k + 2$.

In this case, there are two middle terms and are after k terms.



∴ The middle terms are T_{k+1} and T_{k+2}.

$$T_{k+1} = \frac{T_{n+1}}{2} + 1 = \frac{T_{n+1}}{2} \quad \text{and} \quad T_{k+2} = \frac{T_{n+2}}{2} + 2 = \frac{T_{n+3}}{2}$$

$$\left(\because \quad n = 2k + 1 \; implies \; k = \frac{n-1}{2}\right)$$

∴ The middle terms are $\frac{T_{n+1}}{2}$ and $\frac{T_{n+3}}{2}$.

Thus, in $(a + b)^n$:

(i) **If n is even, there is only one middle term given by** $\frac{T_{n+2}}{2}$.

(ii) **If n is odd, there are two middle terms given by** $\frac{T_{n+1}}{2}$ **and** $\frac{T_{n+3}}{2}$.

Remark. The middle terms may be easily found out by using the following method :

(i) When n is *even,* we add the even number 2 to n and divided by 2 to get the middle term *i.e.,* $\frac{T_{n+2}}{2}$ th term.

(ii) When n is *odd,* we odd numbers 1 and 3 to n and divide by 2 to get the middle terms *i.e.,* $\frac{T_{n+1}}{2}$ th and $\frac{T_{n+3}}{2}$ th terms.

Example 4. *Show that the coefficient of middle term in the expansion of* $(1 + x)^{2x}$ *is equal to the sum of the coefficients of the two middle terms in the expansion of* $(1 + x)^{2n-1}$.

Sol. The index $2n$ in $(1 + x)^{2n}$ is even.

$\therefore$ Middle term $= T_{\frac{2n+2}{2}} = T_{n+1} = {}^{nn}C_n(1^n)x^n = {}^{2n}C_n(1^n)x^n = 2^nC_nx^n$

$\therefore$ Coefficient of middle term in $(1 + x)^{2n} = {}^{2n}C_n$

The index $2n - 1$ in $(1 + x)^{2n-1}$ is odd.

$\therefore$ Middle terms are $T_{\frac{(2n-1)+1}{2}}$ and $T_{\frac{(2n-1)+1+3}{2}}$.

$$T_{\frac{(2n-1)+1}{2}} = T_n = T_{(n+1)+1} = {}^{2n-1}C_{n-1}1^nx^{n-1} = {}^{2n-1}C_{n-1}x^{n-1}$$

and $$T_{\frac{(2n-1)+3}{2}} = T_{n+1} = {}^{2n-1}C_n1^{n-1}x^n = {}^{2n-1}C_nx^n$$

$\therefore$ Sum of coefficient of middle terms in $(1 + x)^{2n-1}$

$= {}^{2n-1}C_{n-1} + {}^{2n-1}C_n = {}^{(2n-1)+1}C_n$ (Using ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$)

$= {}^{2n}C_n =$ coefficient of middle term in $(1 + x)^{2n}$.

WORKING RULES FOR FINDING GENERAL TERM AND MIDDLE TERMS (S)

Rule I. *In the expansion of $(a + b)^n$, the $(r + 1)$th term is equal to ${}^nC_r a^{n-r} b^r$. Here $0 \le r \le n$.*

Rule 2. *In the expansion of $(a + b)^n$, the number of middle terms depend only on the index 'n'.*

Rule 3. *If n is even, there is only one middle term and if n is odd, then there are two middle terms.*

Rule 4. *If n is even, then the middle term is $T_{\frac{n+2}{2}}$.*

Rule 5. *If n is odd, then the middle terms are $T_{\frac{n+2}{2}}$ and $T_{\frac{n+3}{2}}$*

EXERCISE 7.2

SHORT ANSWER TYPE QUESTIONS

1. Find the general term in the expansion of :

(i) $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^6$ (ii) $\left(2x^2 + \frac{1}{x^3}\right)^7$

(iii) $\left(2x^2 + \frac{1}{3x}\right)^{11}$ (iv) $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$.

2. Find the middle term in the expansion of the following:

(i) $\left(\frac{a}{2} + \frac{b}{3}\right)^8$ (ii) $\left(\frac{x}{3} + 9y\right)^{10}$ (iii) $\left(ax - \frac{b}{x}\right)^{12}$

(iv) $\left(x + \frac{1}{x}\right)^{2n}$ (v) $(1 - 2x + x^2)^n$ (vi) $(1 + 3x + 3x^2 + x^3)^{2n}$.

LONG ANSWER TYPE QUESTIONS

3. Prove that the middle term in the expansion of $\left(2x + \frac{3}{x}\right)^{20}$ is

$19 \text{ X } 17 \text{ X } 13 \text{ X } 11 \text{ X } 3^{10} \text{ X } 2^{12}$.

4. Show that the coefficient of the middle term in the expansion of $(1 + a)^8$ is equal to the sum of the coefficients of middle terms in the expansion of $(1 + a)^7$.

Answers

1. (i) ${}^6C_r (-1)^r 2^{r-6} 3^{6-2r} x^{12-3r}, 0 \le r \le 6$

(ii) ${}^7C_r 2^{7-r} x^{14-5r}, 0 \le r \le 7$

(iii) ${}^{11}C_r (-1)^r 2^{11-r} 3^{-r} x^{22-3r}, 0 \le r \le 11$

(iv) ${}^{18}C_r (-1)^r 3^{36-3r} x^{-r/2}, 0 \le r \le 18$

2. (i) $\frac{35}{648}a^4b^4$ (ii) 61236 x^5y^5 (iii) 924 a^6b^6 (iv) $\frac{(2n)!}{(n!)^2}$

(v) $\frac{(2n)!}{(n!)^2}(-1)^n x^n$ (vi) $\frac{(6n)!}{((3n)!)^2}x^{3n}$.

PARTICULAR TERMS

We know that in the binomial expansion of $(a + b)^n$, the value of T_{r+1} is given by ${}^nC_r\, a^{n-r}\, b^r$, where $r = 0, 1, 2, \ldots\ldots. n$.

Sometimes, a particular term satisfying certain conditions is required in the binomial expansion of the type $(a + b)^n$. This can be done by expanding $(a + b)^n$ and then locating the required term. Generally this becomes a tedious task, specially when the index n is large. In such cases, we begin by evaluation the general term T_{r+1} to be the required particular term.

To get the term independent of x, we put the power of x equal to zero and get the value of r for which the term is independent of x. Putting this value of r in T_{r+1}, we get the term independent of x.

Example 5. *In the expansion of* $\left(\frac{4}{7}x - y^2\right)^5$, *find the fourth term.*

Sol. 4^{th} term in $\left(\frac{4}{7}x - y^2\right)^5$, $T_4 = T_{3+1} = {}^5C_3\left(\frac{4}{7}x\right)^{5-3}$

$(-y^2)^3$ ($\because$ T_{r+1} in $(a + b)^n = {}^nC_r\, a^{n-r}b^r$)

$$= \frac{5\times5}{1\times2}.\frac{16}{49}x^2.(-y^6) = -\frac{160}{49}x^2y^6, \qquad \left(\because {}^5C_3 = {}^5C_2 = \frac{5\times4}{1\times2}\right)$$

Example 6. *If a_1, a_2, a_3, a_4, be the coefficients of four consecutive term in the expansion of $(1 + x)^n$, then prove that*

$$\frac{a_1}{a_1 + a_2} + \frac{a_3}{a_3 + a_4} = \frac{2a_2}{a_2 + a_3}.$$

Sol. T_{r+1} in $(1 + x)^n = {}^nC_r 1^{n-r}x^r = {}^nC_r x^r$

$\therefore$ Coefficient of $T_{r+1} = {}^nC_r$

Let a_1, a_2, a_3, a_4 be the coefficients of T_{r+1}, T_{r+2}, T_{r+3}, T_{r+4} respectively.

$\therefore$ $a_1 = {}^nC_r$, $a_2 = {}^nC_{r+1}$, $a_3 = {}^nC_{r+2}$, $a_4 = {}^nC_{r+3}$.

$$\therefore \quad \frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}=\frac{{}^nC_r}{{}^nC_r+{}^nC_{r+1}}+\frac{{}^nC_{r+2}}{{}^nC_{r+2}+{}^nC_{r+3}}=\frac{{}^nC_r}{{}^{n+1}C_{r+1}}+\frac{{}^nC_{r+2}}{{}^{n+1}C_{r+3}}$$

$$=\frac{n!}{r!(n-r)!}\cdot\frac{(r+1)!(n-r)!}{(n+r)!}+\frac{n!}{(r+2)(n-r-2)!}\cdot\frac{(r+3)!(n-r-2)!}{(n+1)!}$$

$$=\frac{r+1}{n+1}+\frac{r+3}{n+1}=\frac{2r+4}{n+1}=\frac{2(r+2)}{n+1}$$

Also, $= \dfrac{2a_2}{a_2+a_3}=\dfrac{2\,{}^nC_{r+1}}{{}^nC_{r+1}+{}^nC_{r+2}}=\dfrac{2\,{}^nC_{r+1}}{{}^{n+1}C_{r+2}}$

$$=\frac{2(n)!}{(r+1)!(n-r-1)!}\cdot\frac{(r+2)!(n-r-1)!}{(n+1)}=\frac{2(r+2)}{n+1}$$

$$\therefore \quad \frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}=\frac{2a_2}{a_2+a_3}.$$

WORKING RULES FOR SOLVING PROBLEMS

Role I. *In the expansion of $(a + b)^n$, the $(r + 1)$th term is equal to ${}^nC_r\, a^{n-r}b^r$. Here r can take values 0, 1, 2, ……, n.*

Role II. *For evaluation ${}^nC_r$, it is useful to write ${}^nC_r$ as ${}^nC_{n-r}$ if $r > \dfrac{n}{2}$.*

Role III. *rth term from the end in $(a + b)^n$ is equal to T_r in $(b + a)^n$.*

Role IV. *To find some particular term, assume T_{r+1} to be that term and find the value of r and hence T_{r+1} by using the given conditions.*

EXERCISE 7.3

SHORT ANSWER TYPE QUESTIONS

1. (i) Find the 3rd term in the expansion of $\left(3x-\dfrac{y^3}{6}\right)^4$.

(ii) Find the 13th term in the expansion of $\left(9x-\frac{1}{3\sqrt{x}}\right)^{18}$.

2. (i) Find the 4th term from the end in the expansion of $\left(\frac{4x}{5}-\frac{5}{2x}\right)^{9}$.

   (ii) Find the $(n + 1)$th term from the end in the expansion of $\left(x-\frac{1}{x}\right)^{3n}$

3. Find the coefficient of:

   (i) x^2 in the expansion of $\left(3x-\frac{1}{x}\right)^{6}$. (ii) x^7 in the expansion of $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^{9}$

LONG ANSWER TYPE QUESTIONS

4. If the coefficient of $(m + 1)$th term in the expansion of $(1 + x)^{2n}$ be equal to that of $(m + 3)$th term, them show that $m = n - 1$.
5. Find the value of r if the coefficients of $(2r + 4)$th and $(r - 2)$th terms in the expansion of $(1 + x)^{18}$ are equal.

Answers

1. (i) $\frac{3}{2}x^2y^2$ (ii) 18564

2. (i) $\frac{10500}{x^3}$ (ii) $\frac{(3n)!}{n!(2n)!}\cdot\frac{1}{x^n}$

3. (i) 1215 (ii) There is no term containing x^7.

SOME APPLICATIONS OF BINOMIAL THEOREM

In this section, we shall learn applying binomial theorem in solving practical problems like computation of powers of numbers etc. We illustrate the procedure by taking some practical problems.

Example 7. *Use the Binomial theorem to evaluate (1001)³.*

Sol. $(1001)^3 = (1000 + 1)^3$

$$= {}^3C_0(1000)^3 \cdot 1^0 + {}^3C_1(1000)^2 \cdot 1^1 + {}^3C_2(1000)^1 \cdot 1^2 +$$

$^3C_3(1000)^0 \cdot 1^3$

$= (1)1000{,}000{,}000(1) + (3)1000{,}000(1) + (3)1000.1 + 1.1.1$

$= 1000{,}000{,}000 + 3000{,}000 + 3000 + 1 = 1003003001.$

Theorem. Using Binomial theorem, prove that:

(i) $^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots + {}^nC_n = 2^n$

(ii) $^nC_0 + {}^nC_2 + {}^nC_4 + \ldots\ldots\ldots\ldots + = 2^{n+1}$

(iii) $^nC_1 + {}^nC_3 + {}^nC_5 + \ldots\ldots\ldots + = 2^{n-1}$.

Proof. For $n \in N$, we have $(1 + x)^n = {}^nC_0 + {}^nC_2x^2 + \ldots\ldots\ldots + {}^nC_nx^n$ .....(1)

When $x= 1$.

(1) $\Rightarrow$ $(1 + 1)^n = {}^nC_0 + {}^nC_1 \cdot 1 + {}^nC_2 \cdot 1^2 + \ldots\ldots\ldots + {}^nC_n \cdot 1^n$

$\Rightarrow$ $^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots\ldots + {}^nC_n = 2^n$. .....(2)

$\therefore$ (i) holds.

When $x = - 1$.

(1) $\Rightarrow$ $(1 - 1)^n = {}^nC_0 + {}^nC_1(- 1) + {}^nC_2(- 1)^2 + \ldots\ldots\ldots + {}^nC_n (- 1)^n$

$\Rightarrow$ $^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots\ldots = 0^n = 0$ .....(3)

(2)+(3) $\Rightarrow$ $2\left({}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots\ldots\ldots\right) = 2^n + 0$

$\Rightarrow$ $^nC_0 + {}^nC_2 + {}^nC_4 + \ldots\ldots\ldots + = 2^{n-1}$.

$\therefore$ (ii) holds.

(2) – (3) $\Rightarrow$ $2\left({}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots\ldots\ldots\right) = 2^n + 0$

$\Rightarrow$ $^nC_1 + {}^nC_3 + {}^nC_5 + \ldots\ldots\ldots + = 2^{n-1}$.

$\therefore$ (iii) holds.

Remark 1. The L.H.S. of (ii) and (iii) will have only finite number of terms, because $^nC_r = 0$ in case $r > n$.

Remark 2. $^nC_0, {}^nC_1, \ldots\ldots{}^nC_n$ are called **binomial coefficients.** $^nC_0, {}^nC_2, {}^nC_4 \ldots\ldots$ are called **even binomial coefficients.** $^nC_1, {}^nC_3, {}^nC_5 \ldots\ldots$ are called **odd binomial coefficients.**

Remark 3. In case of no ambiguity, the binomial coefficients $^nC_0$, $^nC_1$,……. ^nC_n …… are written as C_0, C_1,…..,C_n.

Remark 4. The last term in L.H.S. of (ii) and (iii) above will depend on the fact as to whether n is even or odd.

WORKING RULES FOR SOLVING PROBLEMS

Rule I. *If* $n \in N$ *, then* $^nC_0 + {^nC_1} + {^nC_2} + \ldots\ldots\ldots + {^nC_n} = 2^n$.

Rule II. *If* $n \in N$ *, then*

(i) $^nC_0 + {^nC_2} + {^nC_4} + \ldots\ldots\ldots + {^nC_n} = 2^{n-1}$ *if n is even*

(ii) $^nC_0 + {^nC_2} + {^nC_4} + \ldots\ldots\ldots + {^nC_{n-1}} = 2^{n-1}$ *if n is odd.*

Rule III. *If* $n \in N$ *, then*

(i) $^nC_1 + {^nC_3} + {^nC_5} + \ldots\ldots\ldots + {^nC_{n-1}} = 2^{n-1}$ *if n is even*

(ii) $^nC_1 + {^nC_3} + {^nC_5} + \ldots\ldots\ldots + {^nC_n} = 2^{n-1}$ *if n is odd.*

Example 8. *Evaluate :*

(i) $^7C_0 + {^7C_1} + \ldots\ldots\ldots.{^7C_7}$

(ii) $^8C_0 + {^8C_2} + \ldots\ldots\ldots.{^8C_8}$

(iii) $^{10}C_1 + {^{10}C_3} + \ldots\ldots\ldots.{^{10}C_9}$

(iv) $^{12}C_1 + {^{12}C_2} + \ldots\ldots\ldots.{^{12}C_{11}}$.

Sol. (i) $^7C_0 + {^7C_1} + \ldots\ldots\ldots.{^7C_7}$

= sum of all binomial coefficients in the expansion of $(1 + x)^7 = 2^7 = 128$.

(ii) $^8C_0 + {^8C_2} + \ldots\ldots\ldots.{^8C_8}$

= sum of all even binomial coefficient in the expansion of $(1 + x)^8$

$= 2^{8-1} = 2^7 = 128$.

(iii) $^{10}C_1 + {^{10}C_3} + \ldots\ldots\ldots.{^{10}C_9}$

= sum of all odd binomial coefficients in the expansion of $(1 + x)^{10}$

$= 2^{10-1} = 2^9 = 512$.

(iv) $^{12}C_1 + {^{12}C_2} + \ldots\ldots\ldots.{^{12}C_{11}}$

$= ({}^{12}C_0 + {}^{12}C_1 + {}^{12}C_{12} + \ldots\ldots\ldots + {}^{12}C_{11} + {}^{12}C_{12}) - {}^{12}C_0 - {}^{12}C_{12}$

= sum of all odd binomial coefficient in $(1 + x)^{12} - 1 - 1 = 2^{12} - 2 = 4094$.

EXERCISE 7.4

SHORT ANSWER TYPE QUESTIONS

1. Evaluate:

   (i) ${}^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4$

   (ii) ${}^9C_0 + {}^9C_1 + {}^9C_2 + \ldots\ldots {}^9C_9$

   (iii) ${}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + \ldots\ldots + {}^{10}C_{10}$

   (iv) ${}^8C_1 + {}^8C_2 + {}^8C_3 + \ldots\ldots + {}^8C_7$.

2. Evaluate:

   (i) ${}^{12}C_0 + {}^{12}C_2 + {}^{12}C_4 + \ldots\ldots {}^{12}C_{12}$

   (ii) ${}^9C_1 + {}^9C_3 + {}^9C_5 + \ldots\ldots {}^9C_9$

   (iii) ${}^{50}C_0 + {}^{50}C_2 + {}^{50}C_4 + \ldots\ldots {}^{50}C_{48}$

   (iv) ${}^{22}C_3 + {}^{22}C_5 + {}^{22}C_7 + \ldots\ldots {}^{22}C_{21}$.

3. If C_r denotes the coefficient of x^r in the expansion of $(1 + x)^n$, show that :

   (i) $C_0 + 2C_1 + 2^2C_2 + 2^3C_3 + \ldots\ldots\ldots + 2^nC_n = 3^n$

   (ii) $C_0 + 3C_1 + 3^2C_2 + 3^3C_3 + \ldots\ldots\ldots + 3^nC_n = 4^n$

   (iii) $C_0 - C_1 + C_2 - C_3 + \ldots\ldots\ldots + (-1)^nC_n = 0$.

Answers

1. (i) 16 (ii) 512 (iii) 1023 (iv) 254

2. (i) 2048 (ii) 256 (iii) $2^{49} - 1$ (iv) $2^{21} - 22$.

SUMMARY

1. A **binomial** is an algebraic expression of two terms which are connected by the operations '+' or '-'.
2. The **binomial theorem** for natural numbers states that
$$(a+b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + \dots\dots\dots .+ {}^nC_{n-1} a^1 b^{n-1} + {}^nC_n a^0 b^n,\ n \in N.$$
Here a and b may be any numbers.
3. **General term.** For $0 \le r \le n$, T_{r+1} in the expansion of $(a + b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$.
4. $(r + 1)$th term the end in the expansion of $(a + b)^n$ is same as the $(r + 1)$th term from the beginning in $(b + a)^n$.
5. **Middle terms**
6. (i) If n is an **even** natural number, then there is only one middle term in the expansion of $(a + b)^n$ and is given by $\frac{T_{n+2}}{2}$
7. (ii) If n is an **odd** natural number, then there are two middle terms in the expansion of $(a + b)^n$ and are given by $\frac{T_{n+1}}{2}$ and $\frac{T_{n+2}}{2}$.

TEST YOURSELF

1. Find the value of :
   (i) $\left(1+2\sqrt{x}\right)^5 + \left(1-2\sqrt{x}\right)^5$ (ii) $\left(x+\sqrt{x^2-1}\right)^6 + \left(x-\sqrt{x^2-1}\right)^6$.
2. If the 6th term in the expansion of $\left(\frac{1}{x^{8/3}} + x^2 \log_{10} x\right)^8$ is 5600, find the value of x.
3. The 3rd, 4th and 5th terms in the expansion of $(x + a)^n$ are respectively 84,280 and 560, find the values of x, a and n.
4. If $\left(5+2\sqrt{6}\right)^n = I + f$, where I and n are positive integers and f is a positive fraction less than one, show that I is an odd integer and $(I + f)(1 - f) = 1$.
5. If $\left(7+4\sqrt{3}\right)^n = I + F$, where I is a positive integer and F is a positive fraction less than one, show that $(1 - F)(I + F) = 1$.

6. If the coefficients of three consecutive terms in the expansion of $(1 + x)^n$ be 56, 70 and 56, find the value of n.

Answers

1. (i) $2(1 + 40x + 80x^2)$ (ii) $64x^6 - 96x^4 + 36x^2 - 2$ 2. 10

3. $x = 1$, $a = 2$ and $n = 7$ 6. 8.

SECTION – A

8. BINOMIAL THEOREM (FOR FRACTIONAL INDEX)

LEARNING OBJECTIVES

- Introduction
- Binomial Theorem of Fractional Index
- Some Observations
- Some Particular Expansions
- General Term
- Particular Terms
- Some Applications of Binomial Theorem

INTRODUCTION

Till now, we have been examining the expansion of $(a + b)^n$ for positive integral index n. Now we shall relax the condition on the index n and allow it to be any rational number. Of course, every natural number is also a rational number, so the binomial expansion for rational index n would also be for any positive integral index.

BINOMIAL THEOREM FOR FRACTIONAL I NDEX

For any rational number n,

$$(1+x)^n = 1+nx+\frac{n(n-1)}{1.2}x^2+\frac{n(n-1)(n-2)}{1.2.3}x^3+........\infty \text{ provided } |x| < 1.$$

The restriction on x is not required when n is a natural number. Now we shall see that when n is natural number, then the above expansion coincides with that as given earlier.

Let $n \in N$ and $|x| < 1$, then we have

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + ...$$

$$+ \frac{n(n-1)(n-2).........(n-(n-1)}{1.2.3.........n}x^n + \frac{n(n-1)(n-2)..........(n-(n-1)(n-n)}{1.2.3..........(n+1)}x^{n+1} + ...$$

$$= {}^nC_0 1^n x^0 + {}^nC_1 I^{n-1} x^1 + {}^nC_2 1^{n-2} x^2 + {}^nC_3 1^{n-3} x^3 + ... + {}^nC_n 1^0 x^n .$$

This is the same expansion as would have given by the binomial theorem for positive integral index.

SOME OBSERVATIONS

i. If $n \in N$, then $(1 + x)^n$ is defined for all values of x and if $n \in Q - N$, then $(1+x)^n$ is defined only when $|x| < 1$.

ii. If $n \in N$, then $(1 + x)^n$ contains only $n + 1$ terms and if $n \in Q - N$, then $(1+x)^n$ contain infinitely many terms.

iii. In the expansion of $(1 + x)^n$, the exponent of x goes on increasing through 0.

iv. If $n \in N$, then the coefficient of any term is $(1 + x)^n$ is ${}^nC_k$, when k is the exponent of x.

v. If $n \in N$, then the exact value of $(1 + x)^n$ can be found by adding all terms ($n + 1$ in number) in the expansion of $(1 + x)^n$ and if $n \in Q - N$, then only an approximate value of $(1 + x)^n$ can be found by adding certain finite number of terms in the expansion of $(1 + x)^n$.

WORKING RULES FOR EXPANDING $(1 + x)^n$, n $\in$Q

Step I. (i) *If* $n \in N$ *, then* $(1 + x)^n$ *can be expanded for all values of x and has (n + 1) terms.*

(ii) *If* $n \in Q - N$ *, then* $(1 + x)^n$ *is always '1'.*

Step II. *The first term in* $(1 + x)^n$ *is always '1'.*

Step III. *The second term is the product nx of n and x.*

Step IV. *For the third term, take coefficient as* $\frac{n(n-1)}{1.2}$*, increase the power of x by 1. Thus, the third term is* $\frac{n(n-1)}{1.2}x^2$*. Continue this process repeatedly.*

SOME PARTICULAR EXPANSIONS

For $n \in Q,$ we have:

(i) $(a+b)^n = \left[a\left(1+\frac{b}{a}\right)\right]^n = a^n\left(1+\frac{b}{a}\right)^n$

$$= a^n\left[1+n\left(\frac{b}{a}\right)+\frac{n(n-1)}{1.2}\left(\frac{b}{a}\right)^2+........\right], \text{ provided } \left|\frac{b}{a}\right| < 1 .$$

(ii) $(1+x)^{-n} = 1+(-n)x+\frac{(-n)(-n-1)}{1.2}x^2+\frac{(-n)(-n-1)(-n-2)}{1.2.3}x^3+........$

$$= 1-nx+\frac{n(n-1)}{1.2}x^2-\frac{n(n+1)(n+2)}{1.2.3}x^3+......., \text{ provided } |x| < 1.$$

(iii) $(1-x)^n = (1+(-x))^n = 1+n(-x)+\frac{n(n-1)}{1.2}(-x)^2+\frac{n(n-1)(n-2)}{1.2.3}(-x)^3+.........\,.$

$$= 1-nx+\frac{n(n+1)}{1.2}x^2-\frac{n(n-1)(n-2)}{1.2.3}x^3+......., \text{ provided } |-x| < 1 \text{ i.e., } |x|<1.$$

(iv) $(1-x)^{-n} = (1+(-x))^{-n} = 1 + (-n)(-x) + \frac{(-n)(-n-1)}{1.2}(-x)^2 + \frac{(-n)(-n-1)(-n-2)}{1.2.3}$

$(-x)^3 + ...$

$= \; 1 - nx + \frac{n(n+1)}{1.2}x^2 - \frac{n(n+1)(n+2)}{1.2.3}x^3 + ...,$ provided $|-x| < 1$ i.e., $|x|<1$.

Example 1. *Write the first four terms in the ascending powers of x in the expansions of the following:*

(i) $(1 + 4x)^{-5}$, $|x| < 1/4$ (ii) $(1 - x^2)^{-4}$, $|x| < 1$.

Sol. (i) $|4x| = |4||x| = 4|x| < 4(1/4) = 1$. $\therefore$ $|4x| < 1$

$\therefore$ $(1 + 4x)^{-5}$ can be expanded by *Binomial theorem.*

$\therefore$ $(1 + 4x)^{-5} = 1 + (-5)(4x) + \frac{(-5)(-5-1)}{1.2}(4x)^2 + \frac{(-5)(-5-1)(-5-2)}{1.2.3}(4x)^3 + ...$

$= 1 - 20x + 15(16x^2) - 35(64x^3) + ...$ **=** $1 - 20x + 240x^2 - 2240x^3 + ...$

(ii) $|-x^2| = |-1||x^2| = 1.\, x^2 = |x|^2 < 1$. $\therefore$ $|-x^2| < 1$

$\therefore$ $(1 - x^2)^{-4} = (1 + (-x^2))^{-4}$ can be expanded by *Binomial theorem.*

$\therefore$ $(1 - x^2)^{-4} = 1 + (-4)(-x^2) + \frac{(-4)(-4-1)}{1.2}\left(-x^2\right)^2 + \frac{(-4)(-4-1)(-4-2)}{1.2.3}\left(-x^2\right)^3 + ...$

$= 1 + 4x^2 + 10x^4 + 20x^6 + ...$

EXERCISE 8.1

SHORT ANSWER TYPE QUESTIONS

1. If $|x| < 1$, write the first three terms in the expansion of the following :

   (i) $(1 + x)^{-1}$ (ii) $(1 - x)^{-2}$ (iii) $(1 + x)^{-3}$.

2. Find the coefficient of x^2 in the expansion of $\left(1 - \frac{4}{3}x\right)^{-2}, |x| < \frac{3}{4}$.

3. Find the coefficient of x^6 in the expansion of $(1 + 5x^3)^{-3}$, $|x| < (1/5)^{1/3}$.

4. Write the first three terms in the expansions of the following and also state the conditions of x for which the expansions are valid:

   (i) $\left(1-\frac{2x}{3}\right)^{-1/2}$ (ii) $(1-2x^3)^{11/2}$.

5. Find two values of m such that in the binomial expansion of $(1 - x)^m$, $|x| < 1$, coefficient of x^2 is 3.

Answers

1. (i) $1 - x + x^2 + \ldots\ldots.$ (ii) $1 + 2x + 3x^2 + \ldots\ldots$ (iii) $1 - 3x + 6x^2 + \ldots\ldots$

2. 16/3 3. 150 4. (i) $1+\frac{x}{3}+\frac{x^2}{6}+\ldots..; |x|<\frac{3}{2}$

(ii) $1-11x^3+\frac{99}{2}x^6+\ldots\ldots; |x|<\frac{1}{2^{1/3}}$ 5. -2, 3

GENERAL TERM

For $n \in Q$ and $|x| < 1$, we have $(1 + x)^n$

$$= 1+nx+\frac{n(n-1)}{1.2}x^2+\frac{n(n-1)(n-2)}{1.2.3}x^3+\ldots\ldots..$$

Let T_{r+1} ($r \geq 0$) be the $(r + 1)$th term in the expansion.

$$T_{0+1} = T_1 = 1$$

$$T_{1+1} = T_2 = nx = \frac{n}{1}x^1$$

$$T_{2+1} = T_3 = \frac{n(n-1)}{1.2}x^2$$

$$T_{3+1} = T_4 = \frac{n(n-1)(n-2)}{1.2.3}x^3$$

....

....

$$\therefore \qquad T_{r+1} = \frac{n(n-1)(n-2)\ldots\ldots\ldots(n-(r-1))}{1.2.3\ldots\ldots.r} x^r$$

Remark. T_{r+1} in $(1 + x)^n$ can also be expressed as

$$T_{r+1} = \frac{n(n-1)(n-2)\ldots\ldots\ldots \, r factors}{r!} x^r .$$

Example 2. *Find the 5th term in the expansion of* $(1 - 2x^3)^{11/2}$, $x < 1/2^{1/3}$.

Sol. T_{r+1} in $(1+x)^n = \dfrac{n(n-1)\ldots\ldots\ldots \, r factors}{r!} x^r$

$$\therefore \qquad T_5 = T_{4+1} \text{ in } (1-2x^3)^{11/2} = \frac{\frac{11}{2}\left(\frac{11}{2}-1\right)\left(\frac{11}{2}-2\right)\left(\frac{11}{2}-3\right)}{4!}(-2x^3)^4$$

(Here r = 4 and n = 11/2)

$$= \frac{\frac{11}{2}\times\frac{9}{2}\times\frac{7}{2}\times\frac{5}{2}}{24}.16x^{12} = \frac{1155}{8}x^{12}$$

EXERCISE 8.2

SHORT ANSWER TYPE QUESTIONS

1. Find the 5th term in the expansion of $(1 - 2x)^{3/4}, |x| < 1/2$.
2. Find the 5th term in the expansion of $\dfrac{1}{(1-3x)^{2/3}}, |x| < \dfrac{1}{3}$.
3. Find the 5th term in the expansion of $(2 + 3x)^{-5}, |x| < 2/3$.
4. Find the 8th term in the expansion of $(1 - 3x^2)^{16/3}, |x| < \dfrac{1}{\sqrt{3}}$.

LONG ANSWER TYPE QUESTIONS

5. Find the (r + 1)th term in the expansion of $(1 + x)^{-4}, |x| < 1$.
6. Find the (k + 1)th term in the expansion of $(2 + 3x)^{3/2}, |x| < 2/3$.
7. Find the (r + 1)th term in the expansion of $(2 - 3x^2)^{-2/3}, |x| < \sqrt{\dfrac{2}{3}}$.

Answers

1. $-\frac{45}{128}x^4$ 2. $\frac{110}{3}x^4$ 3. $\frac{2835}{256}x^4$ 4. $\frac{208}{9}x^{14}$

5. $\frac{(-1)^r(r+1)(r+2)(r+3)}{6}x^r$ 6. $\frac{(-1)^k 1.3.5......(5-2k)}{k!}2^{\frac{3-4k}{2}}3^{k+1}x^k$

7. $\frac{2.5.8.....(3r-1)}{2^{3/2}(r!)2^r}x^{2r}$.

PARTICULAR TERMS

Sometimes, a particular term satisfying certain conditions is required in the binomial expansion of the type $(1 + x)^n$. This can be done by expanding $(1 + x)^n$ to certain terms and then locating the required term. Generally this becomes a tedious task. In such cases, we begin by evaluation the general term T_{r+1} and then finding the value of r by assuming T_{r+1} to be required term.

WORKING RULES FOR FINDING PARTICULAR TERMS

Step I. *In the expansion of $(1 + x)^n$, the $(r + 1)$th term is equal to*

$\frac{n(n-1)(n-2)......rfactors}{1.2.3......r}x^r$ *, where r can take values 0, 1, 2, ...*

Step II. *Find the general term T_{r+1} in the expansion of $(1 + x)^n$.*

Step III. *Assume the T_{r+1} is the desired particular term.*

Step IV. *Find the value of r.*

Step V. *Put the value of r in the term T_{r+1}. This gives the required particular term (s).*

Example 3. *In the expansion of $(1 - 2x)^{-1/2}$, $|x| < 1/2$, find the coefficients of x^8 and x^9.*

Sol. T_{r+1} in the expansion of $(1 - 2x)^{-1/2}$

$$= \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)......\left(-\frac{1}{2}-(r-1)\right)}{1.2.3......r}(-2x)^r$$

$$= \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)......\left(-\frac{2r-1}{2}\right)}{r!}(-2)^r x^r$$

$$= \frac{(-1)^r 1.3.5......(2r-1)}{r!2^r}(-1)^r 2^r x^r = \frac{1.3.5......(2r-1)}{r!}x^r$$

[$\because$ $(-1)^r (-1)^r = (-1)^{2r} = 1$]

$\therefore$ Coefficient of $x^3 = \frac{1.3.5......(2r-1)}{r!}$

Putting $r = 8$, we get coefficient of $x^8 = \frac{1.3.5.7.9.11.13.15}{8!} = \frac{6435}{128}$

Putting $r = 9$, we get coefficient of $x^9 = \frac{1.3.5.7.9.11.13.15.17}{9!} = \frac{12155}{128}$.

EXERCISE 8.3

SHORT ANSWER TYPE QUESTIONS

1. Find the coefficient of x^6 in the expansion of $(1-2x)^{-5/2}, |x| < 1/2$.
2. Find the coefficient of x^6 in the expansion $(1-3x)^{10/3}, |x| < 1/3$.
3. Find the coefficient of x^{10} and x^n in the expansion of $(1+x+x^2+......)^2, |x| < 1$.
4. Find the coefficient of x^{10} in the expansion of $\frac{1+2x}{(1-2x)^2}, |x| < \frac{1}{2}$.
5. Find the coefficient of x^{100} in the expansion of $\frac{3-5x}{(1-x)^2}, |x| < 1$.
6. Find the coefficient of x^{2r} in the expansion of $(1-4x^2)^{-1/2}, |x| < 1/2$.
7. Show that the coefficient of x^n in the expansion of $(1-2x)^{-1/2}$, $|x| < \frac{1}{2}$ is $\frac{(2n)!}{2^n (n!)^2}$.
8. Find the first negative term in the expansion of $(1+x)^{7/2}, 0 < x < 1$.

Answers

1. $\frac{15015}{16}$ 2. $\frac{35}{9}$ 3. $11, n+1$ 4. 21504

5. – 197 6. $\frac{1.3.5.......(2r-1)}{r!}2^r$ 8. $-\frac{7}{256}x^5$

SOME APPLICATIONS OF BINOMIAL THE OREM

In this section, we shall study to use of binomial theorem for fractional index in the root extraction of numbers and for approximation of functions.

(i) If x be numerically so small that its cube and higher powers maybe neglected , then $(1 + x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2$ (approximately), because x^3, x^4, x^5, ... are all approximately zero.

Example 4. *If x is nearly equal to 1, prove that* $mx^m - nx^n = (m-n)x^{m+n}$.

Sol. Let $x = 1 + h$.

$\therefore$ h is nearly equal to 0 and so we can neglect $h^2, h^3, ...$

L.H.S. = $mx^m - nx^n = m(1 + h)^m - n(1 + h)^n = m(1 + mh) - n(1 + nh)$

(Neglecting h^2, h^3,......)

$= m + m^2h - n - n^2h = (m - n) + (m^2 - n^2)h = (m - n)\ [1 + (m + n)h\ ]$

L.H.S. = $(m - n)x^{m+n} = (m - n)(1 + h)^{mn} = (m - n)[1 + (m + n)h]$

L.H.S. = R.H.S.

EXERCISE 8.4

SHORT ANSWER TYPE QUESTIONS

1. If x be numerically so small that its square and higher powers may be neglected, then find the binomial expansions for:

   (i) $(1+3x)^{-2}$ (ii) $(1-2x)^{-1/2}$ (iii) $(5+7x)^{1/3}$ (iv) $(4-9x)^{-3/5}$

2. If x be numerically so smart that its cube and higher powers may be neglected, then find the binomial expansions for:

(i) $(1-x)^{-5}$ (ii) $(1+2x)^{1/3}$ (iii) $(4+3x)^{-2}$ (iv) $(3+7x)^{-2./5}$

LONG ANSWER TYPE QUESTIONS

3. Use binomial theorem to evaluate:
   (i) $(1010)^{1/3}$ correct to four places of decimal
   (ii) $(244)^{1/5}$ correct to four places of decimal
   (iii) $(7.60)^{1/3}$ correct to four places of decimal
   (iv) $(1.025)^{-1/3}$ correct to three places of decimal

4. Use binomial theorem to evaluate:
   (i) $(624)^{1/4}$ correct to four places of decimal
   (ii) $(719)^{1/6}$ correct to four places of decimal
   (iii) $\sqrt[7]{129}$ correct to four places of decimal
   (iv) $\sqrt[4]{16.08}$ correct to four places of decimal

5. Use binomial expansion of $(100 + x)^{1/2}$ to find $\sqrt{101}$ correct to 6 places of decimal.

6. If x be numerically so small that its square and higher powers be neglected, then prove that

(i) $\left(\frac{1+x}{1-x}\right)^2 = 1+4x$ (ii) $\left(1+\frac{3x}{4}\right)^{-4}(4-3x)^{1/2} = 2-\frac{27x}{4}$.

Answers

1. (i) $1 - 6x$ (ii) $1 + x$ (iii) $5^{1/3}\left[1+\frac{7}{15}x\right]$ (iv) $\frac{1}{4^{3/5}}\left[1+\frac{27}{20}x\right]$

2. (i) $1 + 5x + 15x^2$ (ii) $1+\frac{2}{3}x-\frac{4}{9}x^2$ (iii) $\frac{1}{16}\left[1-\frac{3}{2}x+\frac{27}{16}x^2\right]$

   (iv) $\frac{1}{3^{2/5}}\left[1-\frac{14}{15}x+\frac{343}{225}x^2\right]$

3. (i) 10.0332 (ii) 3.0024 (iii) 1.9661 (iv) 0.992

4. (i) 4.9980 (ii) 2.9931 (iii) 2.0022 (iv) 2.0025

SUMMARY

1. **The binomial theorem** for fractional index states that $(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + ...\infty$, provided $|x| < 1$ and $n \in Q$
2. **General term.** For $r \geq 0$, T_{r+1} in the expansion of $(1 + x)^n$, $|x| < 1$, $n \in Q$ is given by $T_{r+1} = \frac{n(n-1)(n-2)......(n-r+1)}{1.2.3.....r}x^r$
3. If x be so small that its square and higher powers may be neglected, then $(1+x)^n = 1 + nx$ (approximately).
4. If x be so small that its cube and higher powers may be neglected, then

   $(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2$ (approximately).

TEST YOURSELF

1. Expand the following by using binomial theorem:

   (i) $\left(1 - \frac{x}{2}\right)^{-1/2}, |x| < 2$

   (ii) $\frac{1}{(3-4x^2)^{1/3}}, |x| < \frac{\sqrt{3}}{2}$

   (iii) $(1+x^4)^{-3}, |x| < 1$

   (iv) $\frac{1}{\sqrt{5+4x}}, |x| < \frac{5}{4}$

2. Write the first three terms in the expansion of $\frac{2+x}{(3-2x)^2}, |x| < \frac{3}{2}$.
3. Prove that the coefficient of y^n in the expansion of $\frac{(1+y)^2}{(1-y)^2}, |y| < 1$ in $4n$ for each $n = 1, 2, 3, ...$
4. If x be a quantity numerically so small than x^3 may be neglected in comparison with l^3, show that :

$$\sqrt{\frac{l}{l+x}} + \sqrt{\frac{l}{l-x}} = 2 + \frac{3x^2}{4l^2} (nearly).$$

Answers

1. (i) $1+\frac{x}{4}+\frac{3x^2}{32}+\frac{5x^3}{128}+........$ (ii) $\frac{1}{\sqrt{3}}+\frac{2}{3\sqrt{3}}x^2+\frac{2}{3\sqrt{3}}x^4+\frac{20}{27\sqrt{3}}x^6+......$

(iii) $1-3x^4+6x^8-10x^{12}+.......$ (iv) $\frac{1}{\sqrt{5}}\left(1-\frac{2}{5}x+\frac{6}{25}x^2+\frac{4}{25}x^3+......\right)$

SECTION – B

9. MEASUREMENT OF ANGLES

LEARNING OBJECTIVES

- Introduction
- Angles
- Quadrants
- Systems of Measuring Angles
- Sexagesimal System
- Central System
- Circular System
- Arc, Radius and Angle Relation

INTRODUCTION

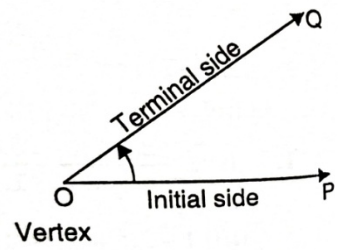
Trigonometry* is that branch of mathematics which deals with the measurement of sides and angles of triangles. The study of trigonometry is useful to engineers, scientists, surveyors, astronomers and others. They is also used in navigation and seismology.

The starting point of trigonometry is *angle* and its *measurement.*

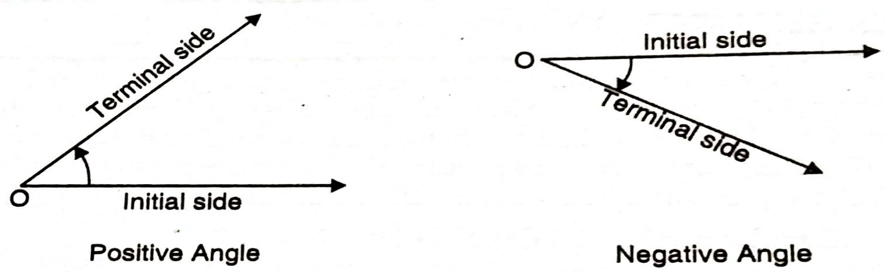
ANGLES

The figures obtained by rotating a given ray about its end point is called an **angle.** The original position of the ray is called the **initial side** of the angle, whereas the final position of the ray is called the **terminal side** of the angle.

An angle is called **positive** if the direction of rotation of ray from the initial side to the terminal side is *anti-clockwise*.

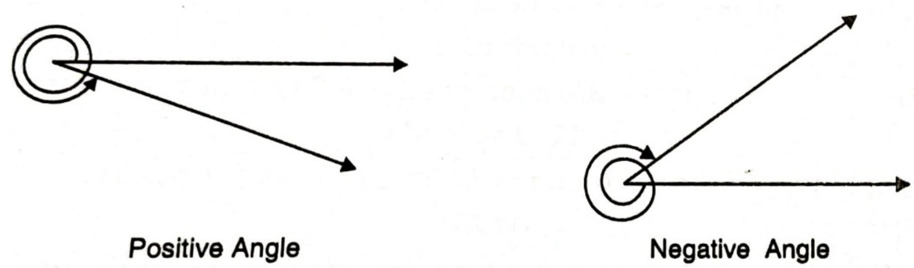


** The word 'trigonometry' is derived from the Greek word 'Trigon' and 'metron'.*



An angle is called **negative** if the direction of rotation of ray from the initial side to the terminal side is *clockwise.*

There is neither lower limit nor upper limit for the magnitude of an angle.

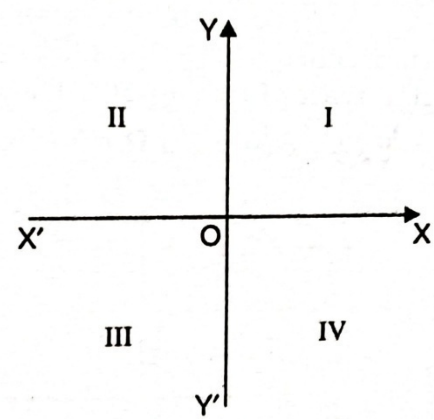


QUADRANTS

Let *X`OX* and *Y`OY* be two perpendicular lines intersecting at *O*. *X`OX* and *Y`OY* are respectively called the *x*-axis and *y*-axis.

The lines *X`OX* and *Y`OY* dividing the plane into four parts, are called **quadrants**.

i. *XOY* is called **first quadrant.**
ii. *X`OY* is called **second quadrant.**
iii. *X`OY`* is called **third quadrant.**
iv. *XOY`* s called **fourth quadrant**



If the vertex of an angle is at *O* and initial side along *x*-axis, then the angle is said to be in **standard position** and it is said to be in that particular quadrant in which the terminal side of the angle lies. For example, the angles 40^0, 220^0 lies in the I and III quadrants respectively.

If the terminal side of an angle in standard form lies along either axis, then the angle is called a **quadrantal angle**

SYSTEMS OF MEASURING ANGLES

There are three systems of measuring angles. These are:

i. Sexagesimal system (or English system)
ii. Centesimal system (or French system)
iii. Circular system.

SEXAGESIMAL SYSTEM

In this system, the unit of measuring an angle is a *degree*. An angle is called a *right angle* when the terminal side and initial side are perpendicular to each other. If a right angle is divided in 90 equal parts, then each part is called a **degree**. One degree is divided in 60 equal parts and each part is called a **minute.** One minute is further divided in 60 equal parts and each part is called a **second.**

$\therefore$ **1 right angle = 90 degrees (written as 90^0)**

1 degree = 60 minutes (written as 60^0)

1 minute = 60 seconds (written as 60^0)

Example 1. *Find (i) number of minutes (ii) number of seconds in 4.5 degrees.*

Sol. (i) 4.5 degrees = 4.5 X 60 minutes ($\because$ 1^o = 60`)

= **270**

(ii) 4.5 degrees = 270 minutes = 270 X 60 seconds (∵ 1\` = 60^0)

= **16,200 seconds.**

Example 2. *Find the quadrant in which the following angles lie:*

(i) *315^0* (ii) *870^0.*

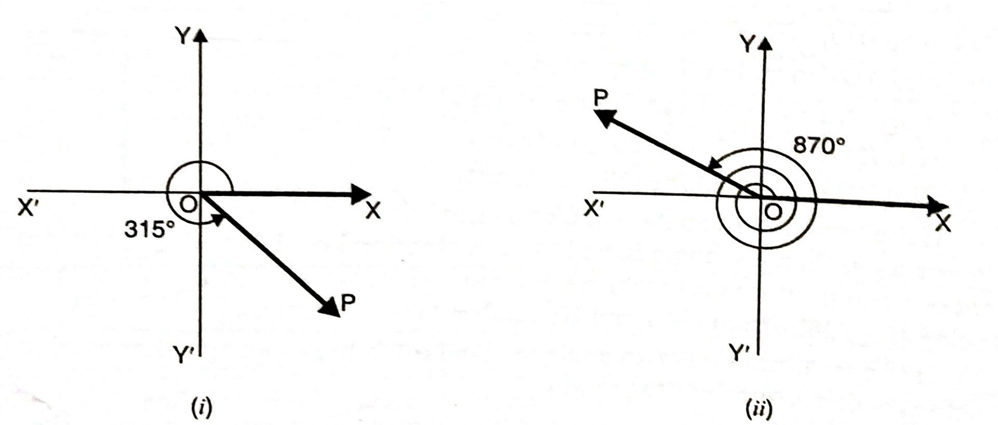
Sol. (i) Angle 315^0 is +ve, so the terminal side revolves in the anti-clockwise direction

Since $315^0 = 270^0 + 45^0$, the terminal side lies in the IV quadrant.

∴ 315^0 is in **IV quadrant.**

(ii) Angle 870^0 is +ve, so the terminal side revolves in the anti-clockwise direction. Since $870^0 = 2 \text{ X } 360^0 + 150^0$, the terminal side makes two complete revolutions and traces further 150^0. The terminal side lies in the II quadrant.

∴ 870^0 is in **II quadrant.**



CENTESIMAL SYSTEM

In this system, the unit of measurement in a **grade.** If a right is divided in 100 equal parts, then each part is called a **grade.** One grade is divided in 100 equal parts and each part is called a **minute. One minute is further divided in 100 equal parts and each part is called second.**

∴ **1 right angle = 100 grades (written as 100^g)**

1 grade = 100 minutes (written as 100')

1 minute = 100 seconds (written as 100'')

Remark 1. $90^0 = 100^0$, because each is equal to one right angles.

Remark 2. Sexagesimal minute, second and centesimal minute, second are different units.

Example 3. *Express* $49^0\ 50^0\ 15^{\ 0}$ in centesimal system.

Sol. $49^0 15`15`` = 49^0 + 50` + 15`` = 49^0 + 50` + \frac{15'}{60} = 49^0 + 50`` + \frac{1'}{4}$

$= 49^0 + 50' \frac{1'}{4} = 49^0 + \frac{201'}{4} = 49^0 + \frac{201^0}{4\times 60} = 49\frac{201^0}{240}$

$= \frac{11961^0}{240} = \frac{11961^0}{240\times 90}$ rt. Angles = 0.553750 rt. Angle

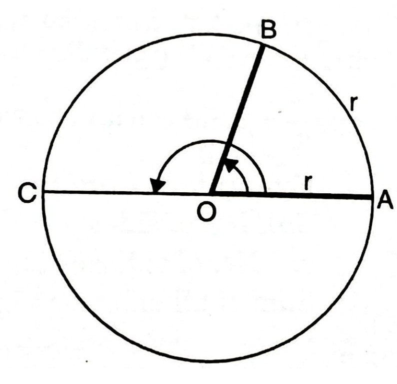
$= (0.553750 \times 100)^g = 55.3750' = 55^g + (0.3750)^g$

$= 55^g + (0.3750\times 100)' = 55^g + 37.50' = 55^g + 37' + (0.50)'$

$= 55^g + 37' + (0.50\times 100)'' = 55^g + 37' + 50' = 55^g 37' 50''$

CIRCULAR SYSTEM

Draw any circle. Let O be its centre and r its radius. Let arc AB be equal to the radius. Product AO to meet the circle at C.



Since angles at the center of a circle are proportional to the arcs on which they stand, we have

$$\frac{\angle AOB}{\angle AOC} = \frac{arcAB}{arcAC}$$

$\therefore$ $$\frac{\angle AOB}{180^{\ 0}} = \frac{r}{\frac{1}{2}(2\pi r)}$$

or $$\angle AOB = \left(\frac{180}{\pi}\right)^0,$$ a constant angle.

$\therefore$ $\angle AOB$ is independent of the radius of the circle. This angle is defined as one *radian.* Thus, one **radian** is the angle subtended at the centre of a circle by a positive arc equal in length to the radius of the circle. The arc is regarded positive if it is measured in anti-clockwise direction. We have also seen that the measure of a radian is independent of the circle used.

In the circular system of measuring angle, the unit of measurement is a *radian*, *i.e.,* the circular measure of an angle is the number of radians it contains.

In the above article, we have $\angle AOB = \left(\frac{180}{\pi}\right)^0$

$\therefore$ 1 radian $= \left(\frac{180}{\pi}\right)^0$ ($\because$ $\angle AOB$ = 1 radian)

$\therefore$ π radians = 180^0 ....(1)

We also have $90^0 = 100^g$.

$\therefore$ $180^0 = 200^g$ .....(2)

By (1) and (2), we have **$180^0 = 200^g = \pi$ radians.**

Radian measures of some commonly used angles are given below:

Angle in degree	0^0	30^0	45^0	60^0	90^0	120^0	135^0	150^0
Angle in radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$

180^0	210^0	225^0	240^0	270^0	300^0	315^0	330^0	360^0
π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π

Remark 1. π radian = $180^0 \Rightarrow$ 1 radian $\frac{180^0}{\pi} = \left(\frac{180}{22/7}\right)^0 = \frac{630^0}{11}$

$$= 57^0 + \frac{3^0}{11} = 57^0 + \left(\frac{3}{11}\times 60\right)' = 57^0 + 16' + \left(\frac{4}{11}\right)'$$

$$= 57^0 + 16' + \left(\frac{4}{11}\times 60\right)' = 57^0 + 16' + 22''(approx.)'$$

$\therefore$ **1 radian = $57^0 16` 22``$ (approx.)**

Remark 2. π radian = $180^0 \Rightarrow 1^0 = \frac{\pi}{180}$ radians $= \frac{22}{7 \times 180}$ radians

= 0.01746 radian (approx.)

$\therefore$ **1^0 = 0.01746 radian (approx.)**

Remark 3. When the angle is measured in the circular measure, then the word 'radian' with the angle is generally omitted. The angle θ radian is also written as θ^0. For example, the angle $\frac{\pi}{6}$ radius can also be written as $\left(\frac{\pi}{6}\right)^c$ or simply as $\frac{\pi}{6}$.

Example 4. *Find in radians the angle of a regular octagon.*

Sol. No. of sides = 8

$\therefore$ No. of exterior angles = 8

Sum of all exterior angles = 360^0

$\therefore$ Each exterior angle $= \frac{360^0}{8} = 45^0$

$\therefore$ Each interior angle = $180^0 - 45^0 = 135^0 = \left(135 \times \frac{\pi}{180}\right)$ radius

$= \frac{3\pi}{4}$ radians.

WORKING RULES FOR SOLVING PROBLEMS

Rule I. *1 right angle = 90^0 , 1^0 = 60` and 1` = 6=``.*

Rule II. *1 right angle = 100^g, 1^g = 100` and 1` = 100``.*

Rule III. *$180^0 = 200^g = \pi$ radius.*

Rule IV. (i) *Radian measure* = $\frac{\pi}{180} \times$ *Degree measure*

(ii) *Degree measure* = $\frac{180}{\pi} \times$ *Radian measure.*

Rule V. *The angle between two consecutive digits in a clock is 30 (= 360/ 12) degrees.*

Rule VI. (i) *The hour-hand subtend an angle of 30 (=360/ 12) degree in one hour.*

(ii) *The minute-hand subtend an angle of 6 (=360/60) degree in one minute.*

Rule VII. *In a regular polygon :*

(i) *all the exterior angles are equal*

(ii) *all the interior angles are equal*

(iii) *all the sides are equal*

(iv) *sum of all exterior angle = 360^0*

(v) *each exterior angle = $\left(\frac{360}{no.of\ sides}\right)^0$*

(vi) *each interior angle = 180^0 – exterior angle.*

Rule VIII. *To find the angle of a regular polygon means to find the interior angle of the regular polygon.*

EXERCISE 9.1

SHORT ANSWER TYPE QUESTIONS

1. Find the quadrant in which the following angles lie :
   (i) 60^0 (ii) 240^0 (iii) 120^0 (iv) 330^0
2. Write the following angles in circular measure:
   (i) 75^0 (ii) 240^0 (iii) $40^0\ 20^0$ (iv) $-37^0\ 30'$.
3. Write the following angles in sexagesimal measure :
   (i) $\left(\frac{\pi}{6}\right)^c$ (ii) $\left(\frac{3\pi}{5}\right)^c$ (iii) $\left(\frac{5\pi}{3}\right)^c$ (iv) $\left(\frac{7\pi}{3}\right)^c$
4. Sketch the following angles :
   (i) 650^0 (ii) -565^0.

   Also find the quadrants in which these angles lie.
5. Find the centesimal measure of the angles whose degree measures are:
   (i) 45^0 (ii) 108^0 (iii) 828^0 (iv) 468^0

LONG ANSWER TYPE QUESTIONS

6. Express the following angles in centesimal system :
   (i) 67^0 35` 32``     (ii) $92^0$ 5` 33``

7. Express the following angles in sexagesimal system :
   (i) $5\pi/7$ radians   (ii) $9\pi/8$ radians

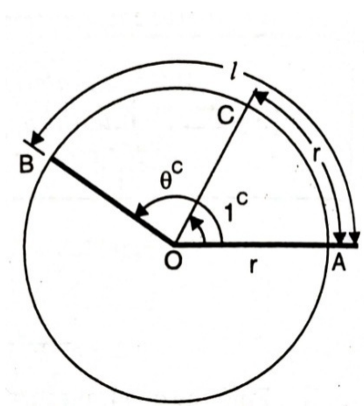
8. The difference of two angles is 20^0 and their sum in 120^g. Find the angles in degrees.

Answers

1. (i) First	(ii) Third	(iii) Second	(iv) Fourth
2. (i) $\frac{5\pi}{12}$ radians	(ii) $\frac{4\pi}{3}$ radians	(iii) $\frac{121\pi}{540}$ radians	(iv) $-\frac{5\pi}{24}$ radians
3. (i) 30^0	(ii) 108^0	(iii) 300^0	(iv) 210^0
4. (i) IV	(ii) II		
5. (i) 50^0	(ii) 120^g	(iii) 920^g	(iv) 520^g
6. (i) 75^g 10` 25``	(ii) 102^g 32` 50``		
7. (i) 128^0 34` $17\frac{1}{7}$`	(ii) 202^0 30`	8. 44^0, 64^0.	

ARC, RADIUS AND ANGLE RELATION

Theorem. If in a circle of radius r, an arc of length 1 subtends an angle θ radius, then prove that $\theta = \frac{1}{r}$.



Proof. Let O be the centre of the circle of radius r.

Let arc $AB = l$ and arc $AC = r$, where units of l and r are same.

Since, angles at the centre of a circle are proportional to the arcs on which they stand, we have

$$\frac{\angle AOB}{\angle AOC} = \frac{arcAB}{arcAC}$$

$\Rightarrow$ $$\frac{\theta\, radian}{1\, radian} = \frac{l}{r}$$

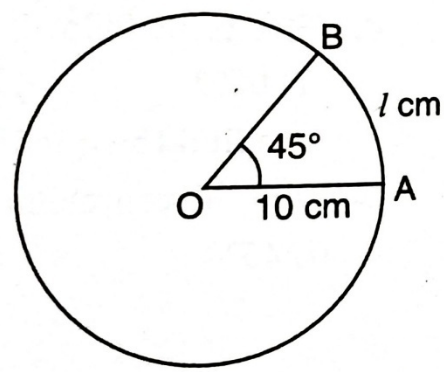
$\Rightarrow$ $$\theta = \frac{1}{r}.$$

Remark. While using the formula $\theta = l/r$, the angle θ must be expressed in radians, l and r must be in the same unit of length.

Example 5. *Find the length of an arc of a circle of radius 10 cm which subtends an angle of 45⁰ at the centre.*

Sol. Let $\angle AOB = 45^0$, OA = 10cm, arc AB = l cm.

We have $45^0 = \frac{\pi}{180} \times 45\, \text{radian} = \frac{\pi}{4}$ radian



Using $\frac{l}{r} = \theta$, we have $\frac{l}{10} = \frac{\pi}{4}$.

$\therefore$ $$l = \frac{10\pi}{4} = \frac{5\pi}{2}$$

$\therefore$ Length of arc = $\frac{5\pi}{2} cm$.

EXERCISE 9.2

SHORT ANSWER TYPE QUESTIONS

1. A wire 121 cm long is bent so as to lie along the arc of a circle of 180 cm radius. Find in degrees the angle subtended at the centre of the arc.
2. (i) Find in degrees the angle subtended at the centre of a circle of diameter 200 cm by an arc of length 22 cm.

   (ii) Find the radius of the circle in which a central angle of 60^0 intercepts an arc of 37.4 cm in length.
3. Find the angle in radian through which a pendulum swings if its length is 75 cm and the tip describes an arc of length :

(i) 10 cm (ii) 15 cm (iii) 21 cm.

4. A railroad curve is to be laid out on a circle. What radius should be used if the track is to change direction by 25^0 in a distance of 40 metres?

LONG ANSWER TYPE QUESTIONS

5. A circular wire 6 cm radius is cut and bent so as to lie along the circumference of a hoop whose radius is 96 cm. Find in radians the angle which is subtended at the centre of the hoop.
6. The radius of a circle is 5 cm. A chord of this circle is equal to the radius. Find the length of the minor arc of the chord.
7. A train is travelling at the rate of 10km/hr on a circular curve of half a kilometer radius. Through what angle in degrees has it turned in a minute?

Answers

1. 38^0 30\` 2. (i) $12^0$ 36\` (ii) 35.7 cm 3. (i) $\frac{2}{15}$radian

(ii) $\frac{1}{5}$radian (iii) $\frac{7}{25}$ radian 4. 91.64 mt 5. $\frac{\pi}{8}$radian

6. 5.24 cm 7. $\left(19\frac{1}{11}\right)^0$

SUMMARY

1. **Trigonometry** is that branch of mathematics which deals with the measurements of sides and angles of triangles.
2. (i) The figure obtained by rotating a given ray about its end point is called an **angle.**

   (ii) An angle is called **positive** if the direction of rotation of ray from initial side to terminal side is anti-clockwise.

   (iii) An angle is called **negative** if the direction of rotation of ray from initial side to terminal side is clockwise.
3. (i) **Sexagesimal system**

   1 right angle = 90^0 (degrees)

   1 degree = 60` (minutes)

   1 minute = 60” (seconds)

   (ii) **Centesimal system**

   1 right angle = 100^g (grades)

   1 grade = 100` (minutes)

   1 minute = 100`` (seconds)

   (iii) **Circular system.** The **circular measure** of an angle is the number of *radians* it contains, where one **radian** is defined as the angle subtended at the centre of a circle by an arc whose length is equal to the radius.

   (iv) $180^o = 200^g = \pi$ radians.

TEST YOURSELF

1. Express the angular measurement of the angle of a regular decagon in degrees and radians.
2. The angle of one regular polygon is to that of another as 3 : 2 and the number of sides is first is twice that in the second. Determine the number of sides of the polygons.

3. The angle in a triangle are in A.P. The number of degrees in the least and the number of radians in the greatest are as 36 : π. Find the angle in degrees.
4. Suppose that the arcs of same length in two circles subtend angle of 60^0 and 75^0 at the centre. Find the ratio of their radii.
5. The moon's distance from the earth is 3,60,000 km and its diameter subtend an angle of 31\` at the observer. Find the diameter of the moon.

Answers

1. 144^0, $\frac{4\pi}{5} radians$ 2. 8, 4 3. 20^0, 60^0, 100^0 4. 5 : 4

5. 3247.62 km.

SECTION – B

10. TRIGONOMETRIC FUNCTIONS

LEARNING OBJECTIVES

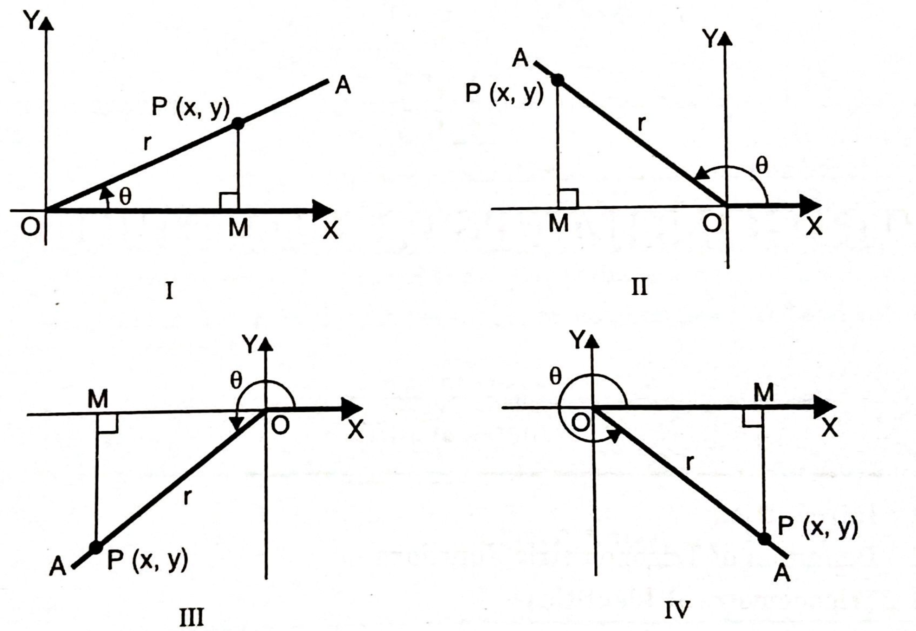
- Introduction
- Definition of Trigonometric Functions
- Trigonometrical Identities
- Elimination
- Signs of Trigonometric Functions
- Domain and Range of Trigonometric Functions
- Expression of any T-Ratio in Terms of a given T-Ratio
- Values of Trigonometric Functions for 0^0, 30^0, 45^0, 60^0, 90^0
- T-Ratios of Allied Angles

INTRODUCTION

In the present chapter, we shall define trigonometric functions of an angle. We shall also learn the methods of finding values of trigonometric functions for some specific angles.

DEFINITION OF TRIGONOMETRIC FUNCTIONS

Let θ be any real number. Construct the angle whose measure is θ radians, with vertex at the origin of a rectangular coordinate system and initial side along the positive x-axis. Let $P(x, y)$ be any point, distinct from the origin, on the terminal side OA of the angle. Let $OP = r$. Some of the possible positions of the terminal side of the angle θ radians are as given in the following figures, r is always taken positive while x and y can be positive or negative depending upon the position of the terminal side OA of the angle XOA.



θ

i. $\frac{y}{r}$ is called the **sine of** θ and is written as **sin** θ**.**

ii. $\frac{x}{r}$ is called the **cosine of** θ and is written as **cos** θ**.**

iii. $\frac{y}{x}$ is called the **tangent of** θ and is written as **tan**θ, provided θ is not an odd multiple of $\frac{\pi}{2}$.

iv. $\frac{x}{y}$ is called the **cotangent of** θ and is written as **cot** θ, provided θ is not an even multiple of $\frac{\pi}{2}$.

v. $\frac{r}{x}$ is called the **secant of** θ and is written as **sec** θ, provided $\theta$is not an odd multiple of $\frac{\pi}{2}$.

vi. $\frac{r}{y}$ is called the **cosecant of** θ and is written as **cosec**θ, provided $\theta$is not an even multiple of $\frac{\pi}{2}$.

The functions sin θ, $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$ and cosesθ are called **trigonometric functions.**

The trigonometric functions $\sin\theta$¸$\cos\theta$¸$\tan\theta$, $\cot\theta$, $\sec\theta$ and cosecθ are also called **trigonometric rations** or as **circular functions.**

Remark 1. sin θ is read the '*sine of angle θ*' and it should never the interpreted as the product of 'sin' and 'θ'.

Remark 2. If the terminal side of the angle θ (*c.f.* Fig. of Art. 10.2) is in the I quadrant, then

$$\sin\theta = \frac{y}{r} = \frac{MP}{OP}, \qquad \cos\theta = \frac{x}{r} = \frac{OM}{OP}, \qquad \tan\theta = \frac{y}{x} = \frac{MP}{OM},$$

$$\cot\theta = \frac{x}{y} = \frac{OM}{MP}, \qquad \sec\theta = \frac{r}{x} = \frac{OP}{OM}, \qquad \cos ec\theta = \frac{r}{y} = \frac{OP}{MP}.$$

Notation $(\cos\ \theta)^2$ is written as $\cos^2\theta$ (read "cos square θ")

$(\tan\ \theta)^3$ is written as $\tan^3\theta$ (read "tan cube θ")

$(\text{cosec}\ \theta)^n$ is written as $\text{cosec}^n\ \theta$ (read "cosec *n*th power θ"), *n* being a positive integer etc.

Caution. $(\cos\ \theta)^2$ should not be written as $\cos\ \theta^2$ or as $\cos^2\ \theta^2$.

TRIGONOMETRICAL IDENTITIES

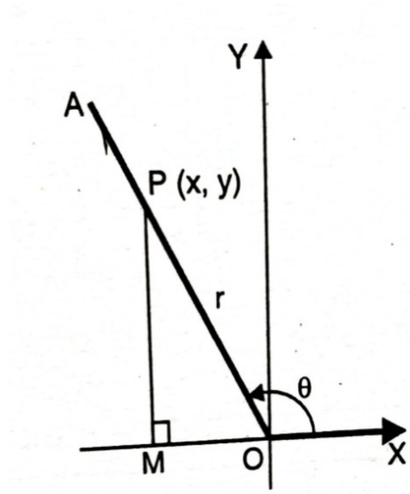
A trigonometrical equations is called a **trigonometrical identity** if it is true for all angles for which the trigonometrical functions involved are defined.

We prove some fundamental trigonometrical identities.

Let θ be any real number. Construct the angle whose measure is θ radians, with vertex at the origin of a rectangular coordinate system and initial side along the positive *x*-axis. Let *P*(*x*,*y*) be any point, distinct from the origin, on the terminal side *OA* of the angle. Let *OP* = *r*.

(i) $\sin\theta\ \text{cosec}\theta$ = $\frac{y}{r}\times\frac{r}{y} = 1$

(ii) $\cos\theta\ \sec\theta = \frac{x}{r}\times\frac{r}{x} = 1$



(iii) $\tan\theta\ \cot\theta = \frac{y}{x}\times\frac{x}{y} = 1$

(iv) $\frac{\sin\theta}{\cos\theta} = \frac{y/r}{x/r} = \frac{y}{x} = \tan\theta$

(v) $\frac{\cos\theta}{\sin\theta} = \frac{x/r}{y/r} = \frac{x}{y} = \cot\theta$

(vi) $\sin^2\theta + \cos^2\theta = (\sin\ \theta)^2 + (\cos\theta)^2 = \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = \frac{y^2+x^2}{r^2} = \frac{r^2}{r^2} = 1$

(Using Pythagoras result)

(vii) $1 + \tan^2\theta = 1 + (\tan\ \theta)^2 = 1 + \left(\frac{y}{x}\right)^2 = \frac{x^2+y^2}{x^2} = \frac{r^2}{x^2} = \left(\frac{r}{x}\right)^2 = (\sec\theta)^2 = \sec^2\theta$

(viii) $1 + \cot^2\theta = 1 + (\cot\ \theta)^2 = 1 + \left(\frac{x}{y}\right)^2 = \frac{y^2+x^2}{y^2} = \frac{r^2}{y^2} = \left(\frac{r}{y}\right)^2 = (\text{cosec}\,\theta)^2 = \text{cosec}^2\theta$

Thus, we have the following fundamental identities:

(i) **sin θ cosec θ = 1**

(ii) **cos θ sec θ = 1**

(iii) **tan θ cot θ = 1**

(iv) **tan θ =** $\frac{\sin\theta}{\cos\theta}$

(v) **cot θ** $= \frac{\cos\theta}{\sin\theta}$

(vi) **$\sin^2\theta + \cos^2\theta = 1$**

(vii) **$1 + \tan^2\theta = \sec^2\theta$**

(viii) **$1 + \cot^2\theta = \text{cosec}^2\theta$.**

Remark. The values of $\tan\theta$, $\cot\theta$, $\sec\theta$ and cosec θ in terms of sin θ and $\cos\theta$ are as follows:

$$\tan\theta = \frac{\sin\theta}{\cos\theta},\quad \cot\theta = \frac{\cos\theta}{\sin\theta},\quad \sec\theta = \frac{1}{\cos\theta},\quad \text{cosec}\,\theta = \frac{1}{\sin\theta}$$

WORKING RULES FOR SOLVING PROBLEMS

Rule I. *Trigonometrical identities are proved by using following methods:*

i. *If one side of the identity is a constant or a very simple expressions, then the other side is simplified and is shown equal to this side.*
ii. *If each side of the identity is neither constant nor a very simple expression, then both sides are simplified separately and are equated.*
iii. *If each side of the identity is neither constant nor a very simple expressions, then the identify can also be proved by using the 'if method'.*

Rule II. *If no formula is directly applicable in solving an expression, then all t-ratios occurring in the expression are changed in terms of 'sine' and 'cosine' and the simplification is carried.*

EXERCISE 10.1

SHORT ANSWER TYPE QUESTIONS

Prove the following identities:

1. $\cos^4 A - \sin^4 A = 1 - 2\sin^2 A$
2. $\sec^4 \theta - \tan^4 \theta = 1 + 2\tan^2 \theta$
3. $\sec^4 \theta - \sec^2 \theta = \tan^4 \theta + \tan^2 \theta$
4. $\sin^4 \theta + \cos^4 \theta = 1 - 2\sin^2 \theta \cos^2 \theta$

LONG ANSWER TYPE QUESTIONS

Prove the following identities (Q. No. 5- 8)

5. $\sin^6 A + \cos^6 A = 1 - 3\sin^2 A \cos^2 A$
6. $\sec^6 A - \tan^6 A + 3\tan^2 A \sec^2 A + 1$
7. $\cos ec^6 A - \cot^6 A = 1 + 3\cot^2 A + 3\cot^4 A$
8. $\dfrac{1+\cos\alpha}{1-\cos\alpha} = \dfrac{\tan^2\alpha}{(\sec\alpha - 1)^2}$.

ELIMINATION

In this section, we shall learn the method of eliminating 'θ' from two given equations involving trigonometric functions of 'θ' . By using given equations involving 'θ' and trigonometrical identities, we shall obtain an equation not involving 'θ' .

Remark. If $ax + by + c = 0$

$a'x + b'y + c' = 0$,

then we have $\dfrac{x}{bc'-cb'} = \dfrac{y}{ca'-ac'} = \dfrac{1}{ab'-ba'}$

i.e., $x = \dfrac{bc'-cb'}{ab'-ba'}$ and $y = \dfrac{ca'-ac'}{ab'-ba'}$.

Example 2. *Eliminate θ from the equations :*

(i) $x = h + a\cos\theta,\ y = k + b\sin\theta$ (ii) $x = a\sec^3\theta,\ y\cot^3\theta$.

Sol. (i) We have $x = h + a\cos\theta$ ....(1) $y = k + b\sin\theta$ .....(2)

(1) $\Rightarrow$ $\cos\theta = \dfrac{x-h}{a}$ (2) $\Rightarrow$ $\sin\theta = \dfrac{y-k}{b}$

Now $\sin^2\theta + \cos^2\theta = 1$

$\therefore$ $\left(\dfrac{x-h}{a}\right)^2 + \left(\dfrac{y-k}{b}\right)^2 = 1$ This is the required eliminant.

(ii) We have $x = a\sec^3\theta$ ....(1) $y = b\cot^3\theta$ .....(2)

(1) $\Rightarrow$ $\sec^3\theta = \dfrac{x}{a}$ $\Rightarrow$ $(\sec^3\theta)^{2/3} = \left(\dfrac{x}{a}\right)^{2/3}$ $\Rightarrow$ $\sec^2\theta = \left(\dfrac{x}{a}\right)^{2/3}$

(2) $\Rightarrow$ $\cot^3\theta = \dfrac{y}{b}$ $\Rightarrow$ $\tan^3\theta = \left(\dfrac{b}{y}\right)$ $\Rightarrow$ $\tan^2\theta = \left(\dfrac{b}{y}\right)^{2/3}$

Now $\sec^2\theta - \tan^2\theta = 1$.

$\therefore \qquad \left(\frac{x}{a}\right)^{2/3} - \left(\frac{b}{y}\right)^{2/3} = 1$. This is the required eliminant.

EXERCISE 10.2

SHORT ANSWER TYPE QUESTIONS

Eliminate θ from the following equations :

1. $x = a \cos^3 \theta, y = b \sin^3 \theta$
2. $x = a \operatorname{cosec}^3 \theta, y = b \cot^2 \theta$

LONG ANSWER TYPE QUESTIONS

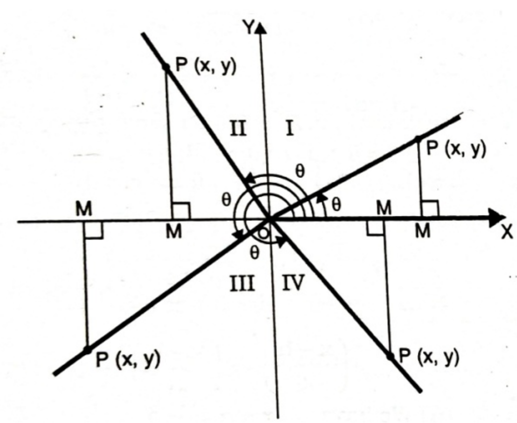
3. $\sin \theta + \cos \theta = a$, $\sin\theta - \cos\theta = b$
4. $x \cos \theta - y \sin \theta = a, x \sin \theta + y \cos \theta = b$
5. $l \tan \theta + m \sec \theta = n, l' \tan \theta + m' \sec \theta = n'$
6. $a \cot \theta + b \operatorname{cosec} \theta = x^2$, $\quad b \cot\theta + a \operatorname{cosec}\theta = y^2$

Answers

1. $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$
2. $\left(\frac{x}{a}\right)^{2/3} - \frac{y}{b} = 1$
3. $a^2 + b^2 = 2$
4. $a^2 + b^2 = x^2 + y^2$
5. $(ln' - l'n)^2 - (m'n - mn')^2 = (lm' - l'm)^2$
6. $x^4 - y^4 = b^2 - a^2$

SIGNS OF TRIGONOMETRIC FUNCTIONS

Let θ be any real number. Construct the angle whose measure is θ radians with vertex at the origin of a rectangular coordinate system and initial side along the positive x - axis. Let $P(x, y)$ be any point, distinct from the origin, on the terminal side of the angle. Let $OP = r$. Some of the possible positions of the terminal side of the angle θ radians are as given in the figure 'r' is always taken positive.



First quadrant. Here *x, y* and *r* are all positive. The ratio of any two of these is positive.

$\therefore$ **All t – functions are positive.**

Second quadrant. Here *x* is negative, *y* and *r* are positive.

$$\sin\theta = \frac{y}{r} = \frac{+ve}{+ve} = +ve, \quad \cos\theta = \frac{x}{r} = \frac{-ve}{+ve} = -ve, \quad \tan\theta = \frac{y}{x} = \frac{+ve}{-ve} = -ve,$$

$$\cot\theta = \frac{x}{y} = \frac{-ve}{+ve} = -ve, \quad \sec\theta = \frac{r}{x} = \frac{+ve}{-ve} = -ve, \quad \cos ec\,\theta = \frac{r}{y} = \frac{+ve}{+ve} = +ve.$$

$\therefore$ **In the second quadrant, sin θ and cosecθ are positive and all other t-functions are negative.**

Third quadrant. Here *x* and *y* are negative, *r* is positive.

$$\sin\theta = \frac{y}{r} = \frac{-ve}{+ve} = -ve, \quad \cos\theta = \frac{x}{r} = \frac{-ve}{+ve} = -ve, \quad \tan\theta = \frac{y}{x} = \frac{-ve}{-ve} = +ve,$$

$$\cot\theta = \frac{x}{y} = \frac{-ve}{-ve} = +ve, \quad \sec\theta = \frac{r}{x} = \frac{+ve}{-ve} = -ve, \quad \cos ec\,\theta = \frac{r}{y} = \frac{+ve}{-ve} = -ve.$$

$\therefore$ **In the third quadrant, tan θ and cotθ are positive and all other t-functions are negative.**

Fourth quadrant. Here *x* and *y* are positive, *y* is negative.

$$\sin\theta = \frac{y}{r} = \frac{-ve}{+ve} = -ve, \quad \cos\theta = \frac{x}{r} = \frac{+ve}{+ve} = +ve, \quad \tan\theta = \frac{y}{x} = \frac{-ve}{+ve} = -ve,$$

$$\cot\theta = \frac{x}{y} = \frac{+ve}{-ve} = -ve, \quad \sec\theta = \frac{r}{x} = \frac{+ve}{+ve} = +ve, \quad \cos ec\,\theta = \frac{r}{y} = \frac{+ve}{-ve} = -ve.$$

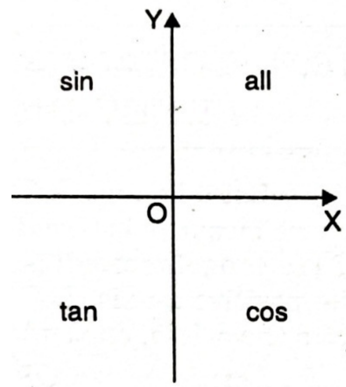
$\therefore$ **In the fourth quadrant, cos θ and secθ are positive and all other t-functions are negative.**

Thus, we see that :

i. In the **first** quadrant, **all** *t*-functions are **positive.**
ii. In the **second** quadrant, only **sine** and its reciprocal **cosecant** are **positive.**

iii. In the **third** quadrant, only **tangent** and its reciprocal **cotangent** are **positive.**

iv. In the **fourth** quadrant, only **cosine** and its reciprocal **secant** are **positive**.



Remark 1. The figure given on the right shows which trigonometric functions (with their reciprocals) are positive in various quadrants.

ALL-SIN-TAN-COS can be easily recalled by remembering that the first letters *A, S, T* and *C* are also the first letters of the phrase 'After school to college'.

Remark 2. When the angle *A* is expressed in degrees then a *t*-ratio of *A* represents the corresponding *t*-ratio of the radian measure of the angle *A*. For example, sine function of 30^0 represents the sine function of $\pi / 6$ and we write $\sin 30^0 = \sin \pi / 6$.

Example 3. *Which of the trigonometric ratios are positive for :*

(i) *240^0* (ii) *-200^0* ?

Sol. (i) 240^0 is in the third quadrant. In the third quadrant, only tan θ and $\cot\theta$ are positive.

$\therefore$ **tan 240^0** and **cot 240^0** are positive.

(ii) – 200^0 lies in the second quadrant.

[$\because$ All angles between – 180^0 and – 270^0 are in the second quadrant]

$\therefore$ **Sin (- 200^0) and cosec (- 200^0)** are positive.

EXERCISE 10.2

SHORT ANSWER TYPE QUESTIONS

1. Which of the following are positive :

(i) sin 240^0 (ii) cos 330^0 (iii) tan 315^0 (iv) sec 315^0 ?

2. Which of the following are negative :

(i) cos 120^0 (ii) cot 210^0 (iii) sec 240^0 (iv) cosec 250^0 ?

3. Which of the following are positive :

(i) cos $(-\pi/3)^c$ (ii) tan $(7\pi/6)^c$ (iii) sec $(2\pi/3)^c$ (iv) cosec $(5\pi/6)^c$?

4. Which of the trigonometric functions are positive for :

(i) 70^0 (ii) 125^0 (iii) -40^0 ?

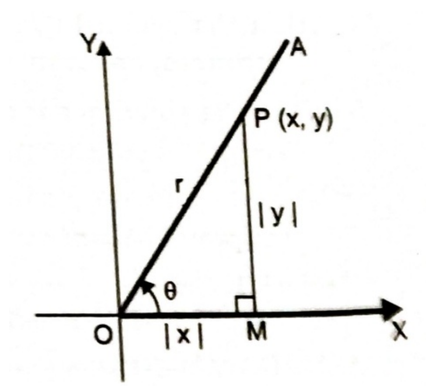
Answers

1. (ii), (iv) 2. (i), (iii). (iv) 3. (i), (ii), (iv) 4. (i) All

(ii) sin 125^0 , cosec 125^0 (iii) cos (-40^0), sec (-40^0).

DOMAIN AND RANGE OF TRIGONOMETRIC FUNCTIONS*

Let θ be any real number. Construct the angle whose measure is θ radius, with vertex at the origin of a rectangular coordinate system and initial line along the positive x – axis. Let $P(x, y)$ be any point, distinct from the origin, one the terminal side OA of the angle.



Let $OP = r.$

(i) $\sin\theta = \dfrac{y}{r}$

For any θ, $-r \leq y \leq r$, $\therefore$ $-1 \leq \dfrac{y}{r} \leq 1$.

$\therefore$ $-1 \leq \sin\theta \leq 1$

$\therefore$ Domain of $\sin\theta$ = **R,** Range of $\sin\theta = [-1, 1]$.

(ii) $\cos\theta = \dfrac{x}{r}$

For any θ, $-r \leq x \leq r$. $\therefore$ $-1 \leq \dfrac{x}{r} \leq 1$. $\therefore$ $-1 \leq \cos\theta \leq 1$.

$\therefore$ Domain of $\cos\theta$ = **R,** Range of $\cos\theta = [-1, 1]$.

(iii) tan θ is defined for any real number θ which is not an **odd** multiple of $\pi/2$ For any such value of θ the ratio y/x can be any real number.

$\therefore$ Domain of tanθ = **R** - $\left[(2n+1)\frac{\pi}{2}; n \in Z\right]$, Range of tan$\theta$= **R**.

(iv) cotθ is defined for any real number θ which is not an **even** multiple of $\pi/2$. For any such value of θ, the ratio x/y can be any real number.

$\therefore$ Domain of cot θ = **R**- $(n\pi : n \in Z)$, Range of cot θ= **R**.

(v) secθ is defined for any real number θ which is not an **odd** multiple of $\pi/2$. For any such value of θ, we have – 1 $\leq x/r \leq$ 1 *i.e.,* $|x/r| \leq 1$.

$$\Rightarrow \quad \frac{1}{|x/r|} \geq 1 \Rightarrow \left|\frac{r}{x}\right| \geq 1 \Rightarrow |\sec\theta| \geq 1.$$

$\therefore$ Domain of secθ = **R** - $\left[(2n+1)\frac{\pi}{2}; n \in Z\right]$, Range of sec$\theta$= **R** – (-1, 1).

(vi) cosecθ is defined for any real number θ which is not an **even** multiple of $\pi 2$. For any such value of θ, we have – 1 $\leq y/r \leq$ 1 *i.e.,* $|y/r| \leq 1$.

$$\Rightarrow \quad \frac{1}{|y/r|} \geq 1 \Rightarrow \left|\frac{r}{y}\right| \geq 1 \Rightarrow |\cos ec\theta| \geq 1.$$

$\therefore$ Domain of cosecθ = **R** - $(n\pi : n \in Z)$, Range of cosec θ= **R** – 1 (-1 , 1).

Remark 1. The maximum (*i.e.,* greater) value of sinθ and cosθ is 1 and the minimum (*i.e.,* least) value is – 1.

$\therefore$ (i) $|\sin\theta| \leq 1$ *i.e.,* $-1 \leq \sin\theta \leq 1$ *i.e.,* $\sin^2\theta \leq 1$.

(ii) $|\cos\theta| \leq 1$ *i.e.,* $-1 \leq \cos\theta \leq 1$ *i.e.,* $\cos^2\theta \leq 1$.

* *If f is a function from A to B, then A is the* ***domain*** *of 'f' and the subset* $(f(x) : x \in A)$ *of B is the* ***range*** *of the function 'f'.*

Remark 2. tanθ and cotθ can take any real number value.

Remark 3. secθ and cosecθ cannot take value in (-1, 1).

$\therefore$ (i) $|\sec\theta| \geq 1$ *i.e.,* $\sec^2\theta \geq 1$.

(ii) $|\text{cosec}\,\theta| \geq 1$ *i.e.,* $\text{cosec}^2\theta \geq 1$.

Example 4. *If the equation 2 $\sin^2\theta$ - $\cos\theta$ + 4 = 0 possible ?*

Sol. Let 2 $\sin^2\theta - \cos\theta + 4 = 0$.

$\therefore$ $2(1 - \cos^2\theta) - \cos\theta + 4 = 0$. $\therefore$ $2\cos^2\theta + \cos\theta - 6 = 0$.

$$\therefore \quad \cos\theta = \frac{-1 \pm \sqrt{1+48}}{4} = \frac{-1 \pm 7}{4} = 2, -\frac{3}{2}$$

Now $|\cos\theta| \le 1$.

$\therefore$ $\cos\theta = 2 \Rightarrow |\cos\theta| = |2| = 2 \ge 1$, which is impossible

and $\cos\theta = -\frac{3}{2} \Rightarrow |\cos\theta| = \left|-\frac{3}{2}\right| = \frac{3}{2} \ge 1$, which is also impossible .

$\therefore$ The given equation is not possible.

EXERCISE 10.4

SHORT ANSWER TYPE QUESTIONS

1. Which of the following are possible :
   (i) $\sin\theta = 1/4$ (ii) $\cos\theta = -2$ (iii) $\tan\theta = -15$
   (iv) $\cot\theta = 1/3$ (v) $\sec\theta = 1/3$ (vi) $\operatorname{cosec}\theta = 2$?
2. (i) Show that $\operatorname{cosec}^2\theta + \sin^2\theta$ can never be less than 2.
   (ii) show that $\tan^2\theta + \cot^2\theta$ can never be less than 2.
3. (i) Show that $\sin\theta + \operatorname{cosec}\theta$ can never be equal in 3/2.
   (ii) show that $\tan\theta + \cot\theta$ can never be equal to 3/2.
4. Is the equation $2\cos^2\theta + \cos\theta - 6 = 0$ possible ?
5. Show that no value of $\sec\theta$ can satisfy the equation $6\sec^2\theta - 5\sec\theta + 1 = 0$
6. Is the equation $\sin\theta = a + \frac{1}{a}$ possible for real value of a ?

Answers

1. (i), (iii), (iv), (vi) 4. No 6. No.

EXPRESSION OF ANY T-RATIO IN TERMS OF A GIVEN T - RATIO

Any trigonometric ratio can be expressed in terms of any other trigonometric ratio by using the following identities:

(i) $\sin\theta \ \text{cosec}\,\theta = 1$ (ii) $\cos\theta \sec\theta = 1$ (iii) $\tan\theta \cot\theta = 1$

(iv) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ (v) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ (vi) $\sin^2\theta + \cos^2\theta = 1$

(vii) $1 + \tan^2\theta = \sec^2\theta$ (viii) $1 + \cot^2\theta = \text{cosec}^2\theta$.

Example 5. *If 4* $\sin\theta = 3$*, find the value of* $\dfrac{\sec\theta + 3\tan\theta}{2\sec\theta - 7\tan\theta}$.

Sol. We have $4\sin\theta = 3$. $\therefore$ $\sin\theta = 3/4$.

$$\frac{\sec\theta + 3\tan\theta}{2\sec\theta - 7\tan\theta} = \frac{\dfrac{1}{\cos\theta} + 3\dfrac{\sin\theta}{\cos\theta}}{\dfrac{2}{\cos\theta} - 7\dfrac{\sin\theta}{\cos\theta}} = \frac{1 + 3\sin\theta}{2 - 7\sin\theta} = \frac{1 + 3\left(\dfrac{3}{4}\right)}{2 - 7\left(\dfrac{3}{4}\right)} = \frac{\dfrac{13}{4}}{-\dfrac{13}{4}} = -1 .$$

Example 6. *If* $\cos A = \dfrac{21}{29}$ *and A lies in the fourth quadrant, find sin A and tan A.*

Sol. We have cos A = 21/29

$\therefore$ $\sin^2 A = 1 - \cos^2 A = 1 - \left(\dfrac{21}{29}\right)^2 = 1 - \dfrac{441}{841} = \dfrac{400}{841}$ $\therefore$ $\sin A = \pm\dfrac{20}{29}$

$\Rightarrow$ $\sin A = -\dfrac{20}{29}$ ($\because$ $\sin A$ is –ve in IV quadrant)

Also $\tan A = \dfrac{\sin A}{\cos A} = \dfrac{-20/29}{21/29} = -\dfrac{20}{21}$.

EXERCISE 10.5

SHORT ANSWER TYPE QUESTIONS

1. If $\sin\theta$ = 1/3 and θ lies in the II quadrant, find the value of $\cos\theta$.
2. If cos θ = 21/29 and θ lies in the I quadrant, find the value of $\sin\theta$.
3. If tan θ = 5 and θ lies in the III quadrant, find the value of $\sec\theta$.
4. If 4 $\sin^2$ 6 = 1, find the value of $\dfrac{2+3\cos^2\theta}{1-2\cos^2\theta}$.
5. If $\sec\theta = \sqrt{2}$ and $\dfrac{3\pi}{2} < \theta < 2\pi$, find the value of $\dfrac{1+\tan\theta+\cos ec\theta}{1+\cot\theta-\cos ec\theta}$.

LONG ANSWER TYPE QUESTIONS

6. If θ lies in the III quadrant and $\cos\theta$ = - 1/2, find the values of other trigonometric functions.
7. If θ lies in the III quadrant and tan θ = 4/3, find the values of other trigonometric functions.
8. If θ lies in the II quadrant and cot x = - 5/12, find the values of other trigonometric functions.

Answers

1. $-\dfrac{2\sqrt{2}}{3}$ 2. $\dfrac{20}{29}$ 3. $-\sqrt{26}$ 4. $-\dfrac{17}{2}$

5. -1

6. $\sin\theta = -\sqrt{3}/2, \tan\theta = \sqrt{3}, \cot\theta = 1/\sqrt{3}, \sec\theta = -2, \cos ec\theta = -2\sqrt{3}$

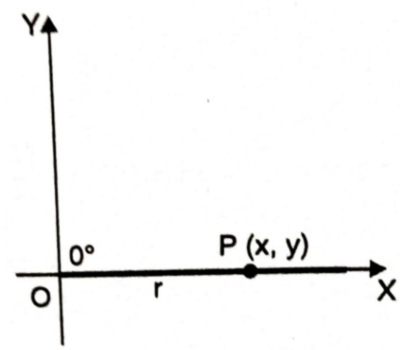
7. $\sin\theta = -4/5, \cos\theta = -3/5, \cot\theta = 3/4, \sec\theta = -5/3, \cos ec\theta = -5/4$

8. $\sin x = 12/13, \cos x = -5/13, \tan x = -12/5, \sec x = -13/5, \cos ecx = 13/12$.

VALUES OF TRIGONOMETRIC FUNCTION S FOR 0^0, 30^0, 45^0, 60^0, 90^0

(i) $\theta = 0^0$. In this case, the terminal side of angle $\theta(=0)^0$ coincides with the initial side *OX*.

Let $P(x, y)$ be any point on the terminal side OX.



Let $OP = r > 0$.

$\therefore \quad x = r, \; y = 0$

$\therefore$ We have **sin 0⁰** = $\frac{y}{r} = \frac{0}{r} = 0$

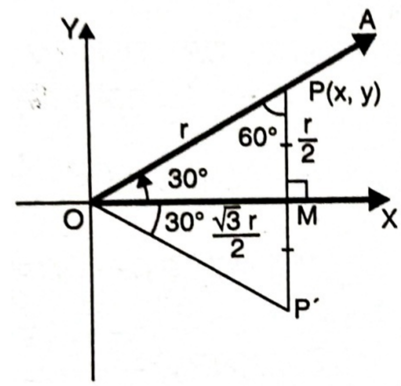
cos 0⁰ = $\frac{x}{r} = \frac{r}{r} = 1$

tan 0⁰ = $\frac{y}{x} = \frac{0}{r} = 0$ **cot 0⁰ is not defined**

sec 0⁰ = $\frac{r}{x} = \frac{r}{r} = 1$ **cosec 0⁰ is not defined.**

(ii) $\theta = 30^0$. Let angle XOA be 30^0.

Let $P(x, y)$ be any point on the terminal side OA.



Let $OP = r > 0$.

Produce PM to P' such that $MP' = MP$.

Δs OMP and OMP' are congruent.

$\therefore \quad \angle P'OM = 30^0$ and so $\Delta OPP'$ is equilateral.

$\therefore \quad OP = PP' = PM + MP'$

$= PM + PM = 2PM$

$\therefore \quad PM = \frac{OP}{2} = \frac{r}{2}$

Now $OM^2 = OP^2 = r^2 - \frac{r^2}{4} = \frac{3r^2}{4}$

$\therefore \quad OM = \frac{\sqrt{3}}{2} r$

$\therefore$ **sin 30⁰** = $\frac{y}{r} = \frac{MP}{OP} = \frac{r/2}{2} = \frac{1}{2}$ **cos 30⁰** = $\frac{OM}{OP} = \frac{\frac{\sqrt{3}}{2}r}{r} = \frac{\sqrt{3}}{2}$

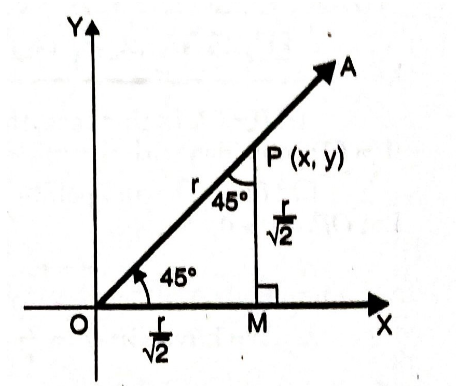
tan 30⁰ = $\frac{MP}{OM} = \frac{r/2}{\frac{\sqrt{3}}{2}r} = \frac{1}{\sqrt{3}}$ **cot 30⁰** = $\frac{OM}{MP} = \frac{\frac{\sqrt{3}}{2}r}{r/2} = \sqrt{3}$

sen 30⁰ = $\frac{OP}{OM} = \frac{r}{\frac{\sqrt{3}}{2}r} = \frac{2}{\sqrt{3}}$ **cosec 30⁰** = $\frac{OP}{MP} = \frac{r}{\frac{r}{2}} = 2.$

(iii) $\theta = 45^0$. Let angle XOA be 45^0.

Let $P(x, y)$ be any point on the terminal side OA. Let $OP = r > 0$.

Triangle OMP is isosceles. Also $OM^2 + MP^2 = OP^2$.



$\therefore$ $2OM^2 = OP^2$ or $OM = \sqrt{\frac{r^2}{2}} = \frac{r}{\sqrt{2}}$

$\therefore$ MP is also $\frac{r}{\sqrt{2}}$.

$\therefore$ $\sin 45^0 = \frac{MP}{OP} = \frac{r/\sqrt{2}}{r} = \frac{1}{\sqrt{2}}$

$\cos 45^0 = \frac{OM}{OP} = \frac{r/\sqrt{2}}{r} = \frac{1}{\sqrt{2}}$

It is easy to see that tan 45⁰ = 1, cot 45⁰ = 1, sec 45⁰ = $\sqrt{2}$, cosec 45⁰ = $\sqrt{2}$.

(iv) $\theta = 60^0$. Let angle XOA be 60^0.

Let $P(x, y)$ be any point on the terminal side OA. Let $OP = r > 0$. Let M' be on OX such that $MM' = OM$.

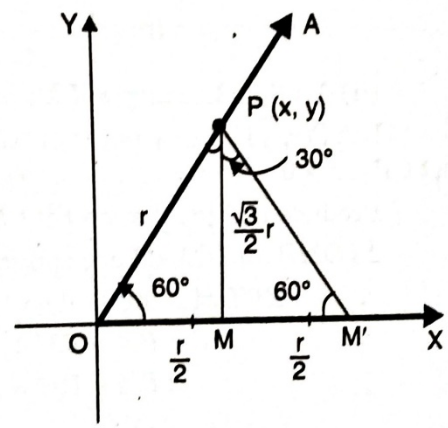
Δs OMP and $M'MP$ are congruent.

$\therefore \angle MM'P = 60^0$ and so $\Delta OM'P$ is equilateral.

$\therefore$ $OP = OM' = OM + MM'$

$= OM + OM = 2OM$

$\therefore \quad OM = \frac{1}{2} OP = \frac{r}{2}$



Also $MP^2 = OP^2 - OM^2 = r^2 - \frac{r^2}{4} = \frac{3r^2}{4}$

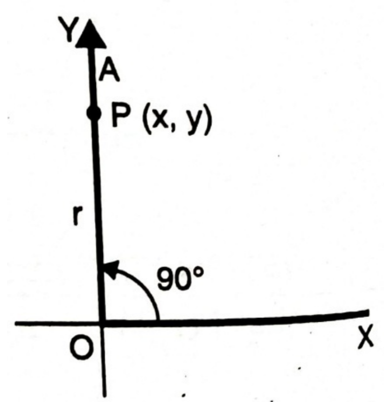
$\therefore \quad MP = \frac{\sqrt{3}}{2} r$

$\therefore$ **sin 60⁰** $= \frac{MP}{OP} = \frac{\frac{\sqrt{3}}{2} r}{r} = \frac{\sqrt{3}}{2}$

cos 60⁰ $= \frac{OM}{OP} = \frac{\frac{r}{2}}{r} = \frac{1}{2}$

It is easy to see that $\tan 60^0 = \sqrt{3}$, $\cot 60^0 = \frac{1}{\sqrt{3}}$, $\sec 60^0 = 2$, $\text{cosec}\, 60^0 = \frac{2}{\sqrt{3}}$.

(v) $\theta = 90^0$. Let angle *XOA* be 90^0. In this case, the terminal side coincide with *OY*. Let *P*(*x, y*) be any point on the terminal side *OY*.



Let $\quad OP = r > 0$

$\therefore \quad x = 0,\ y = r$

$\therefore$ We have $\sin 90^0 = \frac{y}{r} = \frac{r}{r} = 1 \quad \cos 90^0 = \frac{x}{r} = \frac{0}{r} = 0$

tan 90⁰ is not defined $\qquad \cos 90^0 = \frac{x}{y} = \frac{0}{r} = 0$

sec 90⁰ is not defined $\qquad \text{cosec}\ 90^0 = \frac{x}{y} = \frac{r}{r} = 1.$

Remark. The values of *t*- rations for $\theta = 0^0, 30^0, 45^0, 60^0, 90^0$ can be easily recalled, by using the following table:

	0^0	30^0	45^0	60^0	90^0
sin	$\frac{\sqrt{0}}{2} = 0$	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2} = 1$

cos	$\frac{\sqrt{4}}{2}=1$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}}$	$\frac{\sqrt{1}}{2}=\frac{1}{2}$	$\frac{\sqrt{0}}{2}=0$

The values of the remaining *t*-ratios can be found by using the identities:

$$\tan\theta=\frac{\sin\theta}{\cos\theta},\quad \cot\theta=\frac{\cos\theta}{\sin\theta},\quad \sec\theta=\frac{1}{\cos ec\theta},\quad \cos ec\theta=\frac{1}{\sin\theta}.$$

Example 7. *Show that:* $\sec^2 30^0 + \text{cosec}^2 45^0 + \cot^2 60^0 + \sin^2 90^0 = \frac{14}{3}$.

Sol. We have $\sec 30^0 = \frac{2}{\sqrt{3}}$, $\text{cosec } 45^0 = \sqrt{2}$, $\cot 60^0 = \frac{1}{\sqrt{3}}$, $\sin 90^0 = 1$.

$\therefore$ $\sec^2 30^0 + \text{cosec}^2 45^0 + \cot^2 60^0 + \sin^2 90^0$

$$=\left(\frac{2}{\sqrt{3}}\right)^2+\left(\sqrt{2}\right)^2+\left(\frac{1}{\sqrt{3}}\right)^2+(1)^2=\frac{4}{3}+2+\frac{1}{3}+1=\frac{14}{3}.$$

EXERCISE 10.6

SHORT ANSWER TYPE QUESTIONS

1. If $A = 60^0$ and $B = 30^0$, verify that:

   (i) $\sin(A+B)=\sin A\cos B+\cos A\sin B$ (ii) $\cos(A+B)=\cos A\cos B-\sin A\sin B$

   (iii) $\sin(A-B)=\sin A\cos B-\cos A\sin B$ (iv) $\cos(A-B)=\cos A\cos B+\sin A\sin B$.

LONG ANSWER TYPE QUESTIONS

2. Find the value of $\theta(0^0\le\theta\le 90^0)$ from the equations:

   (i) $2\cos^2\theta-5\cos\theta+2=0$ (ii) $2\cos^2\theta=3\sin\theta$

   (iii) $3\sec^4\theta-10\sec^2\theta+8=0$ (iv) $3\tan\theta+\cot\theta=5\cos ec\theta$.

3. Show that :

   (i) $\sin^2 0^0,\sin^2 30^0,\sin^2 45^0,\sin^2 60^0$ are in A.P.

   (ii) $\cos^2 0^0,\cos^2 30^0,\cos^2 45^0,\cos^2 60^0$ are in A.P.

Answers

2. (i) 60^0 (ii) 30^0 (iii) 30^0 or 45^0 (iv) 60^0

T-RATIOS OF ALLIED ANGLES

An angle is said to be **allied to another angle** if either :

(i) their sum is zero

or (ii) their sum or difference is a multiple of $\pi / 2$.

For example, - θ is allied to angle θ $[\because \ -\theta + \theta = 0]$

$\frac{\pi}{2} + \theta$ is allied to angle θ is $\left[\therefore \left(\frac{\pi}{2} + \theta\right) - \theta = \frac{\pi}{2}\right]$

180^0 - θ is allied to angle θ [$\because$ (180^0 - θ) + θ = 180^0 = 2 X 90^0]

The following are the results to find the *t*-ratios of angles called to θ, in terms of *t*-ratios of θ.

Result. (i) If θ is a positive acute angle, then

sin (- θ) = - sin θ, cos (-θ) = cos θ, tan (-θ) = - tan θ.

cot (- θ) = - cot θ, sec (-θ) = sec θ, cosec (-θ) = - cosec θ.

Result. (ii) If θ is a positive acute angle, then

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta, \quad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta, \quad \tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta.$$

$$\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta, \quad \sec\left(\frac{\pi}{2} - \theta\right) = \operatorname{cosec}\theta, \quad \operatorname{cosec}\left(\frac{\pi}{2} - \theta\right) = \sec\theta.$$

Result. (iii) If θ is a positive acute angle, then

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta, \quad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta, \quad \tan\left(\frac{\pi}{2} + \theta\right) = \cot\theta.$$

$$\cot\left(\frac{\pi}{2} + \theta\right) = -\tan\theta, \quad \sec\left(\frac{\pi}{2} + \theta\right) = \operatorname{cosec}\theta, \quad \operatorname{cosec}\left(\frac{\pi}{2} + \theta\right) = \sec\theta.$$

Result. (iv) If θ is a positive acute angle, then

$\sin(\pi-\theta) = \sin\theta$, $\cos(\pi-\theta) = -\cos\theta$, $\tan(\pi-\theta) = -\tan\theta$

$\cot(\pi-\theta) = -\cot\theta$, $\sec(\pi-\theta) = -\sec\theta$, $\operatorname{cosec}(\pi-\theta) = \operatorname{cosec}\theta$.

Result. (v) If θ is a positive acute angle, then

$\sin(\pi+\theta) = -\sin\theta$, $\cos(\pi+\theta) = -\cos\theta$, $\tan(\pi+\theta) = \tan\theta$

$\cot(\pi+\theta) = -\cot\theta$, $\sec(\pi+\theta) = -\sec\theta$, $\operatorname{cosec}(\pi+\theta) = -\operatorname{cosec}\theta$.

Result (vi) If θ is a positive acute angle, then

$\sin\left(\frac{3\pi}{2}=\theta\right) = -\cos\theta$, $\cos\left(\frac{3\pi}{2}=\theta\right) = -\sin\theta$, $\tan\left(\frac{3\pi}{2}=\theta\right) = \cot\theta$,

$\cot\left(\frac{3\pi}{2}=\theta\right) = \tan\theta$, $\sec\left(\frac{3\pi}{2}=\theta\right) = -\operatorname{cosec}\theta$, $\operatorname{cosec}\left(\frac{3\pi}{2}=\theta\right) = -\sec\theta$,

Result. (vii) If θ is a positive acute angle, then

$\sin\left(\frac{3\pi}{2}+\theta\right) = -\cos\theta$, $\cos\left(\frac{3\pi}{2}+\theta\right) = \sin\theta$, $\tan\left(\frac{3\pi}{2}+\theta\right) = -\cot\theta$,

$\cot\left(\frac{3\pi}{2}+\theta\right) = \tan\theta$, $\sec\left(\frac{3\pi}{2}+\theta\right) = \operatorname{cosec}\theta$, $\operatorname{cosec}\left(\frac{3\pi}{2}+\theta\right) = -\sec\theta$,

Result. (viii) If θ is a positive acute angle, then

$\sin(2\pi-\theta) = -\sin\theta$, $\cos(2\pi-\theta) = \cos\theta$, $\tan(2\pi-\theta) = -\tan\theta$,

$\cot(2\pi-\theta) = -\cot\theta$, $\sec(2\pi-\theta) = \sec\theta$, $\operatorname{cosec}(2\pi-\theta) = -\operatorname{cosec}\theta$

Result. (ix) If θ is any angle, then the t-ratios of 2n$\pi+\theta$ are same as that of θ.

Remark 1. Since sin(-θ) = - sinθ, so the trigonometric function sinθ is an *odd* function. Also cos(-θ) = cosθ implies that the trigonometric function cosθ is an *even* function.

Remark 2. Theorems (i) – (ix) also holds good for any θ: $-\infty < \theta < \infty$.

Remark 3. If the angles are expressed in degrees, then (i) in case of allied angles -θ, 180^0 - θ, 180^0 + θ, 360^0 - θ; the *t*-ratio remains the same (ii) in case of allied angles 90^0 - θ, 90^0 + θ, 270^0 - θ, 270^0 + θ, then *t*-ratio is changed as : sin $\leftrightarrow$ cos, tan $\leftrightarrow$ cot, sec $\leftrightarrow$ cosec.

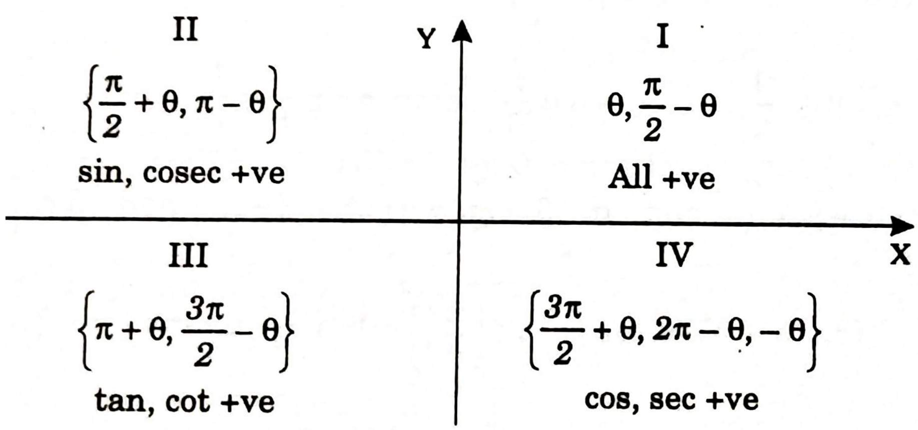
WORKING RULES TO EXPRESS T-RATIOS OF ALLIED ANGLES OF θ IN TERMS OF T-RATIOS OF θ

Step I. *If the angle is negative, use t-ratios of '- θ' to make the angle positive.*

For example, we write sin (-240^0) = - sin 240^0.

Step II. *If the angle is greater than 2π (i.e., 360^0), subtract the greatest possible multiple of 2π from it, remembering that the t-ratios of $(n(2\pi)+\theta)$ are exactly the same as those of θ.*

Step III. *Consider θ lying in the first quadrant (even if it actually does not lie) and find the quadrant in which the allied angle lie. Then determine the sign of the given t-ratio of the allied angle by the rule show in the figure given below :*



Step IV. *(i) Now, if allied angle is $-\theta, \pi-\theta, \pi+\theta, 2\pi-\theta$; t-ratio remains the same i.e., sine remains since, consine remains consines and so on.*

(ii) If allied angle is $\frac{\pi}{2}-\theta, \frac{\pi}{2}+\theta, \frac{3\pi}{2}-\theta, \frac{3\pi}{2}+\theta$; the t-ratio is changed as follows :

since changes to cosine and vice-versa ; tangent changes to cotangent and vice-versa ; secant changes to consecant and vice-versa.

Example 8. *Find the values of :*

(i) *sin* $(90^0 + \theta)$ (ii) *cos* $(180^0 - \theta)$ (iii) *tan* $(270^0 - \theta)$ (iv) *sec* $(180^0+\theta)$.

Sol. (i) $90^0 + \theta$ involves 90^0, the *t*-ratio sine is changed to cosine. Also, assuming θ to be in I quadrant, $90^0 + \theta$ lies in II quadrant and in this quadrant sine is +ve.

$\therefore$ **sin** $(\mathbf{90^0} + \theta) =$ **cos** θ.

(ii) $180^0 - \theta$ involves 180^0, the *t*-ratio cosine remain same. Also, assuming θ to be in I quadrant, $180^0 - \theta$ lies in II quadrant and in this quadrant, cosine is –ve.

$\therefore$ **cos** $(\mathbf{180^0} - \theta) =$ **- cos** θ .

(iii) $270^0 - \theta$ involves 270^0, the *t*-ratio tangent is changed to cotangent. Also, assuming θ to be in I quadrant, $270^0 - \theta$ lies in III quadrant and in this quadrant tangent is +ve.

$\therefore$ **tan** $(\mathbf{270^0} - \theta) =$ **cot** θ .

(iv) $180^0+\theta$ involves 180^0, the *t*-ratio secant remains same. Also, assuming θ to be in I quadrant, $180^0 - \theta$ lies in III quadrant and in this quadrant secant is -ve.

$\therefore$ **sec** $(\mathbf{180^0} - \theta) =$ **sec** θ

Example 9. *If sin* $(\alpha+\beta) = 1$ *and sin* $(\alpha-\beta) = \frac{1}{2}$*, where* $0 \le \alpha, \beta \le \frac{\pi}{2}$*, then find the values of tan* $(\alpha+2\beta)$ *and tan* $(2\alpha+\beta)$.

Sol. We have $0 \le \alpha \le \frac{\pi}{2}$ and $0 \le \alpha + \beta \le \frac{\pi}{2}$.

$$\therefore \quad 0 \le \alpha + \beta \le \frac{\pi}{2} + \frac{\pi}{2} \quad i.e., \quad 0 \le \alpha + \beta \le \pi$$

Also $0 \le \beta \le \frac{\pi}{2} \Rightarrow 0 \ge -\beta \ge -\frac{\pi}{2} \Rightarrow -\frac{\pi}{2} \le -\beta \le 0$

$$\therefore \quad 0 + \left(-\frac{\pi}{2}\right) \le \alpha + (-\beta) \le \frac{\pi}{2} + 0 \quad \text{or} \quad -\frac{\pi}{2} \le \alpha - \beta \le \frac{\pi}{2}$$

$\sin(\alpha+\beta)=1 \Rightarrow \alpha+\beta=\frac{\pi}{2}$, because $\sin\frac{\pi}{2}=1$ and $0\leq\alpha+\beta\leq\pi$

$\sin(\alpha-\beta)=\frac{1}{2} \Rightarrow \alpha-\beta=\frac{\pi}{6}$, because $\sin\frac{\pi}{6}=\frac{1}{2}$, and $-\frac{\pi}{2}\leq\alpha-\beta\leq\frac{\pi}{2}$

Solving $\alpha+\beta=\frac{\pi}{2}$ and $\alpha-\beta=\frac{\pi}{6}$, we get, $\alpha=\frac{\pi}{3}$ and $\beta=\frac{\pi}{6}$.

$\therefore \quad \tan(\alpha+2\beta)=\tan\left(\frac{\pi}{3}+2\left(\frac{\pi}{6}\right)\right)=\tan\frac{2\pi}{3}=\tan\left(\pi-\frac{\pi}{3}\right)$

$$=-\tan\frac{\pi}{3}=-\sqrt{3}$$

and $\quad \tan(2\alpha+\beta)=\tan\left(2\left(\frac{\pi}{3}\right)+\frac{\pi}{6}\right)=\tan\frac{5\pi}{6}=\tan\left(\pi-\frac{\pi}{6}\right)=-\tan\frac{\pi}{6}=-\frac{1}{\sqrt{3}}$

EXERCISE 10.7

SHORT ANSWER TYPE QUESTIONS

Evaluate:

1. (i) $\sin 120^0$ (ii) $\cos 150^0$ (iii) $\tan 240^0$
   (iv) $\cot(-30^0)$ (v) $\sec(-60^0)$ (vi) $\operatorname{cosec}(-90^0)$.
2. (i) $\sin 765^0$ (ii) $\tan 270^0$ (iii) $\cos\frac{5\pi}{6}$
   (iv) $\cos 750^0$ (v) $\tan(-480^0)$ (vi) $\operatorname{cosec}(-1410^0)$.
3. (i) $\sin\frac{31\pi}{3}$ (ii) $\sin\left(-\frac{11\pi}{3}\right)$ (iii) $\sec\frac{25\pi}{6}$
   (iv) $\tan\left(-\frac{15\pi}{4}\right)$ (v) $\cot\left(-\frac{15\pi}{4}\right)$ (vi) $\operatorname{cosec}\frac{11\pi}{4}$.
4. In any quadrilateral *ABCD*, show that:
   (i) $\sin(A+B)+\sin(C+D)=0$ (ii) $\cos(A+B)=\cos(C+D)$.
5. If *ABCD* be a cyclical quadrilateral, show that
   $\cos A+\cos B+\cos C+\cos D=0$

Answers

1. (i) $\sqrt{3}/2$ (ii) $-\sqrt{3}/2$ (iii) $\sqrt{3}$ (iv) $-\sqrt{3}$

   (v) 2 (vi) – 1

2. (i) $1/\sqrt{2}$ (ii) Not defined

   (iii) $-\sqrt{3}/2$ (iv) $\sqrt{3}/2$ (v) $\sqrt{3}$ (vi) 2

3. (i) $\sqrt{3}/2$ (ii) $\sqrt{3}/2$ (iii) $2/\sqrt{3}$ (iv) $1/\sqrt{2}$

   (v) 1 (vi) $\sqrt{2}$

SUMMARY

1. (i) $\sin\theta \ \text{cosec}\theta = 1$ (ii) $\cos\theta \sec\theta = 1$

(iii) $\tan\theta \cot\theta = 1$ (iv) $\frac{\sin\theta}{\cos\theta} = \tan\theta$

(v) $\frac{\cos\theta}{\sin\theta} = \cot\theta$ (vi) $\sin^2\theta + \cos^2\theta = 1$

(vii) $1 + \tan^2\theta = \sec^2\theta$ (viii) $1 + \cot^2\theta = \text{cosec}^2\theta$.

2. (i) $\sin 0^0 = 0$, $\cos 0^0 = 1$, $\tan 0^0 = 0$, $\cot 0^0 =$ not defined, $\sec 0^0 = 1$, $\text{cosec } 0^0 =$ not defined.

(ii) $\sin 30^0 = \frac{1}{2}$, $\cos 30^0 = \frac{\sqrt{3}}{2}$, $\tan 30^0 = \frac{1}{\sqrt{3}}$, $\cot 30^0 = \sqrt{3}$, $\sec 30^0 = \frac{2}{\sqrt{3}}$, $\text{cosec } 30^0 = 2$.

(iv) $\sin 60^0 = \frac{\sqrt{3}}{2}$, $\cos 60^0 = \frac{1}{2}$, $\tan 60^0 = \sqrt{3}$, $\cot 60^0 = \frac{1}{\sqrt{3}}$, $\sec 60^0 = 2$, $\text{cosec } 60^0 = \frac{2}{\sqrt{3}}$.

(v) $\sin 90^0 = 1$, $\cos 90^0 = 0$, $\tan 90^0 =$ not defind, $\cot 90^0 = 0$, $\sec 90^0 =$ not defined, $\text{cosec } 90^0 = 1$.

3. (i) $|\sin\theta| \le 1$ *i.e.,* $\sin^2\theta \le 1$ *i.e.,* $-1 \le \sin\theta \le 1$

(ii) $|\cos\theta| \le 1$ *i.e.,* $\cos^2\theta \le 1$ *i.e.,* $-1 \le \cos\theta \le 1$

(iii) $-\infty < \tan\theta < \infty$

(iv) $-\infty < \cot\theta < \infty$

(v) $|\sec\theta| \ge 1$ *i.e.,* $\sec^2\theta \ge 1$ *i.e.,* $\sec\theta \le -1$ or $\sec\theta \ge 1$

(ii) $|\cos\theta| \le 1$ *i.e.,* $\cos^2\theta \le 1$ *i.e.,* $\text{cosec}\theta \le -1$ or $\text{cosec}\theta \ge 1$.

TEST YOURSELF

1. Prove the following identities:

(i) $$\left(\frac{1}{\sec^2\theta - \cos^2\theta} + \frac{1}{\cos ec^2\theta - \sin^2\theta}\right)\sin^2\theta\cos^2\theta = \frac{1-\sin^2\theta\cos^2\theta}{2+\sin^2\theta\cos^2\theta}$$

(ii) $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}}+\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}=-2\sec\theta, \theta\in\left(\frac{\pi}{2},\pi\right)$.

2. If $\frac{ax}{\cos\theta}+\frac{by}{\sin\theta}=a^2-b^2$ and $\frac{ax\sin\theta}{\cos^2\theta}-\frac{by\cos\theta}{\sin^2\theta}=0$, show that $(ax)^{2/3}+(bx)^{2/3}$ $=(a^2-b^2)^{2/3}$

3. If $\cot\theta=\frac{7}{24}$ and θ lies in the third quadrant: find the values of $\cos\theta$ - $\sin\theta$.

4. Find the value of θ ($270^0 \le \theta \le 360^0$) from the equations :

   (i) $2\sin^2\theta + 3\cos\theta - 3 = 0$ (ii) $2\cos^2\theta - 5\cos\theta + 2 = 0$.

5. Find the solution of the equation $\tan^2 x + \cot^2 x = 2$, where x lies between 0^0 and 180^0.

Answers

3. $\frac{17}{25}$ 4. (i) 300^0 or 360^0 (ii) 300^0 5. 45^0, 135^0.

SECTION – B

11. TRIGONOMETRIC FUNCTIONS OF SUM AND DIFFERENCE OF TWO ANGLES

LEARNING OBJECTIVES

- Introduction
- Trigonometric Functions of Sum of Two Angles
- Trigonometric Functions of Difference of Two Angles

INTRODUCTION

In this chapter, we shall study the methods of finding the values of trigonometric functions for sum and difference of angles. These formulae would help us to find the values such as sin 15^0, cos105^0 etc. The formulae for finding the values of trigonometric functions for sum of angles and for difference of angles are called **addition formulae** and **subtraction formula** respectively.

TRIGONOMETRIC FUNCTIONS OF SUM OF SUM OF TWO ANGLES

We have following formulae :

(i) **sin (A + B) = sin A cos B + cos A sin B**

(ii) **cos (A + B) = cos A cos B – sin A sin B**

(iii) **tan (A + B) =** $\frac{\tan A + \tan B}{1 - \tan A \tan B}$

These are called **addition formulae.**

TRIGONOMETRIC FUNCTIONS OF DIFFERENCE OF TWO ANGLES

We have following formulae:

(i) **sin (A – B) = sin A cos B – cos A sin B**

(ii) **cos (A – B) = cos A cos B + sin A sin B**

(iii) **tan (A – B)** $= \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

There are called **subtraction formulae.**

Theorem I. By using sine and cosine formulae prove that:

(i) **tan (A + B)** $= \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ (ii) **tan (A – B)** $= \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

Proof. (i) $\tan(A + B) = \dfrac{\sin(A+B)}{\cos(A+B)} = \dfrac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$

$$= \dfrac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}} = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$$

$\therefore$ **tan (A + B)** $= \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(ii) $\tan(A - B) = \dfrac{\sin(A-B)}{\cos(A-B)} = \dfrac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}$

$$= \dfrac{\dfrac{\sin A \cos B}{\cos A \cos B} - \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} + \dfrac{\sin A \sin B}{\cos A \cos B}} = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$$

$\therefore$ **tan (A – B)** $= \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

Corollary. (i) **tan (45⁰ + A)*** $= \dfrac{1 + \tan A}{1 - \tan A}$

(ii) **tan (45⁰ - A)** $= \dfrac{1-\tan A}{1+\tan A}$.

Proof. (i) $$\tan(45^0 + A) = \frac{\tan 45^0 + \tan A}{1-\tan 45^0 \tan A} = \frac{1+\tan A}{1-1.\tan A} = \frac{1+\tan A}{1-\tan A}$$

(ii) $$\tan(45^0 - A) = \frac{\tan 45^0 - \tan A}{1+\tan 45^0 \tan A} = \frac{1-\tan A}{1+1.\tan A} = \frac{1-\tan A}{1+\tan A}.$$

Theorem II. By using sine and cosine formulae prove that:

(i) **cot (A + B)** $= \dfrac{\cot A \cot B - 1}{\cot B + \cot A}$ (ii) **cot (A – B)** $= \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$.

Proof. (i) $$\cot(A+B) = \frac{\cot(A+B)}{\cot(A+B)} = \frac{\cos A \cos B - \sin A \sin B}{\sin A \cos B + \cos A \sin B}$$

** tan (45⁰ + A) represents the tangent function of the radian measure of the angle 45⁰ + A.*

$$= \frac{\dfrac{\cos A \cos B}{\sin A \sin B} - \dfrac{\sin A \sin B}{\sin A \sin B}}{\dfrac{\sin A \cos B}{\sin A \sin B} + \dfrac{\cos A \sin B}{\sin A \sin B}} = \frac{\cot A \cot B - 1}{\cot B + \cot A}$$

∴ **cot (A + B) =** $\dfrac{\cot A \cot B - 1}{\cot B + \cot A}$.

(ii) $$\cot(A-B) = \frac{\cot(A-B)}{\cot(A-B)} = \frac{\cos A \cos B + \sin A \sin B}{\sin A \cos B - \cos A \sin B}$$

$$= \frac{\dfrac{\cos A \cos B}{\sin A \sin B} + \dfrac{\sin A \sin B}{\sin A \sin B}}{\dfrac{\sin A \cos B}{\sin A \sin B} - \dfrac{\cos A \sin B}{\sin A \sin B}} = \frac{\cot A \cot B + 1}{\cot B - \cot A}$$

∴ **cot (A - B) =** $\dfrac{\cot A \cot B + 1}{\cot B - \cot A}$.

Theorem III. Prove that :

(i) **sin (A + B) sin (A – B) = sin² A – sin² B**

(ii) **cos (A + B) cos (A – B) = cos² A – sin² B.**

Proof. (i) $\sin (A + B) \sin (A - B)$

$= (\sin A \cos B + \cos A \sin B) (\sin A \cos B - \cos A \sin B)$

$= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B = \sin^2 A (1 - \sin^2 B) - (1 - \sin^2 A) \sin^2 B.$

$= \sin^2 A - \sin^2 A \sin^2 B - \sin^2 B + \sin^2 A \sin^2 B = \sin^2 A - \sin^2 B$

$\therefore$ **$\sin (A + B) \sin (A - B) = \sin^2 A - \sin^2 B$.**

(ii) $\cos (A + B) \cos (A - B) = (\cos A \cos B - \sin A \sin B) (\cos A \cos B + \sin A \sin B)$

$= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B = \cos^2 A (1 - \sin^2 B) - (1 - \cos^2 A) \sin^2 B$

$= \cos^2 A - \cos^2 A \sin^2 B - \sin^2 B + \cos^2 A \sin^2 B = \cos^2 A - \sin^2 B.$

$\therefore$ **$\cos (A + B) \cos (A - B) = \cos^2 A - \sin^2 B$.**

Corollary 1. $\sin (A + B) \sin (A - B) = \cos^2 B - \cos^2 A$.

Proof. We have $\sin (A+ B) \sin (A - B) = \sin^2 A - \sin^2 B$

$= (1 - \cos^2 A) - (1 - \cos^2 B)$

$= \cos^2 B - \cos^2 A.$

Corollary 2. $\cos (A + B) \cos (A - B) = \cos^2 B - \sin^2 A$.

Proof. We have $\cos (A+ B) \cos (A - B) = \cos^2 A - \sin^2 B$

$= (1 - \sin^2 A) - (1 - \cos^2 B)$

$= \cos^2 B - \sin^2 A.$

WORKING RULES FOR SOLVING PROBLEMS

Rule I. (i) $\sin (A + B) = \sin A \cos B + \cos A \sin B$

(ii) $\cos (A + B) = \cos A \cos B - \sin A \sin B$

(iii) $\tan (A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(iv) $\cot (A + B) = \dfrac{\cot A \cot B - 1}{\cot B + \cot A}$.

Rule II. (i) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

(ii) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

(iii) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(iv) $\cot(A - B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

Rule III. (i) $\tan(45^0 + A) = \dfrac{1 + \tan A}{1 - \tan A}$ (ii) $\tan(45^0 - A) = \dfrac{1 - \tan A}{1 + \tan A}$.

Rule IV. (i) $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$

(ii) $\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B$.

Example 1. *Calculate sin 105⁰ and cos 15⁰.*

Sol. Sol. $\sin 105^0 = \sin(60^0 + 45^0) = \sin 60^0 \cos 45^0 + \cos 60^0 \sin 45^0$

$$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

$\cos 15^0 = \cos(45^0 - 30^0) = \cos 45^0 \cos 30^0 + \sin 45^0 \sin 30^0$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

Remark. The values of sin 105⁰ and cos 15⁰ are equal, because

$\text{Sin } 105^0 = \sin(90^0 + 15^0) = \cos 15^0$.

Example 2. *If α, β are the roots of a cos $\theta + b \sin\theta = c$, then show that:*

(i) $\cos(\alpha + \beta) = \dfrac{a^2 - b^2}{a^2 + b^2}$ (ii) $\cos(\alpha - \beta) = \dfrac{2c^2 - (a^2 + b^2)}{a^2 + b^2}$.

Sol. We have $a \cos\theta + b \sin\theta = c$. ....(1)

$\Rightarrow$ $a \cos\theta = c - b \sin\theta$ $\Rightarrow$ $a^2 \cos^2\theta = (c - b\sin\theta)^2$

$\Rightarrow \quad a^2(1-\sin^2\theta) - c^2 - b^2\sin^2\theta + 2bc\sin\theta = 0$

$\Rightarrow \quad (a^2+b^2)\sin^2\theta - 2abc\sin\theta + c^2 - a^2 = 0$ .....(2)

We are given that α, β are roots of (1).

$\therefore$ $\sin\alpha$ and $\sin\beta$ are roots of quadratic equation (2) in $\sin\theta$.

$\therefore \quad \sin\alpha \ \sin\beta = \dfrac{c^2-a^2}{a^2+b^2}$ ....(3)

Also, (1) $\Rightarrow \quad b\sin\theta = c - a\cos\theta$ (Note this step)

$\Rightarrow \quad b^2\sin^2\theta = (c - a\cos\theta)^2$

$\Rightarrow \quad b^2(1-\cos^2\theta) = c^2 + a^2\cos^2\theta - 2ac\cos\theta$

$\Rightarrow \quad (a^2+b^2)\cos^2\theta - 2ac\cos\theta + c^2 - a^2 = 0$ ....(4)

The roots of (4) are $\cos\alpha$ and $\cos\beta$.

$\therefore \quad \cos\alpha\cos\beta = \dfrac{c^2-b^2}{a^2+b^2}$ ....(5)

$\therefore$ **Using (4) and (5)**, we get

(i) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \dfrac{c^2-b^2}{a^2+b^2} - \dfrac{c^2-a^2}{a^2+b^2} = \dfrac{a^2-b^2}{a^2+b^2}$.

(ii) $\cos(\alpha-\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \dfrac{c^2-b^2}{a^2+b^2} + \dfrac{c^2-a^2}{a^2+b^2} = \dfrac{2c^2-(a^2+b^2)}{a^2+b^2}$.

EXERCISE 11.1

SHORT ANSWER TYPE QUESTIONS

Evaluate:

1. (i) $\sin 12^0 \cos 18^0 + \cos 18^0 \sin 12^0$ (ii) $\sin 70^0 \cos 10^0 - \cos 70^0 \sin 10^0$

   (ii) $\cos 40^0 \cos 20^0 - \sin 40^0 \sin 20^0$ (iv) $\cos 68^0 \cos 38^0 + \sin 68^0 \sin 38^0$

   (v) $\dfrac{\tan 69^0 + \tan 66^0}{1 - \tan 69^0 \tan 66^0}$ (vi) $\dfrac{\tan 35^0 + \tan 5^0}{1 - \tan 35^0 \tan 5^0}$

2. (i) $\sin 75^0$ (ii) $\cos 75^0$

(iii) cos 105^0 (iv) sin 15^0

(v) tan 15^0 (vi) tan $13\pi/12$.

LONG ANSWER TYPE QUESTIONS

Show that:

3. (i) $\cos\left(\frac{\pi}{4}-A\right)\cos\left(\frac{\pi}{4}-B\right)-\sin\left(\frac{\pi}{4}-A\right)\sin\left(\frac{\pi}{4}-B\right)=\sin(A+B)$

(ii) $\sin(60^0+A)\cos(30^0-B)+\cos(60^0+A)\sin(30^0-B)=\cos(A-B)$

(iii) $\sin(n+1)\theta\sin(n-1)\theta+\cos(n+1)\theta\cos(n-1)\theta=\cos 2\theta$

(iv) $\frac{\tan(A-B)+\tan B}{1-\tan(A-B)\tan B}=\tan A$ (v) $\cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)=\sqrt{2}\cos x$.

4. $\sin\frac{7\pi}{12}\cos\frac{\pi}{4}-\cos\frac{7\pi}{12}\sin\frac{\pi}{12}=\frac{1}{2}$ (ii) $\sin\frac{\pi}{4}\cos\frac{\pi}{12}+\cos\frac{\pi}{4}\sin\frac{\pi}{12}=\frac{\sqrt{3}}{2}$.

Answers

1. (i) $\frac{1}{2}$ (ii) $\frac{\sqrt{3}}{2}$ (iii) $\frac{1}{2}$ (iv) $\frac{\sqrt{3}}{2}$

(v) -1 (vi) $\frac{1}{\sqrt{3}}$ 2. (i) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ (ii) $\frac{\sqrt{3}-1}{2\sqrt{2}}$

(iii) $\frac{1-\sqrt{3}}{2\sqrt{2}}$ (iv) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ (v) $2-\sqrt{3}$ (vi) $2-\sqrt{3}$

SUMMARY

1. (i) sin $(A + B)$ = sin A cos B + cos A sin B
   (ii) sin $(A - B)$ = sin A cos B - cos A sin B
2. (i) cos $(A + B)$ = cos A cos B - sin A sin B
   (ii) cos $(A - B)$ = cos A cos B + sin A sin B
3. (i) $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$ (ii) $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$
4. (i) $\tan(45^0+A)=\dfrac{1+\tan A}{1-\tan A}$ (ii) $\tan(45^0-A)=\dfrac{1-\tan A}{1+\tan A}$
5. (i) $\cot(A+B)=\dfrac{\cot A\cot B-1}{\cot B+\cot A}$ (ii) $\cot(A-B)=\dfrac{\cot A\cot B+1}{\cot B-\cot A}$.
6. (i) $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B$
   (ii) $\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B$.

TEST YOURSELF

1. If $\sin A=\dfrac{3}{5}$ and $\cos B=\dfrac{9}{41}, 0<A<\dfrac{\pi}{2}, 0<B<\dfrac{\pi}{2}$, find the values of the following:
   (i) sin $(A + B)$ (ii) sin $(A - B)$
   (iii) cos $(A + B)$ (iv) cos $(A - B)$
   (v) tan $(A + B)$ (vi) tan $(A - B)$.
2. Find the value of $\tan(\alpha+\beta)$, given that $\cot\alpha=\dfrac{1}{2}, \alpha\in\left(\pi,\dfrac{3\pi}{2}\right)$ and $\sec\beta=-\dfrac{5}{3},\ \beta\in\left(\dfrac{\pi}{2},\pi\right)$.
3. Show that $\dfrac{\tan(A+B)}{\cot(A-B)}=\dfrac{\tan^2 A-\tan^2 B}{1-\tan^2 A\tan^2 B}$.
4. If tan $(A + B)$ = x and tan $(A - B)$ = y, find the values of tan $2A$ and tan $2B$.

Answers

1. (i) $\dfrac{187}{205}$ (ii) $-\dfrac{133}{205}$ (iii) $-\dfrac{84}{205}$ (iv) $\dfrac{156}{205}$
   (v) $-\dfrac{187}{84}$ (vi) $-\dfrac{133}{156}$

2. $\dfrac{2}{11}$

4. $\dfrac{x+y}{1-xy}, \dfrac{x-y}{1+xy}$

SECTION – B

12. TRANSFORMATION FORMULAE

LEARNING OBJECTIVES

- Introduction
- Transformation of Products into Sum or Difference of T-Functions
- Transformation of Sum or Difference into Product of T-Functions

INTRODUCTIONS

In the present chapter, we shall learn some transformation formulae for writing the product of two trigonometric functions as the sum (or difference) of two trigonometric functions and for writing the sum (or difference) of two trigonometric functions as the product of two trigonometric functions. These transformation formulae will involve only two trigonometric functions namely: since and cosine.

TRANSFORMATION OF PRODUCTS INTO SUM OR DIFFERENCE OF T-FUNCTIONS

In this section, we shall learn the method of expressing the product of *t*-functions as sum or difference of *t*-functions.

Theorem I. If A and B are arbitrary angles, then prove that

(i) $2 \sin A \cos B = \sin (A + B) + \sin (A - B)$

(ii) $2 \cos A \sin B = \sin (A + B) - \sin (A - B)$

(iii) $2 \cos A \cos B = \cos (A + B) + \cos (A - B)$

(iv) $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$.

Proof. (i) sin (A + B) + sin (A- B)

= (sin A cos B + cos A sin$B$) + (*sin A cos B* – cos A sin B) = 2 sin A cos B.

∴ **2 sin A cos B = sin (A + B) + sin (A – B)**

(ii) sin (A + B) – sin (A – B)

= (sin A cos B + cos A sin B) – (sin A cos B – cos A sin B) = 2 sin A cos B.

∴ **2 cos A sin B = sin (A + B) – sin (A – B)**

(iii) cos (A + B) + cos (A – B)

= (cos A cos B – sin A sin B) + (cos A cos B + sin A sin B) = 2 cos A cos B.

∴ **2 cos A cos B = cos (A + B) + cos (A – B)**

(iv) cos (A – B) – cos (A + B)

= (cos A cos B + sin A sin B) – (cos A cos B – sin A sin B) = 2 sin A sin B.

∴ **2 sin A sin B = cos (A – B) – cos (A + B).**

The formulae (i) – (iv) are called **product formulae.**

Caution. In the last formula (iv) *i.e.,* for 2 sin A sin B ; the R.H.S. is

cos (A – B) – cos (A + B) and not cos (A + B) – cos (A – B).

Remark. The above formulae are also called "*A, B*" formulae.

Aid to Memory

1. In the "*A, B*" formulae, 2 must be there with the product. If not, we create it:
   e.g., $\sin A \cos B = \frac{1}{2}(2\sin A \cos B)$, $\frac{1}{5}\cos A \sin B = \frac{1}{10}(2\cos A \sin B)$ etc.
2. (i) 2 sin A cos B = sin (*sum*) + sin (*difference*)
   (ii) 2 cos A cos B = sin (*sum*) – sin (*difference*)
   (iii) 2 cos A cos B = cos (*sum*) + cos (*difference*)
   (iv) 2 sin A sin B = cos (*difference*) – cos (*sum*).

Here *sum* stands for A + B and *difference* for A – B.

Example 1. *Show that :* $\frac{2\cos 2A + 1}{2\cos 2A - 1} = \tan(60^0 + A)\tan(60^0 - A)$

Sol. R.H.S. = tan (60^0 + A) tan (60^0 – A)

$$= \frac{\sin(60^0 + A)\sin(60^0 - A)}{\cos(60^0 + A)\cos(60^0 - A)} = \frac{2\sin(60^0 + A)\sin(60^0 - A)}{2\cos(60^0 + A)\cos(60^0 - A)}$$

$$= \frac{\cos[(60^0 + A) - (60^0 - A)] - \cos[(60^0 + A) + (60^0 - A)]}{\cos[(60^0 + A) + (60^0 - A)] + \cos[(60^0 + A) - (60^0 - A)]} = \frac{\cos 2A - \cos 120^0}{\cos 120^0 + \cos 2A}$$

$$= \frac{\cos 2A - \left(-\frac{1}{2}\right)}{\left(-\frac{1}{2}\right) + \cos 2A} = \frac{2\cos 2A + 1}{2\cos 2A - 1} = L.H.S.$$

EXERCISE 12.1

SHORT ANSWER TYPE QUESTIONS

1. Change the following products as sum or difference of *t*-ratios:

(i) $2\sin 7\theta \cos 2\theta$ (ii) $2\cos 5\theta \sin \theta$

(iii) $\frac{1}{2}\cos 2\theta \cos \theta$ (iv) $\frac{1}{7}\sin 8\theta \sin 2\theta$.

LONG ANSWER TYPE QUESTIONS

Show that :

2. (i) $\sin(45^0 + A)\sin(45^0 - A) = \frac{1}{2}\cos 2A$ (ii) $\sec\left(\frac{\pi}{4} + \theta\right)\sec\left(\frac{\pi}{4} - \theta\right) = 2\sec 2\theta$

3. (i) $2\sin(2\theta + \phi)\cos(\theta - 2\phi) = \sin(3\theta - \phi) + \sin(\theta + 3\phi)$

(ii) $\cos(60^0 + \alpha)\sin(60^0 - \alpha) = \frac{1}{4}\left(\sqrt{3} - 2\sin 2\alpha\right)$.

Answers

1. (i) $\sin 9\theta + \sin 5\theta$ (ii) $\sin 6\theta - \sin 4\theta$

(iii) $\frac{1}{4}[\cos 3\theta + \cos \theta]$ (iv) $\frac{1}{14}[\cos 6\theta - \cos 10\theta]$.

TRANSFORMATION OF SUM OR DIFFERENCE INTO PRODUCT OF T-FUNCTIONS

In this section, we shall learn the method of expressing the sum or difference of *t*-functions into product *t*-functions.

Theorem II. If C and D are arbitrary angles, then prove that

i. $\mathbf{\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}}$

ii. $\mathbf{\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}}$

iii. $\mathbf{\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}}$

iv. $\mathbf{\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}}$

Proof. Let $A = \frac{C+D}{2}$ and $B = \frac{C-D}{2}$

$\therefore \quad C + D = 2A \quad$ and $\quad C - D = 2B$

Solving for C and D, we get

$$C = A + B \quad \text{and} \quad D = A - B.$$

(i) $\sin C + \sin D = \sin(A + B) + \sin(A - B)$

$$= (\sin A \cos B + \cos A \sin B) + (\sin A \cos B - \cos A \sin B)$$

$$= 2 \sin A \cos B = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}$$

$\therefore \quad \mathbf{\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}}$**.**

(ii) $\sin C - \sin D = \sin(A + B) - \sin(A - B)$

$$= (\sin A \cos B + \cos A \sin B) - (\sin A \cos B - \cos A \sin B)$$

$$= 2 \cos A \sin B = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}$$

$\therefore$ **sin C – sin D = 2 cos** $\frac{C+D}{2}$ **sin** $\frac{C-D}{2}$**.**

(iii) cos C + cos D = cos $(A + B)$ + cos $(A - B)$

= (cos A cos B – sin A sin B) + (cos A cos B + sin A sin B)

= 2 cos A cos B = 2 cos $\frac{C+D}{2}$ sin $\frac{C-D}{2}$.

$\therefore$ **cos C + cos D = 2 cos** $\frac{C+D}{2}$ **cos** $\frac{C-D}{2}$**.**

(iv) cos C – cos D = cos $(A + B)$ – cos $(A - B)$

= (cos A cos B – sin A sin B) + (cos A cos B + sin A sin B)

= - 2 sin A sin B = - 2 sin $\frac{C+D}{2}$ sin $\frac{C-D}{2}$.

$$= -2\sin\frac{C+D}{2}\ \sin\left(-\frac{D-C}{2}\right)$$

$$= -2\sin\frac{C+D}{2}\times\left(-\sin\frac{D-C}{2}\right) = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$$

$\therefore$ **cos C + cos D = 2 cos** $\frac{C+D}{2}$ **cos** $\frac{C-D}{2}$**.**

Caution. The R.H.S. of (iv) contains sin $\frac{D-C}{2}$ and not sin $\frac{C-D}{2}$**.**

Remark 1. The above formulae are called '*C, D'* formulae.

Remark 2. In each of the four *'C, D'* formulae, both *t*-ratios on the L.H.S. are same.

If however, we have sin C and cos D, then either 'sin' is changed into 'cos' or 'cos' into 'sin' as below:

$$\sin C + \cos D = \sin C + \sin\left(\frac{\pi}{2} - D\right) \quad \text{or} \quad \sin C + \cos D = \cos\left(\frac{\pi}{2} - C\right) + \cos D$$

Example 2. *Express as product of t-ratios:*

(i) $\sin 4\theta + \sin 3\theta$ (ii) $\sin 7\theta - \sin 4\theta$

(iii) *cos* $9\theta + \cos\theta$ (iv) *cos* 3θ - *cos* 7θ .

Sol. (i) $\sin 4\,\theta + \sin 3\theta = 2\sin\frac{4\theta+3\theta}{2}\cos\frac{4\theta-3\theta}{2} = 2\sin\frac{7\theta}{2}\cos\frac{\theta}{2}$

(ii) $\sin 7\theta - \sin 4\theta = 2\cos\frac{7\theta+4\theta}{2}\sin\frac{7\theta-4\theta}{2} = 2\cos\frac{11\theta}{2}\sin\frac{3\theta}{2}$

(iii) $\cos 9\theta + \cos 3\theta = 2\cos\frac{9\theta+3\theta}{2}\cos\frac{9\theta-3\theta}{2} = 2\cos 6\theta\cos 3\theta$

(iv) $\cos 3\theta - \cos 7\theta = 2\sin\frac{3\theta+7\theta}{2}\sin\frac{7\theta-3\theta}{2} = 2\sin 5\theta\sin 2\theta$.

Example 3. *If b sin*β = α *sin* $(2\alpha+\beta)$, *prove that (b + a) cot* $(\alpha+\beta)$ = *(b – a) cot*α

Sol. We have *b sin*β = α *sin* $(2\alpha+\beta)$.

$\therefore \quad \frac{\sin(2\alpha+\beta)}{\sin\beta} = \frac{b}{a}$

$\Rightarrow \quad \frac{\sin(2\alpha+\beta)+\sin\beta}{\sin(2\alpha+\beta)-\sin\beta} = \frac{b+a}{b-a}$

(By applying componendo and dividendo rule)

$\Rightarrow \quad \frac{2\sin(\alpha+\beta)\cos\alpha}{2\cos(\alpha+\beta)\sin\alpha} = \frac{b+a}{b-a} \quad \Rightarrow \quad \frac{\cot\alpha}{\cot(\alpha+\beta)} = \frac{b+a}{b-a}$

$\Rightarrow$ **(b + a) cot (**$\alpha+\beta$**) = (b – a) cot** α **.**

EXERCISE 12.2

SHORT ANSWER TYPE QUESTIONS

1. Express the following as product of *t*-ratios:

   (i) $\sin 9\theta + \sin 5\theta$ (ii) $\frac{1}{2}(\sin 8\theta - \sin 2\theta)$

   (iii) $\cos 4\theta + \cos\theta$ (iv) $\frac{1}{2}(\cos 2\theta - \cos 20\theta)$.

2. Show that:

   (i) $\frac{\sin x - \sin y}{\cos x + \cos y} = \tan\frac{x-y}{2}$ (ii) $\frac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} = \tan 4x$

(iii) $\dfrac{\sin x+\sin 3x}{\cos x+\cos 3x}=\tan 2x$

(iv) $\dfrac{\cos 9x-\cos 5x}{\sin 17x-\sin 3x}=-\dfrac{\sin 2x}{\cos 10x}$

LONG ANSWER TYPE QUESTIONS

Show that (3 – 7)

3. (i) $\sin 3x+\sin 2x-\sin x=4\sin x\cos\dfrac{x}{2}\cos\dfrac{3x}{2}$

   (ii) $\cos 55^0+\cos 65^0+\cos 175^0=0$

4. (i) $\cos 3A\cos 2A+\sin 4A\sin A=\cos A\cos 2A$

   (ii) $\sin\dfrac{11\theta}{4}\sin\dfrac{\theta}{4}+\sin\dfrac{7\theta}{4}\sin\dfrac{3\theta}{4}=\sin 2\theta\sin\theta$

5. (i) $(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2=4\sin^2\dfrac{\alpha-\beta}{2}$

   (ii) $\cos^2 2x-\cos^2 6x=\sin 4x\sin 8x$

6. $\dfrac{\cos\theta+\cos 3\theta-\cos 2\theta}{\sin\theta+\sin 3\theta-\sin 2\theta}=\cot 2\theta$

7. $\dfrac{\cos 4x+\cos 3x+\cos 2x}{\sin 4x+\sin 3x+\sin 2x}=\cot 3x$

Answers

1. (i) $2\sin 7\theta\cos 2\theta$ (ii) $\cos 5\theta\sin 3\theta$

   (iii) $2\cos\dfrac{5\theta}{2}\cos\dfrac{3\theta}{2}$ (iv) $\sin 11\theta\sin 9\theta$.

SUMMARY

1. **Transformation of a product into sum or difference**
   (i) $2 \sin A \cos B = \sin (A + B) + \sin (A - B)$
   (ii) $2 \cos A \sin B = \sin (A + B) - \sin (A - B)$
   (iii) $2 \cos A \cos B = \cos (A + B) + \cos (A - B)$
   (iv) $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$.

2. **Transformation of sum or difference into product**
   (i) $\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}$
   (ii) $\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}$
   (iii) $\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}$
   (iv) $\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}$.

TEST YOURSELF

1. If three angles A, B and C are in A.P., then show that $\frac{\sin A - \sin C}{\cos C - \cos A} = \cot B$.
2. Show that $\frac{\cos 2A \cos 3A - \cos 2A \cos 7A + \cos A \cos 10A}{\sin 4A \sin 3A - \sin 2A \sin 5A + \sin 4A \sin 7A} = \cot 5A \cot 6A$
3. Show that $\frac{\sin(\theta + \phi) - 2\sin\theta + \sin(\theta - \phi)}{\cos(\theta + \phi) - 2\cos\theta + \cos(\theta - \phi)} = \tan\theta$
4. Show that
$$\sin(\beta - \gamma) + \sin(\gamma - \alpha) + \sin(\alpha - \beta) + 4\sin\frac{\beta - \gamma}{2}\sin\frac{\gamma - \alpha}{2}\sin\frac{\alpha - \beta}{2} = 0.$$
5. Show that $\cot 4x(\sin 5x + \sin 3x) = \cot x(\sin 5x - \sin 3x)$
6. Show that $\frac{\sin(A - C) + 2\sin A + \sin(A + C)}{\sin(B - C) + 2\sin B + \sin(B + C)} = \frac{\sin A}{\sin B}$

SECTION – B

13. TRIGONOMETRIC FUNCTIONS OF MULTIPLE AND SUB-MULTIPLE ANGLES

LEARNING OBJECTIVES

- Introduction
- Trigonometric Functions of Multiple Angles
- Trigonometric Functions of Sub-multiple Angles
- Trigonometric Functions of 18^0 and 36^0.

INTRODUCTIONS

In the chapter, we shall learn the methods of finding the values of trigonometric functions of multiple angles $2A$, $3A$ etc. and sub-multiple angles $A/2$, $A/3$ etc. in terms of trigonometric functions of angle A. These formulae would lead us to evaluate the values of trigonometric functions of angle like 18^0, 36^0, $7\frac{1^0}{2}$, $142\frac{1^0}{2}$ etc.

TRIGONOMETRIC FUNCTIONS OF MULTIPLE ANGLES

The angles $2A$, $3A$, ……. Are called **multiple angles** of A.

Theorem I. Prove that :

(i) **sin 2A = 2sin A cos A**

(ii) **cos 2A =** $\begin{cases} \cos^2 A - \sin^2 A \\ 1 - 2\sin^2 A \\ 2\cos^2 A - 1 \end{cases}$

(iii) **tan 2A =** $\frac{2\tan A}{1-\tan^2 A}$.

Proof. (i) sin 2A = sin (A + A) = sin A cos A + cos A sin A = 2 sin A cos A

∴ **sin 2A = 2 sin A cos A.**

(ii) cos 2A = cos (A + A) = cos A cos A – sin A sin A = $\cos^2 A - \sin^2 A$

∴ **cos 2A = cos² A – sin² A.** ...(1)

(1) ⇒ cos 2A = (1 – $\sin^2 A$) – $\sin^2 A$ = 1 – 2 $\sin^2 A$

∴ **cos 2A = 1 – 2 sin² A.**

(1) ⇒ cos 2A = $\cos^2 A$ – (1 – $\cos^2 A$) = 2 $\cos^2 A$ – 1.

∴ **cos 2 A = 2 cos² A – 1.**

(iii) $$\tan 2A = \tan(A+A) = \frac{\tan A + \tan A}{1-\tan A\tan A} = \frac{2\tan A}{1-\tan^2 A}$$

∴ **tan 2A =** $\frac{2\tan A}{1-\tan^2 A}$

Corollary I. Prove that

(i) **1 – cos 2A = 2 sin² A** (ii) **1 + cos 2A = 2 cos² A.**

Proof. (i) We have cos 2A = 1 – 2 $\sin^2 A$. ∴ **1 – cos 2A = 2 sin² A.**

(ii) We have cos 2A = 2 $\cos^2 A$ – 1 ∴ **1 + cos 2A = 2 cos² A.**

Aid to memory

(i) 1 – cos (*double angle*) = 2 sin² (*angle*)

(ii) 1 + cos (*double angle*) = 2 cos² (*angle*).

Corollary 2. Prove that:

(i) **sin² A =** $\frac{1-\cos 2A}{2}$ (ii) **cos² A =** $\frac{1+\cos 2A}{2}$

Proof. (i) $\frac{1-\cos 2A}{2} = \frac{1-(1-2\sin^2 A)}{2} = \frac{2\cos^2 A}{2} = \sin^2 A$

$\therefore$ **$\sin^2 A$ =** $\dfrac{1-\cos 2A}{2}$.

(ii) $\dfrac{1+\cos 2A}{2} = \dfrac{1+(2\cos^2 A-1)}{2} = \dfrac{2\cos^2 A}{2} = \cos^2 A$

$\therefore$ **$\cos^2 A$ =** $\dfrac{1+\cos 2A}{2}$.

Aid to memory

(i) $\sin^2 (angle) = \dfrac{1-\cos(double\ angle)}{2}$ (ii) $\cos^2 (angle) = \dfrac{1+\cos(double\ angle)}{2}$

Theorem II. Prove that

(i) **sin 2A =** $\dfrac{2\tan A}{1+\tan^2 A}$ (ii) **cos 2A =** $\dfrac{1-\tan^2 A}{1+\tan^2 A}$.

Proof. (i) $\sin 2A = 2\sin A\cos A = \dfrac{2\sin A\cos A}{1} = \dfrac{2\sin A\cos A}{\cos^2 A+\sin^2 A}$

$$= \dfrac{\dfrac{2\sin A\cos A}{\cos^2 A}}{\dfrac{\cos^2 A}{\cos^2 A}+\dfrac{\sin^2 A}{\cos^2 A}} = \dfrac{2\tan A}{1+\tan^2 A}$$

$\therefore$ **sin 2A =** $\dfrac{2\tan A}{1+\tan^2 A}$

(ii) $\cos 2A = \cos^2 A - \sin^2 A = \dfrac{\cos^2 A-\sin^2 A}{1} = \dfrac{\cos^2 A-\sin^2 A}{\cos^2 A+\sin^2 A}$

$$= \dfrac{\dfrac{\cos^2 A}{\cos^2 A}-\dfrac{\sin^2 A}{\cos^2 A}}{\dfrac{\cos^2 A}{\cos^2 A}+\dfrac{\sin^2 A}{\cos^2 A}} = \dfrac{1-\tan^2 A}{1+\tan^2 A}$$

$\therefore$ **cos 2A =** $\dfrac{1-\tan^2 A}{1+\tan^2 A}$.

Caution. In the denominator of sin 2A, we have $1 + \tan^2 A$, whereas in the denominator of tan 2A, there is $1 - \tan^2 A$.

Theorem III. Prove that

(i) **sin 3A = 3 sin A – 4 sin³ A** (ii) **cos 3A = 4 cos³ A – 3 cos A**

(iii) **tan 3A =** $\frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

Proof. (i) $\sin 3A = \sin(A + 2A) = \sin A \cos 2A + \cos A \sin 2A$

$= \sin A(1 - 2\sin^2 A) + \cos A \,.\, 2\sin A \cos A$

$= \sin A - 2\sin^3 A + 2\sin A\cos^2 A$

$= \sin A - 2\sin^3 A + 2\sin A(1 - \sin^2 A)$

$= \sin A - 2\sin^3 A + 2\sin A - 2\sin^3 A$

$= 3\sin A - 4\sin^3 A.$

∴ **sin 3A = 3 sin A – 4 sin³ A.**

(ii) $\cos 3A = \cos(A + 2A) = \cos A\cos 2A - \sin A \sin 2A$

$= \cos A(2\cos^2 A - 1) - \sin A \,.\, 2\sin A\cos A$

$= 2\cos^3 A - \cos A - 2\sin^2 A\cos A$

$= 2\cos^3 A - \cos A - 2(1 - \cos^2 A)\cos A$

$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$

$= 4\cos^3 A - 3\cos A$

∴ **cos 3A = 4 cos³ A – 3 cos A.**

(iii) $\tan 3A = \tan(A + 2A) = \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A}$

$$= \frac{\tan A + \frac{2\tan A}{1 - \tan^2 A}}{1 - \tan A \frac{2\tan A}{1 - \tan^2 A}} = \frac{\tan A(1 - \tan^2 A) + 2\tan A}{1 - \tan^2 A - 2\tan^2 A}$$

$$= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

$\therefore$ **tan 3A =** $\dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

TRIGONOMETRIC FUNCTIONS OF SUB-MULTIPLE ANGLES

The angles *A/2, A/3, ……* are called **sub-multiple angles** of *A*.

Theorem. Prove that:

(i) $\sin\dfrac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{2}}$ (ii) $\cos\dfrac{A}{2} = \pm\sqrt{\dfrac{1+\cos A}{2}}$

(iii) $\tan\dfrac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{1+\cos A}}$ (iv) $\tan\dfrac{A}{2} = \dfrac{\pm\sqrt{1+\tan^2 A}-1}{\tan A}$

(v) $\tan\dfrac{A}{2} = \dfrac{1-\cos A}{\sin A}$ (vi) $\cot\dfrac{A}{2} = \dfrac{1+\cos A}{\sin A}$

Proof.(i) We have $\cos A = \cos 2\left(\dfrac{A}{2}\right) = 1 - 2\sin^2\dfrac{A}{2}$

$\Rightarrow$ $2\sin^2\dfrac{A}{2} = 1-\cos A$ $\Rightarrow$ $\sin^2\dfrac{A}{2} = \dfrac{1-\cos A}{2}$

$\therefore$ $\sin\dfrac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{2}}$

(ii) We have $\cos A = \cos 2\left(\dfrac{A}{2}\right) = 2\cos^2\dfrac{A}{2} - 1$

$\Rightarrow$ $2\cos^2\dfrac{A}{2} = 1+\cos A$ $\Rightarrow$ $\cos^2\dfrac{A}{2} = \dfrac{1+\cos A}{2}$

$\therefore$ $\cos\dfrac{A}{2} = \pm\sqrt{\dfrac{1+\cos A}{2}}$

(iii) We have $\cos A = \cos 2\left(\dfrac{A}{2}\right) = \dfrac{1-\tan^2\dfrac{A}{2}}{1+\tan^2\dfrac{A}{2}}$

$$\Rightarrow \qquad \frac{\cos A + 1}{\cos A - 1} = \frac{\left(1 - \tan^2 \frac{A}{2}\right) + \left(1 + \tan^2 \frac{A}{2}\right)}{\left(1 - \tan^2 \frac{A}{2}\right) - \left(1 + \tan^2 \frac{A}{2}\right)}$$

(Using componendo and dividend formula)

$$= \frac{2}{-\tan^2 \frac{A}{2}} = -\frac{1}{\tan^2 \frac{A}{2}}$$

$$\Rightarrow \qquad \tan^2 \frac{A}{2} = -\frac{\cos A - 1}{\cos A + 1} = \frac{1 - \cos A}{1 + \cos A}$$

$$\therefore \qquad \tan \frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{1 + \cos A}}$$

Alternatively, $\tan \frac{A}{2} = \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = \frac{\pm\sqrt{\frac{1 - \cos A}{2}}}{\pm\sqrt{\frac{1 + \cos A}{2}}} = \pm\sqrt{\frac{1 - \cos A}{1 + \cos A}}$

$$\therefore \qquad \tan \frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{1 + \cos A}}$$

(iv) We have $\tan A = \tan 2\left(\frac{A}{2}\right) = \frac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}}$

$$\Rightarrow \qquad \tan A\left(1 - \tan^2 \frac{A}{2}\right) = 2 \tan \frac{A}{2}$$

$$\Rightarrow \qquad (\tan A) \tan^2 \frac{A}{2} + 2 \tan \frac{A}{2} - \tan A = 0$$

$$\Rightarrow \qquad \tan \frac{A}{2} = \frac{-2 \pm \sqrt{4 + 4 \tan^2 A}}{2 \tan A} = \frac{\pm\sqrt{1 + \tan^2 A} - 1}{\tan A}$$

$$\therefore \qquad \tan \frac{A}{2} = \frac{\pm\sqrt{1 + \tan^2 A} - 1}{\tan A}$$

(v) $$\tan\frac{A}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=\frac{2\sin^2\frac{A}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}}=\frac{1-\cos A}{\sin A}$$

$\therefore$ $$\tan\frac{A}{2}=\frac{1-\cos A}{\sin A}$$

(vi) $$\cot\frac{A}{2}=\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}=\frac{2\cos^2\frac{A}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}}=\frac{1+\cos A}{\sin A}$$

$\therefore$ $$\cot\frac{A}{2}=\frac{1+\cos A}{\sin A}.$$

WORKING RULES FOR SOLVING PROBLEMS

Rule I. (i) $\sin 2A = 2\sin A\cos A = \dfrac{2\tan A}{1+\tan^2 A}$

(ii) $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1 = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

(iii) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

Rule II. (i) $\sin^2 A = \dfrac{1-\cos 2A}{2}$ (ii) $\cos^2 A = \dfrac{1+\cos 2A}{2}$

Rule III. (i) $\sin 3A = 3\sin A - 4\sin^3 A$ (ii) $\cos 3A = 4\cos^2 A - 3\cos A$

(iii) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

Rule IV. (i) $\sin\dfrac{A}{2}=\pm\sqrt{\dfrac{1-\cos A}{2}}$ (ii) $\cos\dfrac{A}{2}=\pm\sqrt{\dfrac{1+\cos A}{2}}$

(iii) $\tan\dfrac{A}{2}=\pm\sqrt{\dfrac{1-\cos A}{1+\cos A}}$ (iv) $\tan A=\dfrac{\pm\sqrt{1+\tan^2 A}-1}{\tan A}$

(v) $\tan\dfrac{A}{2}=\dfrac{1-\cos A}{\sin A}$ (vi) $\cot\dfrac{A}{2}=\dfrac{1+\cos A}{\sin A}$.

Example 1. *Find sin 2A, cos 2A, tan 2A if:* $\sin A = \frac{4}{5}$

Sol. We have $\sin A = \frac{4}{5}$.

$$\therefore \quad \cos^2 A = 1 - \sin^2 A = 1 - \left(\frac{4}{5}\right)^2 = \frac{9}{25} \quad \textit{i.e.,} \quad \cos A = \pm\frac{3}{5}$$

$$\therefore \quad \sin 2A = 2\sin A\cos A = 2\left(\frac{4}{5}\right)\left(\pm\frac{3}{5}\right) = \pm\frac{24}{25}$$

$$\cos 2A = 1 - 2\sin^2 A = 1 - 2\left(\frac{4}{5}\right)^2 = 1 - \frac{32}{25} = -\frac{7}{25}$$

Alternatively, $\cos 2A = 2\cos^2 A - 1 = 2\left(\pm\frac{3}{5}\right)^2 - 1 = \frac{18}{25} - 1 = -\frac{7}{25}$.

We have $\tan A = \frac{\sin A}{\cos A} = \frac{4/5}{\pm 3/5} = \pm\frac{4}{3}$.

$$\therefore \quad \tan 2A = \frac{2\tan A}{1 - \tan^2 A} = \frac{2(\pm 4/3)}{1 - (\pm 4/3)^2}$$

$$= \frac{\pm 8/3}{1 - (16/9)} = \frac{\pm 8/3}{-7/9} = \mp\frac{24}{7}$$

Alternatively, $\tan 2A = \frac{\sin 2A}{\cos 2A} = \frac{\pm 24/25}{-7/25} = \mp\frac{24}{7}$.

Remark. The sign '∓' means that if tan *A* = 4/3 then tan 2A = -24/7 and tan 2*A*= 24/7.

Example 2. *Prove that* $(3\sin A - \sin 3A)^{2/3} + (3\cos A + \cos 3A)^{2/3} = 4^{2/3}$.

Sol. L.H.S. = $(3\sin A - \sin 3A)^{2/3} + (3\cos A + \cos 3A)^{2/3}$

$$= \left[3\sin A - (3\sin A - 4\sin^3 A)\right]^{2/3} + \left[3\cos A + (4\cos^3 A - 3\cos A)\right]^{2/3}$$

$$= (4\sin^3 A)^{2/3} + (4\cos^3 A)^{2/3} = 4^{2/3}(\sin^2 A + \cos^2 A)$$

$$= 4^{2/3}(1) = 4^{2/3} = \text{R.H.S.}$$

TRIGONOMETRIC FUNCTIONS OF 18^0 AND 36^0

Theorem. Find the values of trigonometric functions of 18^0 and 36^0.

Proof. $\theta = 18^0 \quad \Rightarrow \quad 5\theta = 90^0 \Rightarrow \quad 2\theta - 90^0 - 3\theta$

$\Rightarrow \quad \sin 2\theta = \sin(90^0 - 3\theta) = \cos 3\theta$

$\Rightarrow \quad 2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta \quad \Rightarrow \quad \cos\theta(4(1-\sin^2\theta) - 3 - 2\sin\theta) = 0$

$\Rightarrow \quad 4\sin^2\theta + 2\sin\theta - 1 = 0 \qquad [\because \ \cos 18^0 \neq 0]$

$\Rightarrow \quad \sin\theta = \dfrac{-2 \pm \sqrt{4+16}}{8} = \dfrac{-1 \pm \sqrt{5}}{4} \quad \Rightarrow \quad \sin 18^0 = \dfrac{\sqrt{5}-1}{4} \qquad (\because \ 18^0 \text{ is +ve })$

$$\cos 18^0 = \sqrt{1-\sin^2 18^0} = \sqrt{1 - \frac{5+1-2\sqrt{5}}{16}} = \sqrt{\frac{10+2\sqrt{5}}{16}} = \frac{\sqrt{10+2\sqrt{5}}}{4}.$$

Now, the other *t*- functions of 18^0 can be easily found out.

The identity $\cos 2\theta = 1 - 2\sin^2\theta$ implies

$$\cos 36^0 = 1 - 2\left(\frac{5+1-2\sqrt{5}}{16}\right) = \frac{\sqrt{5}+1}{4}.$$

Also, $\sin 36^0 = \sqrt{1-\cos^2 36^0} = \sqrt{1 - \dfrac{5+1+2\sqrt{5}}{16}} = \dfrac{\sqrt{10-2\sqrt{5}}}{4}.$

The other *t*- functions of 36^0 can now be found out easily.

Corollary 1. (i) **sin 72^0** = sin (90^0 – 18^0) = cos 18^0 = $\dfrac{\sqrt{10+2\sqrt{5}}}{4}$

(ii) **cos 72^0** = cos (90^0 – 18^0) = sin 18^0 = $\dfrac{\sqrt{5}-1}{4}$

Corollary 2. (i) **sin 54^0** = sin (90^0 – 36^0) = cos 36^0 = $\dfrac{\sqrt{5}+1}{4}$

(ii) **cos 54^0** = cos (90^0 – 36^0) = sin 36^0 = $\frac{\sqrt{10-2\sqrt{5}}}{4}$.

Example 3. *Show that cos 36^0 cos 72^0 cos 108^0 cos 144^0 =* $\frac{1}{16}$.

Sol. L.H.S. = cos 36^0 cos 72^0 cos 108^0 cos 144^0

= cos 36^0 cos (90^0 - 18^0) cos(90^0 + 18^0) cos(180^0 - 36^0)

= cos 36^0 sin 18^0 (- sin 18^0) (-cos 36^0)

= sin^2 18^0 cos^2 36^0 = $\left(\frac{\sqrt{5}-1}{4}\right)^2\left(\frac{\sqrt{5}+1}{4}\right)^2 = \left(\frac{5-1}{16}\right)^2$

= $\left(\frac{1}{4}\right)^2 = \frac{1}{16} =$ R.H.S.

EXERCISE 13.1

SHORT ANSWER TYPE QUESTIONS

1. Show that $\frac{\sin x}{1+\cos x} = \tan\frac{x}{2}$,
2. Show that $\frac{1-\cos 2A}{1+\cos 2A} = \tan^2 A$.
3. If cos A= 4/5, find the value of cos 2A.
4. If tan A = 2/3, find the value of tan 2A.
5. Show that $\frac{1-\tan^2\left(45^0+A\right)}{1+\tan^2\left(45^0+A\right)} = -\sin 2A$.

LONG ANSWER TYPE QUESTIONS

6. Show that sin 4A = 4 sin A cos^3 A – 4 cos A sin^3 A.
7. Show that cos 4A = 1 – 8 sin^2 A cos^2 A.
8. If $\cos\theta = \frac{1}{2}\left(a+\frac{1}{a}\right)$, then show that $\cos 2\theta = \frac{1}{2}\left(a^2+\frac{1}{a^2}\right)$.
9. (i) Find the values of sin 2A and cos 2A, when sin A = 3/5.

   (ii) Find the values of sin 2A and cos 2A, when cos A = 20/29.

10. Show that $\tan 4A = \dfrac{4\tan A(1-\tan^2 A)}{1-6\tan^2 A+\tan^4 A}$.

11. Show that $\dfrac{\tan^2\theta}{\tan^2\theta-1}+\dfrac{\cos ec^2\theta}{\sec^2\theta-\cos ec^2\theta}=-\sec 2\theta$.

12. Show that cos A cos $2A$ cos $4A$ cos $8A$ = $\dfrac{\sin 16A}{16\sin A}$.

Answers

3. 7/25 4. 12/5

9. (i) $\pm$24/25, 7/25 (ii) $\pm$840/841, -41/841.

SUMMARY

1. (i) $\sin 2A = \begin{cases} 2\sin A\cos A \\ \dfrac{2\tan A}{1+\tan^2 A} \end{cases}$ (ii) $\cos 2A = \begin{cases} \cos^2 A - \sin^2 A \\ 1 - 2\sin^2 A \\ 2\cos^2 A - 1 \\ \dfrac{1-\tan^2 A}{1+\tan^2 A} \end{cases}$

   (iii) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

2. (i) $\sin 3A = 3\sin A - 4\sin^3 A$ (ii) $\cos 3A = 4\cos^3 A - 3\cos A$

   (iii) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

3. $\sin^2 A = \dfrac{1-\cos 2A}{2}$ $\cos^2 A = \dfrac{1+\cos 2A}{2}$

4. $\sin 18^0 = \dfrac{\sqrt{5}-1}{4}$ $\cos 36^0 = \dfrac{\sqrt{5}+1}{4}$

TEST YOURSELF

1. Show that $8\cos^3\dfrac{\pi}{9} - 6\cos\dfrac{\pi}{9} = 1$.
2. Show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.
3. Show that $\tan 82\dfrac{1^0}{2} = \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$.
4. Show that $\sin\dfrac{\pi}{5}\sin\dfrac{2\pi}{5}\sin\dfrac{3\pi}{5}\sin\dfrac{4\pi}{5} = \dfrac{5}{16}$.
5. Find the value of $\tan 22^0, 30'$.
6. Show that $\sin(B-C) + \sin(C-A) + \sin(A-B) = -4\dfrac{B-C}{2}\sin\dfrac{C-A}{2}\sin\dfrac{A-B}{2}$.
7. Show that $\tan 6^0 \tan 42^0 \tan 66^0 \tan 78^0 = 1$.

Answer

5. $\sqrt{2}-1$

SECTION – B

14. RELATIONS BETWEEN THE SIDES AND THE TRIGONOMETRIC RATIOS OF THE ANGLES OF A TRIANGLE

LEARNING OBJECTIVES

- Introduction
- Sine Formula
- Cosine Formulae
- Projection Formulae
- Napier's Analogy
- Half-Angle Formulae

INTRODUCTIONS

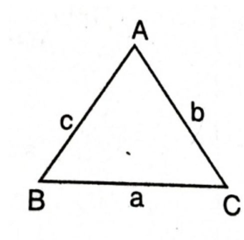
Every triangle contains three sides and three angles. In the present chapter, we shall study some relations between the sides and the trigonometric ratios of the angles of a triangle. These relations will be found very useful in finding the areas of triangles, polygons and also in solution of triangles.

Let ABC be a triangle. The angles of ΔABC corresponding to the vertices A, B, C are denoted by A, B, C themselves. The sides opposite to the angles A, B, C are denoted by a, b, c respectively. The sides and angle of a triangle are called its **elements.**

SINE FORMULA

Statement. If *ABC* is a triangle with sides a = BC, b = CA, c = AB, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$



Remark 1. Since formula is also known as the **law of sines.**

Remark 2. We have $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

Let each ratio be equal to k.

$\therefore$ $a = k \sin A,\ b = k \sin B,\ c = k \sin C.$

These equalities helps us to replace any relation involving sides of a triangle by a corresponding relation involving *sines* of the corresponding angles.

Example 1. *In any triangle ABC, show that* $\dfrac{\tan \frac{A-B}{2}}{\tan \frac{A+B}{2}} = \dfrac{a-b}{a+b}$

Sol. By **law of sines,** $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$ (say)

$\therefore$ $a = k \sin A,\ b = k \sin B,\ c = k \sin C$

R.H.S. $= \dfrac{a-b}{a+b} = \dfrac{k \sin A - k \sin B}{k \sin A + k \sin B} = \dfrac{\sin A - \sin B}{\sin A + \sin B}$

$$= \frac{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}} = \frac{\left(\sin\frac{A-B}{2}\right) / \left(\cos\frac{A-B}{2}\right)}{\left(\sin\frac{A+B}{2}\right) / \left(\cos\frac{A+B}{2}\right)}$$

$$= \frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}} = L.H.S$$

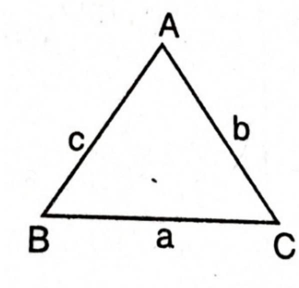
COSINE FORMULAE

Statement. If ABC is a triangle with sides a = BC, b = CA, c = AB, then

(i) **cos A =** $\frac{b^2+c^2-a^2}{2abc}$

(ii) **cos B =** $\frac{c^2+a^2-b^2}{2ca}$

(iii) **cos C =** $\frac{a^2+b^2-c^2}{2abc}$.



Remark 1. Cosine formula is also known as the **law of cosines.**

Remark 2. $\cos C = \frac{a^2+b^2-c^2}{2abc}$ implies $\mathbf{c^2 = a^2 + b^2 - 2ab\cos C}$**.**

Similarly, $\cos A = \frac{b^2+c^2-a^2}{2abc}$ implies $\mathbf{a^2 = b^2 + c^2 - 2abc\cos A}$

and $\cos B = \frac{c^2+a^2-b^2}{2ca}$ implies $\mathbf{b^2 = c^2 + a^2 - 2ca\cos B}$**.**

Example 2. *Deduce cosine formulae by using sine formula.*

Sol. The cosine formulae are :

$$\cos A = \frac{b^2+c^2-a^2}{2bc}, \cos B = \frac{c^2+a^2-b^2}{2ca} \text{ and } \cos C = \frac{a^2+b^2-c^2}{2ab}.$$

Let us prove the formula for cos A.

$$\frac{b^2+c^2-a^2}{2bc} = \frac{(k\sin B)^2+(k\sin C)^2-(k\sin A)^2}{2(k\sin B)(k\sin C)}$$ (By using **sine formula**)

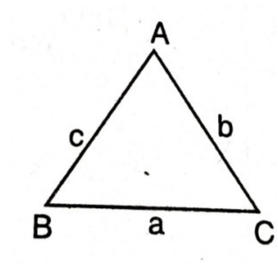
$$= \frac{\sin^2 B+\sin^2 C-\sin^2 A}{2\sin B\sin C} = \frac{\sin^2 B+\sin(C+A)\sin(C-A)}{2\sin B\sin C}$$

$$= \frac{\sin^2 B+\sin B*(C-A)}{2\sin B\sin C} = \frac{\sin B+\sin(C-A)}{2\sin C}$$

$$= \frac{\sin(A+C)+\sin(C-A)}{2\sin C} = \frac{2\sin C\cos A}{2\sin C} = \cos A.$$

$\therefore \qquad CosA = \dfrac{b^2+c^2-a^2}{2bc}$

PROJECTION FORMULAE



Statement. If ABC is a triangle with sides a = BC, b = CA, c = AB, then

(i) **a = b cos C + c cos B**

(ii) **b = c cos A + a cos C**

(iii) **c = a cos B + b cos A.**

Example 3. *Deduce the projection formula from (i) laws of sines and (ii) laws of cosines.*

Sol. Let *ABC* be a triangle with *a* = *BC*, *b* = *CA*, *c* = *AB*.

(i) The **law of sines** is $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$ (say).

Now, $b \cos C + c \cos B = (k \sin B) \cos C + (k \sin C) \cos B$

$$= k \sin (B + C) = k \sin (180^0 - A) = k \sin A = a.$$

$\therefore$ **a = b cos C + c cos B**

Similarly, we can prove other **projection formulae.**

(ii) The **laws of cosines** are

$$\cos A = \frac{b^2+c^2-a^2}{2bc}, \quad \cos B = \frac{c^2+a^2-b^2}{2ca} \quad \text{and} \quad \cos C = \frac{a^2+b^2-c^2}{2ab}$$

Now, $b \cos C + c \cos B = b\left(\dfrac{a^2+b^2-c^2}{2ab}\right) + c\left(\dfrac{c^2+a^2-b^2}{2ca}\right)$

$$= \frac{a^2+b^2-c^2}{2a} + \frac{c^2+a^2-b^2}{2a}$$

$$= \frac{a^2 + b^2 - c^2 + c^2 + a^2 - b^2}{2a} = \frac{2a^2}{2a} = a\,.$$

∴ **a = b cos C + c cos B.**

Similarly, we can prove other **projection formulae.**

NAPIER'S ANALOGY

Theorem. If ABC is a triangle with sides a = BC, b = CA, c = AB, then prove that

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}, \qquad \tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2}$$

And $$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}.$$

Proof. Let us establish the first relation.

R.H.S. $= \frac{b-c}{b+c}\cot\frac{A}{2} = \frac{k\sin B - k\sin C}{k\sin B + k\sin C}\cot\frac{A}{2}$ (By using **law of sines**)

$$= \frac{\sin B - \sin C}{\sin B + \sin C}\cot\frac{A}{2} = \frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{3}}{2\sin\frac{B+C}{2}\cot\frac{B-C}{2}}\cot\frac{A}{2}$$

$$= \frac{\sin\left(\frac{180^0 - A}{2}\right)}{\sin\left(\frac{180^0 - A}{2}\right)}\tan\frac{B-C}{2}\cot\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}\tan\frac{B-C}{2}\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}$$

$$= \tan\frac{B-C}{2} = \text{L.H.S.}$$

The proof of other relations are exactly similar to that of first relation.

Remark. The above relations are also called **law of tangents.**

WORKING RULES FOR SOLVING PROBLEMS

Rule I. $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Rule II. (i) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ (ii) $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$

(iii) $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Rule III. (i) $a = b \cos C + c \cos B$ (ii) $b = c \cos A + a \cos C$

(iii) $c = a \cos B + b \cos A$

EXERCISE 14.1

SHORT ANSWER TYPE QUESTIONS

In any triangle ABC, show that:

1. $a \sin (B - C) + b \sin (C - A) + c \sin (A - B) = 0$
2. (i) $\sin \dfrac{A - B}{2} = \dfrac{a - b}{c} \cos \dfrac{C}{2}$ (ii) $\sin \dfrac{B - C}{2} = \dfrac{b - c}{a} \cos \dfrac{A}{2}$

   (iii) $\sin \dfrac{C - A}{2} = \dfrac{c - a}{b} \cos \dfrac{B}{2}$
3. $\dfrac{a^2 \sin(B - C)}{\sin A} + \dfrac{b^2 \sin(C - A)}{\sin B} + \dfrac{c^2 \sin(A - B)}{\sin C} = 0$
4. $a \cos A + b \cos B + c \cos C = 2a \sin B \sin C = 2b \sin C \sin A = 2c \sin A \sin B$.

LONG ANSWER TYPE QUESTIONS

5. If the angles of a triangle are as 1 : 2: 3, show that the corresponding sides area as $1 : \sqrt{3} : 2$.
6. (i) If $a \cos A = b \cos B$, then the triangle is either isosceles or right angled.

   (ii) If $\cot \dfrac{C}{2} = \dfrac{a + b}{c}$, show that the triangle ABC is right angled.

7. In any triangle *ABC*, show that :

   (i) $b^2 = (c-a)^2\cos^2\frac{B}{2} + (c+a)^2\sin\frac{B}{2}$ (ii) $c^2 = (a-b)^2\cos^2\frac{C}{2} + (a+b)^2\sin^2\frac{C}{2}$

8. In a triangle *ABC*, if $\frac{\cos B}{b} = \frac{\cos C}{c}$, prove that the triangle is isosceles.

9. The sides of a triangle are $x^2 + x + 1$, $2x+1$ and $x^2 - 1$, where $x > 1$. Find the greatest angle.

Answer

9. 120^0.

HALF-ANGLE FORMULAE

Theorem. If ABC is a triangle with sides a = BC, b = CA, c = AB, then prove that

(i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ (ii) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

(iii) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$,

Where s is the semi-perimeter of ΔABC, i.e., $s = \frac{a+b+c}{2}$**.**

Proof. (i) $\sin\frac{A}{2} = \sqrt{\sin^2\frac{A}{2}}$ $\left[\because \frac{A}{2} < 90^0 \Rightarrow \sin\frac{A}{2} > 0\right]$

$$= \sqrt{\frac{1-\cos A}{2}} = \sqrt{\frac{1}{2}\left(1\frac{b^2+c^2-a^2}{2bc}\right)} = \sqrt{\frac{2bc - b^2 - c^2 + a^2}{4bc}}$$

$$= \sqrt{\frac{a^2 - (b-c)^2}{4bc}} = \sqrt{\frac{(a+b-c)(a-b+c)}{4bc}}$$

$$= \sqrt{\frac{(a+b+c-2c)(a+c+b-2b)}{4bc}} = \sqrt{\frac{(2s-2c)(2s-2b)}{4bc}}$$

$$= \sqrt{\frac{(s-b)(s-c)}{bc}} \qquad [\because a+b+c=2s]$$

$$\therefore \qquad \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

(ii) $$\cos\frac{A}{2} = \sqrt{\cos^2\frac{A}{2}} \qquad \left[\because \frac{A}{2} < 90^0 \Rightarrow \cos\frac{A}{2} > 0\right]$$

$$= \sqrt{\frac{1-\cos A}{2}} = \sqrt{\frac{1}{2}\left(1+\frac{b^2+c^2-a^2}{2bc}\right)} = \sqrt{\frac{2bc+b^2+c^2-a^2}{4bc}}$$

$$= \sqrt{\frac{(b+c)^2-a^2}{4bc}} = \sqrt{\frac{(b+c+a)(b+c-a)}{4bc}}$$

$$= \sqrt{\frac{2s(2s-a-a)}{4bc}} = \sqrt{\frac{s(s-a)}{bc}}$$

$$\therefore \qquad \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}.$$

(iii) $$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \frac{\sqrt{\frac{(s-b)(s-c)}{bc}}}{\sqrt{\frac{s(s-a)}{bc}}} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\therefore \qquad \tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Remark. The other **half angle formulae** are:

$$\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}, \qquad \sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}, \qquad \cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

$$\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}, \qquad \tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

Corollary. In any triangle *ABC*, we have

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = 2\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}$$

$$= \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

$\therefore$ $\quad \sin A = \dfrac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$.

Similarly, $\sin B = \dfrac{2}{ca}\sqrt{s(s-a)(s-b)(s-c)}$

And $\quad \sin C = \dfrac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$.

Example 4. *In any triangle ABC, show that:*

(i) $2a\sin\dfrac{B}{2}\sin\dfrac{C}{2} = (b+c-a)\sin\dfrac{A}{2}$ (ii) $2a\cos\dfrac{B}{2}\cos\dfrac{C}{2} = (a+b+c)\sin\dfrac{A}{2}$.

Sol. (i) L.H.S. = $2a\sin\dfrac{B}{2}\sin\dfrac{C}{2} = 2a\sqrt{\dfrac{(s-c)(s-a)}{ca}}\sqrt{\dfrac{(s-a)(s-b)}{ab}}$

$$= \frac{2a(s-a)}{a}\sqrt{\frac{(s-c)(s-b)}{bc}}$$

$$= (2s-2a)\sin\frac{A}{2} = (a+b+c-2a)\sin\frac{A}{2}$$

$$= (b+c-a) = \sin\frac{A}{2} = \text{R.H.S.}$$

(ii) L.H.S. = $2a\cos\dfrac{B}{2}\cos\dfrac{C}{2} = 2a\sqrt{\dfrac{s(s-b)}{ca}}\sqrt{\dfrac{s(s-c)}{ab}} = \dfrac{2as}{a}\sqrt{\dfrac{(s-b)(s-c)}{bc}}$

$$= 2s\sin\frac{A}{2} = (a+b+c)\sin\frac{A}{2} = \text{R.H.S.}$$

WORKING RULES FOR SOLVING PROBLEMS

Rule I. (i) $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$ (ii) $\sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ac}}$

(iii) $\sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$

Rule II. (i) $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$ (ii) $\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}}$

(iii) $\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$

Rule III. (i) $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ (ii) $\tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$

(iii) $\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

EXERCISE 14.2

SHORT ANSWER TYPE QUESTIONS

1. In a triangle *ABC*, if $a = 13$, $b = 14$, $c = 15$, find:

   (i) $\sin\frac{A}{2}$ (ii) $\cos\frac{A}{2}$ (iii) $\tan\frac{A}{2}$

2. In a triangle *ABC*, if $a = 24$, $b = 36$, $c = 45$, find:

   (i) $\sin\frac{B}{2}$ (ii) $\cos\frac{B}{2}$ (iii) $\tan\frac{B}{2}$.

3. In a triangle *ABC*, if $a = 13$, $b= 14$, $c = 15$, find :

   (i) $\tan\frac{B}{2}$ (ii) $\tan\frac{C}{2}$.

LONG ANSWER TYPE QUESTIONS

4. In any triangle *ABC,* if $3\tan\frac{A}{2}\tan\frac{C}{2}=1$, show that *a, b, c* are in A.P.

5. If in a triangle *ABC*, sin *A*, sin *B*, sin *C* are in A.P. show that $3\tan\frac{A}{2}\tan\frac{C}{2}=1$.

6. In any triangle *ABC*, if $b + c = 3a$, show that $\cot\frac{B}{2}\cot\frac{C}{2}=2$.

Answers

1. (i) $\frac{1}{\sqrt{5}}$ (ii) $\frac{2}{\sqrt{5}}$ (iii) $\frac{1}{2}$

2. (i) 0.444 (ii) 0.895 (iii) 0.496

3. (i) $\frac{4}{7}$ (ii) $\frac{2}{3}$

SUMMARY

If ABC is a triangle with sides a = BC, b = CA, c = AB, then:

1. $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ **(Sine formula)**

2. (a) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

   (b) $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$ **(Cosine formulae)**

   (c) $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

3. (a) $a = b \cos C + c \cos B$

   (b) $b = c \cos A + a \cos C$ **(Projection formulae)**

   (c) $c = a \cos B + b \cos A$

4. (a) $\tan \dfrac{B - C}{2} = \dfrac{b - c}{b + c} \cot \dfrac{A}{2}$ (b) $\tan \dfrac{C - A}{2} = \dfrac{c - a}{c + a} \cot \dfrac{B}{2}$

   (c) $\tan \dfrac{A - B}{2} = \dfrac{a - b}{a + b} \cot \dfrac{C}{2}$.

TEST YOURSELF

1. In any triangle *ABC*, show that :

   (i) $\dfrac{b^2 - c^2}{\cos B + \cos C} = \dfrac{c^2 - a^2}{\cos C + \cos A} + \dfrac{a^2 - b^2}{\cos A + \cos B} = 0$

   (ii) $\dfrac{b^2 - c^2}{a \sin(B - C)} = \dfrac{c^2 - a^2}{b \sin(C - A)} + \dfrac{a^2 - b^2}{c \sin(A - B)}$.

2. If in ΔABC, $\dfrac{\sin A}{\sin C} = \dfrac{\sin(A - B)}{\sin(B - C)}$, show that *a, b, c* are in A.P.

3. In any triangle *ABC*, if a^2, b^2, c^2 are in A.P., show that cot *A*, cot *B*, cot *C* are also in A.P.

4. If in a triangle *ABC*, $\dfrac{b + c}{11} = \dfrac{c + a}{12} = \dfrac{a + b}{13}$, then show that $\dfrac{\cos A}{7} = \dfrac{\cos B}{19} = \dfrac{\cos C}{25}$.

SECTION – B

15. AREA OF A TRIANGLE

LEARNING OBJECTIVES

- Hero's Formula
- Area of a Triangle
- Area of a Triangle when One Side and Two Angles are Given
- Area of a Triangle when Two Sides and One Angle are Given
- Area of a Triangle when All Sides are Given

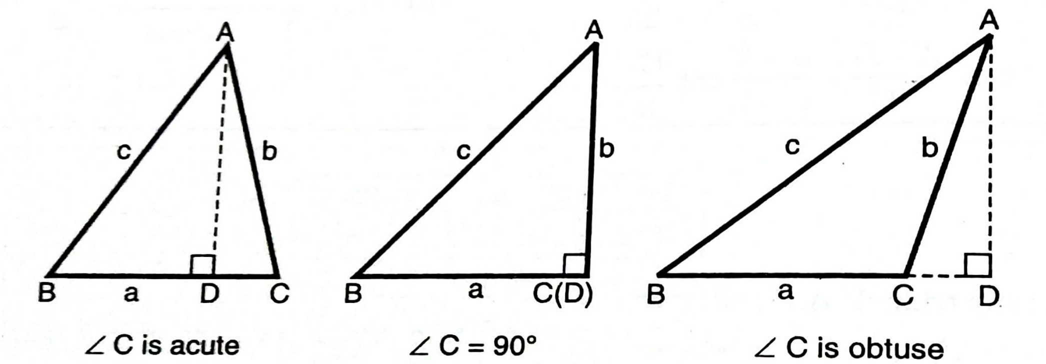
HERO'S FORMULA

Theorem. In any triangle ABC, prove that the area (Δ) of triangle ABC is given by

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)},$$

where s is the semi-perimeter of the triangle.

Proof. Let *ABC* be the triangle with $a = BC$, $b = CA$, $c = AB$.



The angle C is either acute or right angle or obtuse

From A draw $AD \perp BC$ (produced if necessary).

We have $\Delta = \frac{1}{2}$ base x height $= \frac{1}{2} BC \times AD = \frac{1}{2} a \,.\, AD = \frac{1}{2} a \,.\, \frac{AD}{AB} \,.\, AB$

$$= \frac{1}{2} a \;.\; \sin B.\ c \;= \frac{1}{2} ac\ \sin B = \frac{1}{2} ac \,.\, 2\sin\frac{B}{2}\cos\frac{B}{2}$$

$$= ac\sqrt{\frac{(s-c)(s-a)}{ca}}\,.\sqrt{\frac{s(s-b)}{ca}} = \sqrt{s(s-a)(s-b)(s-c)}\,.$$

$\therefore$ $\Delta = \sqrt{s(s-a)(s-b)(s-c)}\,.$

This formula is known as **Hero's formula.**

Remark. In the above theorem, we have also proved that $\Delta = \frac{1}{2} ac \sin B$.

Similarly, we can prove that $\Delta = \frac{1}{2} ab \sin C$ and $\Delta = \frac{1}{2} bc \sin A$.

$\therefore$ $\Delta = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ac \sin B$.

Corollary. Prove that

$$\Delta = \frac{a^2 \sin B \sin C}{2\sin A} = \frac{b^2 \sin C \sin A}{2\sin B} = \frac{c^2 \sin A \sin B}{2\sin C}\,.$$

Proof. We have $\Delta = \frac{1}{2} bc \sin A$ ....(1)

By **sine formula,** $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$\therefore$ $b = a\frac{\sin B}{\sin A}$ and $c = a\frac{\sin C}{\sin A}$

$\therefore$ (1) implies $\Delta = \frac{1}{2}\left(a\frac{\sin B}{\sin A}\right)\left(a\frac{\sin C}{\sin A}\right)\sin A = \frac{a^2 \sin B \sin C}{2\sin A}$

Similarly, $\Delta = \frac{b^2 \sin C \sin A}{2\sin B}$ and $\Delta = \frac{c^2 \sin A \sin B}{2\sin C}$.

AREA OF A TRIANGLE

We shall consider the method of finding the area of a triangle in the following three cases :

I. **When one side and two angles are given.**
II. **When two sides and one angle are given.**
III. **When all sides are given.**

AREA OF A TRIANGLE WHEN ONE SIDE AND TWO ANGLES ARE GIVEN

Let ABC be a triangle with sides $a = BC$, $b = CA$, $c = AB$. Let one side and two angles be known. The third angle is found by using the fact that $A + B + C = 180^0$.

The area (Δ) of the triangle ABC is found by using the formula:

$$\Delta = \frac{a^2 \sin B \sin C}{2 \sin B} = \frac{b^2 \sin A \sin C}{2 \sin B} = \frac{c^2 \sin A \sin B}{2 \sin C}$$

Example 1. *Find the area of the triangle ABC, when* $a = 2(\sqrt{3}+1), B = 45^0, C = 60^0$

Sol. We have

$$a = 2(\sqrt{3}+1), B = 45^0, C = 60^0 .$$

$$\therefore \qquad A = 180^0 - (B + C) = 180^0 - (45^0 + 60^0) = 75^0$$

Using $\Delta = \dfrac{a^2 \sin B \sin C}{2 \sin A}$, we have

$$\Delta = \frac{(2\sqrt{3}+1)^2 \sin 45^0 \sin 60^0}{2 \sin 75^0} = \frac{4(\sqrt{3}+1)^2 (1/\sqrt{2})(\sqrt{3}/2)}{2 \sin(45^0 + 30^0)}$$

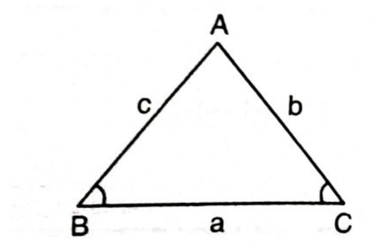
$$= \frac{(\sqrt{3}+1)^2 \sqrt{3/2}}{\frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}.\frac{1}{2}} = \frac{2\sqrt{3}(\sqrt{3}+1)^2}{\sqrt{3}+1} = 2\sqrt{3}(\sqrt{3}+1)$$ sq. units.

AREA OF A TRIANGLE WHEN TWO SIDES AND ONE ANGLE ARE GIVEN

Let *ABC* be a triangle with sides *a* = *BC*, *b* = *AC*, *c* = *AB*. Let one side and two angles be known. The given angle may or may not be the angle included between the sides. We shall consider these possibilities separately.

Case I. The given angle is the angle included between the given sides.

To be specific, let the elements *b, c, A* be given. The area Δ of the triangle is found by using the formula :



$$\Delta = \frac{1}{2} bc \sin A$$

Example 2. *Find the area of the triangle ABC, when a = 23.3 cm, b = 21.5 cm, C = 121⁰.*

Sol. We have a = 23.3 cm, b = 21.5 cm, C = 121^0.

Using $\Delta = \frac{1}{2} ab \sin C$, we have

$$\Delta = \frac{1}{2}(23.3)(21.5)\sin 121^0$$

$$= 250.475 \sin (180^0 - 121^0) = 250.475 \sin 59^0$$

= 250.475 (0.8572) = **214.707 sq. units.**

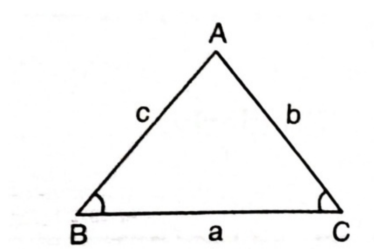
Case II. The given angle is not the angle included between the given sides.

To be specific, let the elements *b, c, B* (opposite to side *b*) be given.

By **law of sines,**

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \qquad \sin C = \frac{c}{b} \sin B$$



If sin $C = 1$ then $C = 90^0$

If $0 < \sin C < 1$ then there are two supplementary values of C *i.e.*, if one value is C_1 (say), then the other value is $180^0 = C_1 = C_2$, say. If $B + C_1$, and $B + C_2$ are both less than 180^0, then two triangles are possible. If either $B + C_1$ or $B + C_2$ is not less than 180^0, then that corresponding value of C is rejected, because in that case, triangle cannot be sketched. The angle A is found by the relation $A + B + C = 180^0$. The area (Δ) of the triangle ABC is found by using the formula.

$$\Delta = \frac{1}{2} bc \sin A \; .$$

AREA OF A TRIANGLE WHEN ALL SIDES ARE GIVEN

Let ABC be a triangle with sides $a = BC$, $b = CA$, $c = AB$. Let all sides be known. The area of the triangle is found by using **Hero's formula**.

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}\,, \text{ where } s = \frac{1}{2}(a+b+c).$$

Example 3. *Find the area of the triangle ABC, when $a = 28.16$, $b = 60.15$, $c = 51.17$.*

Sol. We have $a = 28.16$, $b = 60.15$, $c = 51.17$.

$$\therefore \qquad s = \frac{a+b+c}{2} = \frac{28.16 + 60.15 + 51.17}{2} = 69.74$$

Hero's formula is

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\therefore \qquad \Delta = \sqrt{69.74(69.74 - 28.16)(69.74 - 60.15)(69.74 - 51.17)}$$

$$= \sqrt{(69.74)(41.58)(9.59)(18.57)} \; = \mathbf{718.6 \text{ sq. units.}}$$

EXERCISE 15.1

LONG ANSWER TYPE QUESTIONS

1. Find the area of the triangle ABC when :
   (i) $c = 23$ cm, $A = 20^0$ and $C = 15^0$
   (ii) $c = 23$ cm, $A = 20^0$ and $B = 15^0$
   (iii) $a = 12$ cm, $B = 65^0$ and $C = 35^0$
   (iv) $b = 34.9$ cm, $A = 37' 10^0$ and $C = 62^0 30'$.

2. Find the area of the triangle ABC when :
   (i) $a = 5$, $b = 6$, $C = 30^0$
   (ii) $a = 112$, $b = 219$, $c = 20^0$
   (iii) $b = 27$, $c = 14$, $A = 43^0$
   (iv) $a = 14.27$ cm, $c = 17.23$ cm, $B = 86^0\ 14'$
   (v) $a = 123$, $b = 96.2$, $A = 41^0 50'$.

3. Find the area of the triangle ABC when :
   (i) $a = 3$, $b = 4$, $c = 5$
   (ii) $a = 4$, $b = 13$, $c = 15$.

Answers

1. (i) 200 sq. cm (ii) 41 sq. cm
   (iii) 38 sq. cm (iv) 331 sq. cm
2. (i) 7.5 sq. units (ii) 4600 sq. units
   (iii) 130 sq. units (iv) 122.7 sq. cm
   (v) 5660 sq. units
3. (i) 6 sq. units (ii) 24 sq. units.

SUMMARY

1. In the triangle ABC with sides *a, b, c,* we have

   (i) $\Delta = \frac{1}{2} bc \sin A$ (ii) $\Delta = \frac{1}{2} ac \sin B$ (iii) $\Delta = \frac{1}{2} ab \sin C$.

2. In the triangle ABC with sides, *a, b, c,* we have

   (i) $\Delta = \frac{a^2 \sin B \sin C}{2 \sin A}$ (ii) $\Delta = \frac{b^2 \sin C \sin A}{2 \sin B}$ (iii) $\Delta = \frac{c^2 \sin A \sin B}{2 \sin C}$

3. In the triangle ABC with sides *a, b, c,* we have

   $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$.

   This formula is known as **Hero's formula.**

TEST YOURSELF

1. If the area of a triangle is 75 sq. cm and two of its sides are 20 cm and 15 cm, find the angle between these sides.
2. In any triangle *ABC*, show that :

   (i) $\Delta = s(s-a) \tan \frac{A}{2}$ (ii) $\Delta = s(s-b) \tan \frac{B}{2}$

   (iii) $\Delta = s(s-c) \tan \frac{C}{2}$.

3. In any triangle *ABC*, show that :

   (i) $abcs \, . \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \Delta^2$ (ii) $\frac{abc}{s} \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} = \Delta$

Answer

1. 30^0.

SECTION – B

16. SOLUTION OF TRIANGLE

LEARNING OBJECTIVES

- Introduction
- Solution of Triangle
- Solution of Triangle when one Sides and Two Angles are Given
- Solution of Triangle when Two Sides and One Angle are Given
- Solution of Triangle when all Sides are Given
- Solution of Triangle when all Angles are Given

INTRODUCTION

We know that the three sides and the three angles of a triangle are called the **elements** of the triangle under consideration. When any three elements (with at least one side) of a triangle are given, then the remaining elements can be found by using trigonometric formulae. The process of determining remaining elements of a triangle is called **'solution of triangle'.**

The three angles of a triangle are not independent because their sum is always 180^0. In the process of solving triangles, we shall require the values of *t*-ratios for other angles in terms of *t*- ratios for all possible angles of a triangle.

We have already learnt the methods of finding the values of ***t-ratios*** for acute angles like 0^0, 15^0, 30^0, 45^0, …….. We also know the method of finding the *t*-ratios for other angles in terms of *t*-ratios for acute angles.

In general, the values of *t*-ratios for any desired angle are found by using **trigonometrical tables.**

SOLUTION OF TRIANGLES

We shall consider the solution of triangles in the following four possible cases :

I. **When one side and two angles are given**
II. **When two sides and one angle is given**
III. **When all sides are given**
IV. **When all angles are given**

Remark. In case of a right triangle *i.e.,* when one angle is given to be 90^0, we have an added advantage of making use of *Pythagorean result.* We have stream-lined the process of solving triangles by avoiding solving right triangles and oblique triangles separately, because there is no technical difference between the methods, except for the availability of *Pythagorean result* for right triangles.

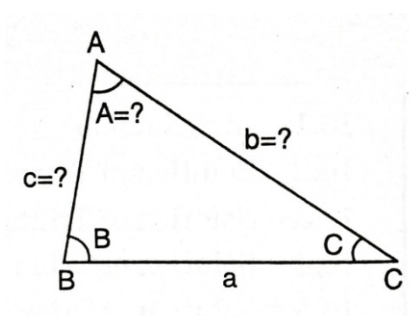
SOLUTION OF TRIANGLE WHEN ONE SIDE AND TWO ANGLES ARE GIVEN

Let ABC be a triangle with sides $a = BC$, $b = CA$, $c = AB$. Let one side and two angles of ΔABC be known. To be specific, let the elements a, B, C be known. The elements to be determined are b, c, A.

Now $A + B + C = 180^0$ implies $A = 180^0 - (B + C)$.

$\therefore$ A is known.

By **law of sines** $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



$\therefore$ $$b = a\frac{\sin B}{\sin A}$$

and $$c = a\frac{\sin C}{\sin A}.$$

$\therefore$ The triangle ABC is solved.

Remark. In case of a right triangle, the third side can also be found out by using *Pythagorean result.*

WORKING RULES FOR SOLVING TRIANGLES

Step I. *Find the third angle by subtracting the given angles from 180^0.*

Step II. *Put the values of all angle and the known side in the 'sine formula':*

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$

Step III. *Let the known side be 'a'.*

$$\therefore \qquad b=a\frac{\sin B}{\sin A} \quad \text{and} \quad c=a\frac{\sin C}{\sin A}.$$

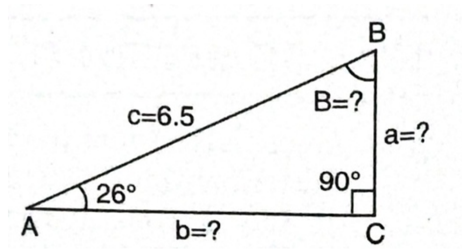
Example 1. *In a right triangle ABC, A = 26^0, C = 90^0, and c = 6.5. solve the triangle.*

Sol. We have $A = 26^0$, $C = 90^0$, $c = 6.5$.

To find B. $A + B + C = 180^0 \Rightarrow B = 180^0 - (A + C)$

$= 180^0 - (26^0 + 90^0) = 64^0$.

To find a, b. By **sine formula :**



$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

$$\therefore \qquad \frac{a}{\sin 26^0}=\frac{b}{\sin 64^0}=\frac{c}{\sin 90^0}$$

$\Rightarrow \qquad a = 6.5 \sin 26^0$ ......(1) and $b = 6.5 \sin 64^0$ .....(2)

($\because \sin 90^0 = 1$)

(1) $\Rightarrow$ $a = 6.5 \text{ X } 0.4384 = 2.8496$

(2) $\Rightarrow$ $b = 6.5 \text{ X } 0.8988 = 5.8422$

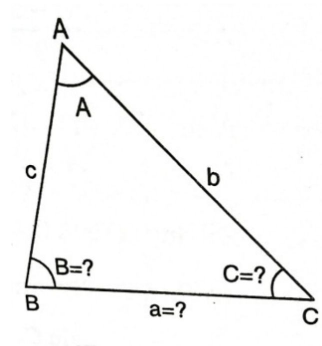
$\therefore$ Remaining elements are $B =$ **64^0**, $a =$ **2.8496** and $b =$ **5.8422**.

SOLUTION OF TRIANGLE WHEN TWO SIDES AND ONE ANGLE ARE GIVEN

Let *ABC* be a triangle with sides $a = BC$, $b = CA$, $c = AB$. Let two sides and one angle be given. The given angle may or may not be the angle included between given sides. We shall consider these possibilities separately.

Case I. The given angle is the angle included between the given sides.

To be specific, let the elements *A, b, c* be given.



To find B, C. By **law of tangents,**

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2} \qquad \text{(Assuming } b \geq c\text{)}$$

By using, the table of *natural tangents,* we shall find the value of $\frac{B-C}{2}$ and so the value of $B - C$.

Also, $B + C = 180^0 - A$. $\therefore$ *B* and *C* are known.

To find a. By **law of sines,**

$$\frac{a}{\sin A}=\frac{b}{\sin B}\left(or\ \frac{c}{\sin C}\right).$$

$$\therefore \qquad a=b\frac{\sin A}{\sin B} \qquad \left(\text{or equivalently } c\ \frac{\sin A}{\sin C}\right)$$

$\therefore$ The triangle *ABC* is solved.

Remark 1. If $b < c$, we use the formula, $\tan\frac{C-B}{2}=\frac{c-b}{c+b}\cot\frac{A}{2}$.

Remark 2. In case of a right triangle, the triangle can also be solved in a slightly different way. The process is explained in the examples.

WORKING RULES FOR SOLVING TRIANGLES

Let the known elements be A, b and c.

Step I. *If b $\geq$ c, put the values of A, b and c in the 'law of tangents' :*

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}\ . \qquad ...(1)$$

If b < c, put the values of A, b and c in the 'law of tangents' :

$$\tan\frac{C-B}{2}=\frac{c-b}{c+b}\cot\frac{A}{2}\ . \qquad ....(1`)$$

Step II. *Using trigonometrical tables, find the value of B – C (or C- B).*

Step III. *Put the value of A in the relation A + B + C = 180^0 and get the value of B + C. This equation and that obtained in* **Step II,** *give the values of angles B and C.*

Step IV. *To find side 'a', use :* $\frac{a}{\sin A}=\frac{b}{\sin B}\left(or\ \frac{c}{\sin C}\right)$

$\therefore \quad a=b\frac{\sin A}{\sin B}$ $\qquad\left(\text{or equivalently } a=c\ \frac{\sin A}{\sin C}\right)$.

Example 2. *Solve the right triangle ABC, given that C = 90^0, a = 50.4, b = 26.2.*

Sol. We have C = 90^0, a = 50.4, b = 26.2.

To find A, B. By **law of tangents,** $\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$.

$\therefore$

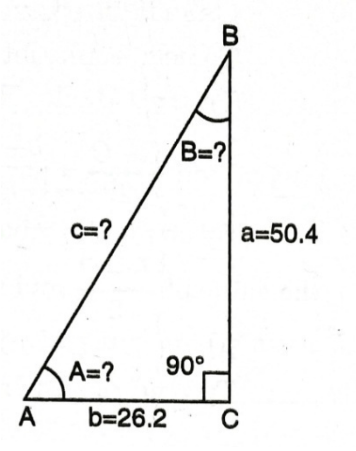
$$\tan\frac{A-B}{2}=\frac{50.4-26.2}{50.4+26.2}\cot\frac{90^0}{2}=\frac{24.2}{76.6}\times 1=0.3159$$

$\therefore \quad \frac{A-2}{2}=17^0 32`$ (From tables)

$\therefore \quad A-B = 35^0\ 4`$ ....(1)

Also $A + B + C = 180^0$

$\therefore \quad A + B = 180^0 - C = 180^0 = 90^0 = 90^0$



$\therefore \quad A + B = 90^0$

Solving (1) and (2), we get $A = 52^0\ 32`$ and $B = 27^0\ 28`$.

To find c. By **sine formula :** $\frac{a}{\sin A} = \frac{c}{\sin C}$ **.**

$$\Rightarrow \quad c = a\frac{\sin C}{\sin A} = 50.4 \times \frac{\sin 90^0}{\sin 62^0 32`}$$

$$= 50.4 \times \frac{1}{0.8873} = 56.3$$

$\therefore$ Remaining elements are c = **56.8,** $\boldsymbol{A = 62^0\ 32`}$**,** $\boldsymbol{B = 27^0\ 28`}$

WORKING RULES FOR SOLVING TRIANGLES

Let the known elements be b, c and B.

Step I. *To find angle 'C', use :* $\frac{b}{\sin B} = \frac{c}{\sin C}$.

$$\Rightarrow \quad \sin C = \frac{c}{b}\sin B$$

Step II. *Sin C is either > 1 or = 1 or between 0 and 1.*

(i) *If sin C > 1, then there is* ***no*** *triangle.*

(ii) *If sin C = 1, then C = 90⁰ and there is* **exactly one** *triangle.*

(iii) *If 0 < sin C < 1, then there exist two values say C_1 and C_2 of C satisfying $C_1 + C_2 = 180^0$. If $B + C_1$ and $B + C_2$ are both less than 180^0, then there are two triangles with given elements. If either $B + C_1$ or $B + C_2$ is not less than 180^0, then reject that value of C.*

Step III. *To find angle A, use : $A = 180^0 - (B + C)$. If there is one value of C, Then A has one value. If there are two values of C, say C_1 and C_2, then A has two values, say A_1 and A_2 respectively.*

Step IV. *To find side 'a', use :*

$$\frac{a}{\sin A} = \frac{b}{\sin B} \qquad \left(or \frac{c}{\sin C}\right)$$

$\therefore \quad a = b\dfrac{\sin A}{\sin B} \quad \left(\text{or equivalently } c\,\dfrac{\sin A}{\sin C}\right)$

If there are two values of C, namely C_1 and C_2, then 'a' has two values, say a_1 and a_2 respectively and are given by

$$a_1 = b\frac{\sin A_1}{\sin B}, a_2 = b\frac{\sin A_2}{\sin B}$$

$$\left(\text{or equivalently } a_1 = c\,\frac{\sin A_1}{\sin C_1},\ a_2 = c\,\frac{\sin A_2}{\sin C_2}\right)$$

SOLUTION OF TRIANGLE WHEN ALL SIDES ARE GIVEN

Let *ABC* be a triangle with sides $a = BC$, $b = CA$, $c = AB$. Let all the three sides *a, b, c* be known.

We have $$s = \frac{a+b+c}{2}$$

To find A. $$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

By using the tables of *natural tangents,* we find the values of *A*.

To find B. $$\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$$

By using the tables of *natural tangents,* we find the value of *B.*

To find C. $A + B + C = 180^0$ implies $C = 180^0 - (A + B)$.

$\therefore$ The triangle is solved.

Remark 1. When one or more sides of a triangle are irrational number, then the angles *A* and *B* should be found by *cosine formulae :* $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$.

Remark 2. If the values of *a, b* and *c* are small numbers, than the angles can be easily found by using '*cosine formulae*'.

WORKING RULES FOR SOLVING TRIANGLES

Step I. *Find* $s = \frac{a+b+c}{2}$.

Step II. *Put the values of s, a, b, c in any of the formulae :*

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}, \tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Find the value of A by using 'trigonometrical tables'.

If the values of a, b, c are small numbers then use $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ *c.*

Step III. *Find angle B by following the same method.*

Step IV. *Find the third angle by subtracting the angles A and B from* 180^0.

Example 3. *Find the greatest angle of the triangle ABC in which a = 2, b =* $\sqrt{6}$ *and c =* $\sqrt{3}$ *- 1.*

Sol. The greatest side is $b = \sqrt{6}$. $\therefore$ The greatest angle is *B*.

By **cosine formula,**

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{\left(\sqrt{3}-1\right)^2 + (2)^2 - \left(\sqrt{6}\right)^2}{2\left(\sqrt{3}-1)(2)\right)}$$

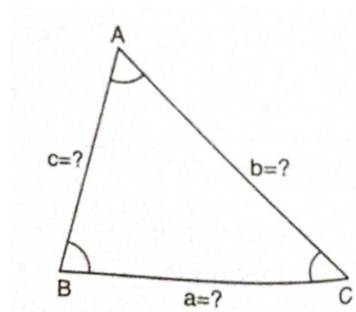
$$= \frac{3 + 1 - 2\sqrt{3} + 4 - 6}{4\left(\sqrt{3}-1\right)} = \frac{2 - 2\sqrt{3}}{4\left(\sqrt{3}-1\right)} = \frac{-2\left(\sqrt{3}-1\right)}{4\left(\sqrt{3}-1\right)} = -\frac{1}{2}$$

$\therefore$ $B =$ **120^0**.

SOLUTION OF TRIANGLE WHEN ALL ANGLES ARE GIVEN

Let *ABC* be a triangle with sides $a = BC$, $b = CA$ and $c = AB$. Let all the three angles *A,B,C* be known.

By **sine formula,** $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$



$\therefore$ *a, b, c* are in the ratio of sin *A*, sin *B*, sin *C*.

In this case, we will not be able to find the actual values of *a, b,* and *c,* rather the ratio of the sides can be determined.

Remark. In the above case, it is sufficient to know only two angles, because the third can be found by using fact $A + B + C = 180^0$.

WORKING RULES FOR SOLVING TRAINGLES

Step I. *If only two angles are given, then find the third angle by subtracting the given angles from* 180^0.

Step II. *Put the values of the angles in the 'sine formula' :*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Step III. *The sides of the triangle are in the ratio sin A : sin B : sin C.*

Example 4. *The angles of a triangle are in the ratio 1 : 2 : 7. Show that the ratio of the greatest side to the least side is* $\sqrt{5}+1 : \sqrt{5}-1$.

Sol. Let the triangle *ABC* be with sides $a = BC$, $b = CA$, $c = AB$.

The angles are in the ratio 1 : 2 : 7.

$\therefore \quad \frac{A}{1}=\frac{B}{2}=\frac{C}{2}=\frac{A+B+C}{1+2+7}=\frac{180^0}{10}=18^0$

$\therefore \quad A = 18^0$, $B = 2(18^0) = 36^0$, $C = 7(18^0) = 126^0$

$\therefore$ The greatest and the least sides are c and a respectively.

By **law of sines,** $\frac{a}{\sin A}=\frac{c}{\sin C}$.

$$\therefore \quad \frac{c}{a}=\frac{\sin C}{\sin A}=\frac{\sin 126^0}{\sin 18^0}=\frac{\sin\left(90^0+36^0\right)}{\sin 18^0}=\frac{\cos 36^0}{\sin 18^0}=\frac{\frac{\sqrt{5}+1}{4}}{\frac{\sqrt{5}-1}{4}}=\frac{\sqrt{5}+1}{\sqrt{5}-1}$$

$\therefore$ The greatest side and the least sides are in the ratio $\sqrt{5}+1 : \sqrt{5}-1$.

EXERCISE 16.1

LONG ANSWER TYPE QUESTIONS

Type I

1. Solve the triangle ABC, given that b = 4.5, A= 39^0, C = 90^0.
2. Solve the triangle ABC, given that b = 302, A = 50^0 10`, C = 72^0.

Type II

3. Solve the triangle ABC, given that a = 123.4, b = 234.5, c = 90^0.
4. If $a=\sqrt{3}+1, b=\sqrt{3}-1$ and C = 60^0, find the other side and the angles of the triangle ABC.

Type III

5. If a = 5, b = 7 and sin A = 3/4, solve the triangle ABC, if possible.
6. If a = 6, b = 8 and A = 30^0, find c of the triangle ABC.

Type IV

7. The sides of a triangle are $x^2+x+1, 2x+1$ and $x^2 - 1$, where $x > 1$. Find the greatest angle.
8. Solve the triangle ABC, given that a = 25, b = 26, c = 27.

Type V

9. If the angle of a triangle are in the ratio 1 : 2 :3, show that the sides are in the ratio $\sqrt{1}:\sqrt{3}:\sqrt{4}$.

10. In the triangle ABC, $A = 45^0$, $B = 75^0$, $C = 60^0$, show that $a + c\sqrt{2} = 2b$.

Answers

1. $B = 51^0$, $a = 3.644$, $c = 5.79$

2. $B = 57^0\ 50'$, $a = 274$, $c = 339$

3. $A = 27^0\ 45'\ 20''$, $B = 62^0\ 14'\ 40''$, $c = 265$

4. $A = 105^0$, $B = 15^0$, $c = \sqrt{6}$

5. Not possible

6. $4\sqrt{3} - 2\sqrt{5}$ or $4\sqrt{3} + 2\sqrt{5}$

7. 120^0

8. $A = 56^0\ 15'\ 4''$, $B = 59''\ 51'\ 10''$, $C = 63^0\ 53'\ 46''$.

SUMMARY

1. When any three elements (with atleast one side) of a triangle are given, than the process of determining remaining elements is called the **solution of triangle.**
2. The solution of triangle is possible in the following cases :
   (i) When one side and two angles are given.
   (ii) When two sides are one angle is given.
   (iii) When all sides are given.
3. In solving triangles the following formulae are used :
   (i) Sine formula
   (ii) Cosine formulae
   (iii) Napier analog.

TEST YOURSELF

1. Solve the triangle ABC, given that $c = 72$, $A = 56^0$, $B = 65^0$.
2. Solve the triangle ABC, given that $a = 18$, $A = 25^0$, $B = 180^0$.
3. Solve the triangle ABC, given that $a = 40$, $c = 40\sqrt{3}$, $B = 30^0$.
4. If the sides a and b of a triangle ABC are in the ratio 7 : 3 and the included angle C is 60^0, find A and B.
5. In the triangle ABC if $b = 14$, $c = 11$ and $A = 60^0$, find B and C.
6. Solve the triangle ABC, given that $a = 843.2$, $c = 1020$, $C = 90^0$.

Answers

1. $C = 59^0$, $a = 69.63$, $b = 76.12$
2. $C = 47^0$, $b = 40.51$, $c = 31.15$
3. $A = 30^0$, $C = 120^0$, $b = 40$
4. $A = 94^0\ 43'$, $B = 25^0\ 17'$
5. $B\ 71^0\ 44'\ 30''$, $C = 48^0\ 15'\ 30''$
6. $B = 571.1$, $A = 55^0\ 45'$, $B = 34^0\ 15'$.

Trigonometrical Tables
NATURAL SINES

Degrees	0′ 0°.0	6′ 0°.1	12′ 0°.2	18′ 0°.3	24′ 0°.4	30′ 0°.5	36′ 0°.6	42′ 0°.7	48′ 0°.8	54′ 0°.9	Mean Differences 1	2	3	4	5
0	.000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	.0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	.0349	0366	0384	0401	0419	0436	0454	0417	0488	0506	3	6	9	12	15
3	.0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3	6	9	12	15
4	.0698	0715	0732	0750	0767	0785	0802	0819	0637	0854	3	6	9	12	15
5	.0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	3	6	9	12	14
6	.1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	3	6	9	12	14
7	.1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3	6	9	12	14
8	.1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3	6	9	12	14
9	.1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3	6	9	12	14
10	.1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	3	6	9	12	14
11	.1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3	6	9	11	14
12	.2079	2096	2113	2130	2147	2164	2181	2198	2215	2232	3	6	9	11	14
13	.2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3	6	8	11	14
14	.2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3	6	8	11	14
15	.2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	3	6	8	11	14
16	.2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3	6	8	11	14
17	.2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14
18	.3090	3107	3123	3140	3158	3173	3190	3206	3223	3239	3	6	8	11	14
19	.3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3	5	8	11	14
20	.3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3	5	8	11	14
21	.3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3	5	8	11	14
22	.3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3	5	8	11	14
23	.3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3	5	8	11	14
24	.4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3	5	8	11	13
25	.4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	3	5	8	11	13
26	.4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3	5	8	10	13
27	.4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3	5	8	10	13
28	.4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3	5	8	10	13
29	.4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	3	5	8	10	13
30	.5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	3	5	8	10	13
31	.5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2	5	7	10	12
32	.5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2	5	7	10	12
33	5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2	5	7	10	12
34	5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2	5	7	10	12
35	.5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	2	5	7	10	12
36	.5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2	5	7	9	12
37	.6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2	5	7	9	12
38	.6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2	5	7	9	11
39	.6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2	4	7	9	11
40	.6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2	4	7	9	11
41	.6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	.6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	6	9	11
43	.6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	2	4	6	8	11
44	.6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10

NATURAL SINES

Degrees		0′ $0^{0}.0$	6′ $0^{0}.1$	12′ $0^{0}.2$	18′ $0^{0}.3$	24′ $0^{0}.4$	30′ $0^{0}.5$	36′ $0^{0}.6$	42′ $0^{0}.7$	48′ $0^{0}.8$	54′ $0^{0}.9$	Mean Differences 1 2 3	4 5
45		.7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	2 4 6	8 10
46		.7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	2 4 6	8 10
47		.7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	2 4 6	8 10
48		.7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	2 4 6	8 10
49		.7547	7558	7570	7581	7593	7604	7615	7627	7638	7649	2 4 6	8 9
50		.7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	2 4 6	7 9
51		.7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	2 4 5	7 9
52		.7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	2 4 5	7 9
53		.7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	2 3 5	7 9
54		.8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	2 3 5	7 8
55		.8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	2 3 5	7 8
56		.8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	2 3 5	6 8
57		.8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	2 3 5	6 8
58		.8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	2 3 5	6 8
59		.8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	1 3 4	6 7
60		.8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	1 3 4	6 7
61		.8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	1 3 4	6 7
62		.8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	1 3 4	5 7
63		.8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	1 3 4	5 6
64		.8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	1 3 4	5 6
65		.9063	9070	9078	9085	9092	9100	9107	9114	9212	9128	1 2 4	5 6
66		.9135	9143	9150	9157	9164	9717	9178	9184	9191	9198	1 2 3	5 6
67		.9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	1 2 3	4 6
68		.9272	9278	9285	9291	9298	9304	9311	9311	9323	9330	1 2 3	4 5
69		.9336	9342	9348	9354	9361	9367	9373	9373	9385	9391	1 2 3	4 5
70		.9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	1 2 3	4 5
71		.9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	1 2 3	4 5
72		.9511	9561	9521	9527	9532	9537	9542	9548	9548	9558	1 2 3	3 4
73		.9563	9568	9573	9578	9583	9588	9593	9598	9598	9608	1 2 2	3 4
74		.9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	1 2 2	3 4
75		.9659	9664	9668	9673	9677	9681	9685	9690	9694	9699	1 1 2	3 4
76		.9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	1 1 2	3 3
77		.9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	1 1 2	3 3
78		.9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	1 1 2	2 3
79		.9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	1 1 2	2 3
80		.9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	0 1 1	2 2
81		.9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	0 1 1	2 2
82		.9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	0 1 1	2 2
83		.9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	0 1 1	1 2
84		.9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	0 1 1	1 2
85		.9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	0 0 1	1 1
86		.9976	9977	9978	9978	9980	9981	9982	9983	9984	9985	0 0 1	1 1
87		.9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	0 0 0	1 1
88		.9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	0 0 0	0 0
89		.9998	9999	9999	9999	9999	1.000	1.000	1.000	1.000	1.000	0 0 0	0 0
90		1.000											

NATURAL COSINES

[*Numbers in difference columns to be subtracted, not added*]

Degrees	0′ $0^0.0$	6′ $0^0.1$	12′ $0^0.2$	18′ $0^0.3$	24′ $0^0.4$	30′ $0^0.5$	36′ $0^0.6$	42′ $0^0.7$	48′ $0^0.8$	54′ $0^0.9$	Mean Differences 1	2	3	4	5
0	1.000	1.000	1.000	1.000	1.000	1.000	.9999	9999	9999	9999	0	0	0	0	0
1	.9998	9998	9998	9997	9997	9997	9996	9996	9995	9995	0	0	0	0	0
2	.9994	9993	9993	9992	9991	9990	9990	9989	9988	9987	0	0	0	1	1
3	.9986	9985	9984	9983	9982	9981	9980	9979	9978	9977	0	0	1	1	1
4	.9976	9974	9973	9972	9971	9969	9968	9966	9965	9963	0	0	1	1	1
5	.9962	9960	9959	9957	9956	9954	9952	9951	9949	9947	0	1	1	1	2
6	.9945	9943	9942	9940	9938	9936	9934	9932	9930	9928	0	1	1	1	2
7	.9925	9923	9921	9919	9917	9914	9912	9910	9907	9905	0	1	1	2	2
8	.9903	9900	9898	9895	9893	9890	9888	9885	9882	9880	0	1	1	2	2
9	.9877	9874	9871	9869	9866	9863	9860	9857	9854	9851	0	1	1	2	2
10	.9848	9845	9842	9839	9836	9833	9829	9826	9823	9820	1	1	2	2	3
11	.9816	9813	9810	9806	9803	9799	9796	9792	9789	9785	1	1	2	2	3
12	.9781	9778	9774	9770	9767	9763	9759	9755	9751	9748	1	1	2	3	3
13	.9744	9740	9736	9732	9728	9724	9720	9715	9711	9707	1	1	2	3	3
14	.9703	9699	9694	9690	9686	9681	9677	9673	9668	9664	1	1	2	3	4
15	.9659	9655	9650	9646	9641	9636	9632	9627	9622	9617	1	2	2	3	4
16	.9613	9608	9603	9598	9593	9588	9583	9578	9573	9568	1	2	2	3	4
17	.9563	9558	9553	9548	9542	9537	9532	9527	9521	9516	1	2	3	3	4
18	.9511	9505	9500	9494	9489	9483	9478	9472	9466	9461	1	2	3	4	5
19	.9455	9449	9444	9438	9432	9426	9421	9415	9409	9403	1	2	3	4	5
20	.9397	9391	9385	9379	9373	9367	9361	9345	9348	9342	1	2	3	4	5
21	.9336	9330	9323	9317	9311	9304	9298	9291	9285	9278	1	2	3	4	5
22	.9272	9265	9259	9252	9245	9239	9232	9225	9219	9212	1	2	3	4	6
23	.9205	9198	9191	9184	9178	9171	9164	9157	9150	9143	1	2	3	5	6
24	.9135	9128	9121	9114	9107	9100	9092	9085	9078	9070	1	2	4	5	6
25	.9063	9056	9048	9041	9033	9026	9018	9011	9003	8996	1	3	4	5	6
26	.8988	8980	8973	8965	8957	8949	8942	8934	8926	8918	1	3	4	5	6
27	.8910	8902	8894	8886	8878	8870	8862	8854	8846	8838	1	3	4	5	7
28	.8829	8821	8813	8805	8796	8788	8780	8771	8763	8755	1	3	4	6	7
29	.8746	8738	8729	8721	8712	8704	8695	8686	8678	8669	1	3	4	6	7
30	.8660	8652	8643	8634	8625	8616	8607	8599	8590	8581	1	3	4	6	7
31	.8572	8563	8554	8545	8536	8526	8517	8508	8499	8490	2	3	5	6	8
32	.8480	8471	8462	8453	8443	8434	8425	8415	8406	8396	2	3	5	6	8
33	.8387	8377	8368	8358	8348	8339	8329	8320	8310	8300	2	3	5	6	8
34	.8290	8281	8271	8261	8251	8242	8231	8221	8211	8202	2	3	5	7	8
35	.8192	8181	8171	8161	8151	8141	8131	8121	8111	8100	2	3	5	7	8
36	.8090	8080	8070	8059	8049	8039	8028	8018	8007	7997	2	3	5	7	9
37	.7986	7976	7965	7955	7944	7934	7923	7912	7902	7891	2	4	5	7	9
38	.7880	7879	7859	7848	7837	7826	7815	7804	7793	7782	2	4	5	7	9
39	.7771	7760	7749	7738	7727	7716	7705	7694	7683	7672	2	4	6	7	9
40	.7660	7649	7638	7627	7615	7604	7593	7581	7570	7559	2	4	6	8	9
41	.7547	7536	7524	7513	7501	7490	7478	7466	7455	7443	2	4	6	8	10
42	.7431	7420	7408	7396	7385	7373	7361	7349	7337	7325	2	4	6	8	10
43	.7314	7302	7290	7278	7266	7254	7242	7230	7218	7206	2	4	6	8	10
44	.7193	7181	7169	7157	7145	7133	7120	7108	7096	7083	2	4	6	8	10

NATURAL CONSINES

[*Numbers in difference columns to be subtracted, not added*]

Degrees	0′ $0^0.0$	6′ $0^0.1$	12′ $0^0.2$	18′ $0^0.3$	24′ $0^0.4$	30′ $0^0.5$	36′ $0^0.6$	42′ $0^0.7$	48′ $0^0.8$	54′ $0^0.9$	Mean Differences 1 2 3	4 5
45	.7071	7059	7046	7034	7022	7009	6997	6984	6972	6959	2 4 6	8 10
46	.6947	6934	6921	6909	6896	6884	6871	6858	6845	6833	2 4 6	8 11
47	.6820	6807	6794	6782	6769	6756	6743	6730	6717	6704	2 4 6	9 11
48	.6691	6678	6665	6652	6639	6626	6613	6600	6587	6574	2 4 7	9 11
49	.6561	6547	6534	6521	6508	6494	6481	6468	6455	6441	2 4 7	9 11
50	.6428	6414	6401	6388	6374	6361	6347	6334	6320	6307	2 4 7	9 11
51	.6293	6280	6266	6252	6239	6225	6211	6198	6184	6170	2 5 7	9 11
52	.6157	6143	6129	6115	6101	6088	6074	6060	6046	6032	2 5 7	9 12
53	.6018	6004	5990	5976	5962	5948	5934	5920	5906	5892	2 5 7	9 12
54	.5878	5864	5850	5835	5821	5807	5793	5779	5764	5750	2 5 7	9 12
55	.5736	5721	5707	5693	5678	5664	5650	5635	5621	5606	2 5 7	10 12
56	.5592	5577	5563	5548	5534	5519	5505	5490	5476	5461	2 5 7	10 12
57	.5446	5432	5417	5402	5388	5373	5358	5344	5329	5314	2 5 7	10 12
58	.5299	5284	5270	5255	5240	5225	5210	5195	5180	5165	2 5 7	10 12
59	.5150	5135	5120	5105	5090	5075	5060	5045	5030	5015	3 5 8	10 13
60	.5000	4985	4970	4955	4939	4924	4909	4894	4879	4863	3 5 8	10 13
61	.4848	4833	4818	4802	4787	4772	4756	4741	4726	3923	3 5 8	10 13
62	.4695	4679	4664	4648	4633	4617	4602	4586	4571	3762	3 5 8	10 13
63	.4540	4524	4509	4493	4478	4462	4446	4431	4415	3600	3 5 8	10 13
64	.4384	4368	4352	4337	4321	4305	4289	4274	4258	3437	3 5 8	11 13
65	.4226	4210	4195	4179	4163	4147	4131	4115	4099	4083	3 5 8	11 14
66	.4067	4051	4035	4019	4003	3987	3971	3955	3939	3923	3 5 8	11 14
67	.3907	3891	3875	3859	3843	3827	3811	3795	3778	3762	3 5 8	11 14
68	.3746	3730	3714	3697	3681	3665	3647	3633	3616	3600	3 5 8	11 14
69	.3584	3567	3551	3535	3518	3502	3486	3469	3453	3437	3 5 8	11 14
70	.3420	3404	3387	3371	3355	3338	3322	3305	3829	3272	3 5 8	11 14
71	.3256	3239	3223	3206	3190	3173	3156	3140	3123	3107	3 6 8	11 14
72	.3090	3074	3057	3040	3024	3007	2290	2974	2957	2940	3 6 8	11 14
73	.2924	2907	2890	2874	2857	2840	2823	2807	2790	2773	3 6 8	11 14
74	.2756	2740	2723	2706	2689	2672	2656	2639	2622	2605	3 6 8	11 14
75	.2588	2571	2554	2538	2521	2504	2487	2470	2453	2436	3 6 8	11 14
76	.2419	2402	2385	2368	2351	2334	2317	2300	2284	2267	3 6 8	11 14
77	.2250	2233	2215	2198	2181	2164	2147	2130	2113	2096	3 6 9	11 14
78	.2079	2062	2045	2028	2011	1994	1977	1959	1942	1925	3 6 9	11 14
79	.1908	1891	1874	1857	1840	1822	1805	1788	1771	1754	3 6 9	11 14
80	.1736	1719	1702	1685	1668	1650	1633	1616	1599	1582	3 6 9	12 14
81	.1564	1547	1530	1513	1495	1478	1461	1444	1426	1409	3 6 9	12 14
82	.1392	1374	1357	1340	1323	1305	1288	1271	1253	1236	3 6 9	12 14
83	.1219	1201	1184	1167	1149	1132	1115	1097	1080	1063	3 6 9	12 14
84	.1045	1028	1011	0993	0976	0958	0941	0924	0906	0889	3 6 9	12 14
85	.0872	0854	0873	0819	0802	0785	0767	0750	0732	0715	3 6 9	12 15
86	.0698	0680	0663	0645	0628	0610	0593	0576	0558	0641	3 6 9	12 15
87	.0523	0506	0488	0471	0454	0436	0419	0401	0384	0360	3 6 9	12 15
88	.0349	0332	0314	0297	0279	0262	0244	0227	0209	0192	3 6 9	12 15
89	.0175	0157	0140	0122	0105	0087	0070	0052	0035	0017	3 6 9	12 15
90	1.000											

NATURAL TANGENTS

Degrees		0′ $0^0.0$	6′ $0^0.1$	12′ $0^0.2$	18′ $0^0.3$	24′ $0^0.4$	30′ $0^0.5$	36′ $0^0.6$	42′ $0^0.7$	48′ $0^0.8$	54′ $0^0.9$	Mean Differences 1 2 3	4 5
0		.0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3 6 9	12 15
1		.0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3 6 9	12 15
2		.0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	3 6 9	12 15
3		.0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	3 6 9	12 15
4		.0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	3 6 9	12 15
5		.0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	3 6 9	12 15
6		.1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	3 6 9	12 15
7		.1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	3 6 9	12 15
8		.1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	3 6 9	12 15
9		.1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	3 6 9	12 15
10		.1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	3 6 9	12 15
11		.1944	1962	1980	1998	2016	2935	2053	2071	2089	2107	3 6 9	12 15
12		.2126	2144	2180	2180	2199	2217	2235	2254	2272	2290	3 6 9	12 15
13		.2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	3 6 8	12 15
14		.2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	3 6 8	12 15
15		.2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	3 6 9	13 16
16		.2667	2886	2905	2924	2943	2962	2981	3000	3019	3038	3 6 9	13 16
17		.3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3 6 10	13 16
18		.3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3 6 10	13 16
19		.3443	3463	3482	3502	3522	3541	3561	3381	3600	3602	3 7 10	13 16
20		.3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3 7 10	13 17
21		.3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	3 7 10	13 17
22		.4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	3 7 10	14 17
23		.4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	3 7 10	14 17
24		.4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4 7 11	14 18
25		.4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4 7 11	14 18
26		.4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	4 7 11	15 18
27		.5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	4 7 11	15 19
28		.5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	4 8 11	15 19
29		.5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	4 8 12	15 19
30		.5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	4 8 12	16 20
31		.6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	4 8 12	16 20
32		.6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	4 8 12	16 20
33		.6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	4 8 13	17 21
34		.6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	4 9 13	17 21
35		.7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	4 9 13	18 23
36		.7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	5 9 14	18 23
37		.7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	5 9 14	18 23
38		.7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	5 9 14	19 24
39		.8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	5 10 14	20 24
40		.8391	8421	8451	8481	8511	8541	8871	8601	8632	8662	5 10 15	20 25
41		.8693	8724	8754	8785	8816	8847	8878	8910	8632	8972	5 10 16	21 26
42		.9004	9036	9067	9099	9131	9163	9195	9228	8941	9293	5 11 16	21 27
43		.9325	9358	9391	9424	9457	9490	9523	9556	9260	9623	6 11 17	22 28
44		.9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	6 1117	23 29

NATURAL TANGENTS

Degrees		0′ $0^0.0$	6′ $0^0.1$	12′ $0^0.2$	18′ $0^0.3$	24′ $0^0.4$	30′ $0^0.5$	36′ $0^0.6$	42′ $0^0.7$	48′ $0^0.8$	54′ $0^0.9$	Mean Differences 1 2 3	4 5
45		1.0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	6 12 18	24 30
46		1.0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	6 12 18	25 31
47		1.0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	6 13 19	25 32
48		1.1106	1145	1184	1224	1263	1303	1343	1483	1423	1463	7 13 20	27 33
49		1.1504	1544	1585	1626	1667	1708	1750	1833	1833	1875	7 14 21	28 34
50		1.1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	7 14 22	29 36
51		1.2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	8 15 23	30 38
52		1.2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	8 16 24	31 39
53		1.3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	8 16 25	33 41
54		1.3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	9 17 26	34 43
55		1.4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	9 18 27	36 45
56		1.4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	10 19 29	38 48
57		1.5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	10 20 30	40 50
58		1.6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	11 21 32	43 53
59		1.6643	6079	6775	6842	6909	6977	7045	7113	7182	7251	11 23 34	45 56
60		1.7321	7391	7461	7532	7603	7675	7747	7820	7893	7966	12 24 36	48 60
61		1.8040	8115	8190	8265	8341	8418	8495	8572	8650	8728	13 37 55	51 64
62		1.8807	8887	8967	9047	9128	9210	9292	9375	9458	9542	14 27 41	55 68
63		1.9626	9711	9797	9883	9970	2.0057	2.0145	2.0233	2.0323	2.0413	15 29 44	58 73
64		2.0503	0594	0686	0778	0872	0965	1060	1155	1251	1348	16 31 47	63 78
65		2.1445	1543	1642	1742	1842	1943	2045	2148	2251	2355	17 34 51	68 85
66		2.2460	2566	2673	2781	2889	2998	3109	3220	3332	3445	18 37 55	73 92
67		2.3559	3673	3789	3906	4023	4142	4262	4384	4504	4627	20 40 60	79 99
68		2.4751	4876	5002	5129	5257	5386	5517	5649	5782	5916	22 43 65	87 108
69		2.6051	6187	6325	6464	6605	6746	6889	7034	7179	7326	24 47 71	95 119
70		2.7475	7625	7776	7929	8083	8239	8397	8556	8716	8878	26 52 78	104 131
71		2.9042	9208	9375	9544	9714	9887	3.0061	3.0237	3.0415	3.0595	29 58 87	116 145
72		3.0777	0961	1146	1334	1524	1716	1910	2106	2305	2506	32 64 96	129 161
73		3.2709	2914	3122	3332	3544	3759	3977	4197	4420	4646	36 72 108	144 180
74		3.4874	5105	5339	5567	5816	6059	6305	6554	6806	7062	41 81 1122	163 204
75		3.7321	7583	7848	8118	8391	8867	8947	9232	9520	9812	46 93 139	186 232
76		4.0108	0408	0713	1022	1335	1653	1976	2303	2635	2972	53 107 160	213 267
77		4.3315	3662	4015	4374	4737	5107	5483	5864	6252	6646		
78		4.7046	7453	7867	8288	8716	9152	9594	5.0045	5.0504	5.0970		
79		5.1446	1929	2422	2924	3435	3955	4486	5026	5578	6140	Mean differences cease to be sufficiently accurate.	
80		5.6713	7297	7894	8502	9124	9758	6.0405	6.1066	6.1742	6.2432		
81		6.3138	3859	4596	5350	6122	6912	7720	8548	9395	7.0264		
82		7.1154	2066	3002	3962	4947	5958	6996	8062	9158	8.0285		
83		8.1443	2636	3863	5126	6427	7769	9152	9.2052	9.2052	9.3572		
84		9.5144	9.677	9.845	10.02	10.20	10.39	10.58	10.78	10.99	11.20		
85		11.43	11.66	11.91	12.16	12.43	12.71	13.00	13.30	13.62	13.95		
86		14.30	14.67	15.06	15.46	15.89	16.35	16.83	17.34	17.89	18.46		
87		19.08	19.74	20.45	21.20	21.20	22.90	23.86	24.90	26.03	27.27		
88		28.64	30.14	31.82	33.69	35.80	38.19	40.92	44.07	47.74	52.08		
89		5729	63.66	71.62	81.85	95.49	114.6	143.2	191.0	286.5	573.0		
90		∞											

SECTION – C

17. CARTESIAN COORDINATES (TWO DIMENSIONS)

LEARNING OBJECTIVES

- Introduction
- Definition
- Cartesian Coordinates
- Distance Formula
- Area of a Triangle
- Condition for Collinearity of Three Points
- Section Formulae
- Centroid of a Triangle
- Incentre of a Triangle

INTRODUCTION

The geometry which we have already studied in our earlier classes was based upon certain concept like that of points, lines are planes. We accepted certain axioms and developed results by using the methods of deductive logic. Moreover, the tools of algebra were also not made use of in studying geometry. This approach to geometry was initiated by Greek mathematician *Euclid.* He wrote his treatise on geometry named '*Elements*´(Vol. I – XIII) around 300 B.C. The approach of Euclid was named '*synthetic approach to geometry*'. This approach to geometry continued for about 2000 years.

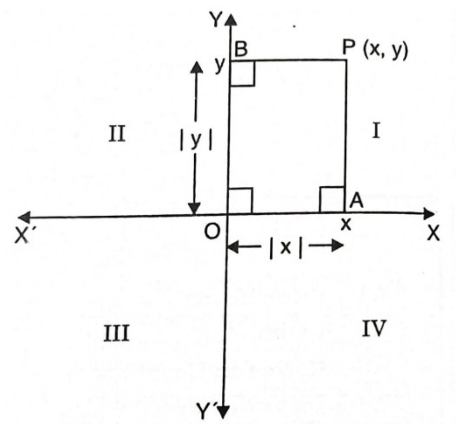
In 1637, a French philosopher and mathematician *Rene Descartes* (1596 – 1650) published his work on geometry in the book named *La Geometrie*. He incorporated the use of tools of algebra in studying geometry by establishing 1- 1 correspondence between the points in a plane and the ordered pairs of real numbers. He simplified the proofs of geometrical results by introducing the processes of algebra in geometry. The approach of *Descartes* was named *'analytic approach to geometry'*.

DEFINITION

Coordinate geometry is that branch of mathematics which treats geometry algebraically.

CARTESIAN COORDINATES

Let *X'OX* and *Y'OY* be two perpendicular straight liens intersecting at *O*. The line *X'OX* is taken horizontal. The point *O* is called the **origin.** The horizontal line *X'OX* and vertical line *Y'OY* are respectively called the **x-axis** and the **y-axis**. Taking *O* as the origin, the number scale is made on both axes. The axes divides the plane in four parts called **quadrants.** The quadrants *XOY*, *X'OY*, *X'OY'* and *XOY'* are respectively called I, II, III, and IV quadrants.



The axes *X'OX* and *Y'OY* are called **rectangular coordinate axes** or simply **coordinate axes,** provided there is no involvement of oblique axes in the discussion.

Let *P* be any point in the plane. Draw $PA \perp Y'OY$.

Let *x* and *y* be the numbers corresponding to the points *A* and *B* on the axes *X'OX* and *Y'OY* respectively.

$\therefore \qquad OA = |x|^* \quad$ and $\quad OB = |Y|^*$.

Thus, we see that for a point *P* in the plane, there correspond an ordered pair (*x, y*) of real numbers.

Conversely, let (*x, y*) be any ordered pair of real numbers. Let *A* and *B* be the points on the axes *X'OX* and *Y'OY* corresponding to the real numbers *x* and *y* respectively. Let the perpendiculars at *A* and *B* meet at *P*. The point *P* is unique for a given ordered pair (*x, y*) of real numbers. Thus, we see that there is 1-1 correspondence between the points in a plane and the ordered pairs of real numbers.

This correspondence is called **Cartesian coordinate system** after the name of *Rene Desartes*.

The real number x is called the ***x* – coordinate** of P or the **abscissa** of P. Similarly, the real number y is called the ***y* – coordinate** of P or the **ordinate** of P. The real numbers x and y are not reversible in (x, y). The ordered pairs (x, y) and (y, x) represents the same point on the plane if and only if $x = y$.

The coordinates x and y of P are written as (x, y).

The reader would find it interesting to note that if:

(i) P is on x-axis, then $y = 0$. (ii) P is on y-axis, then $x = 0$.

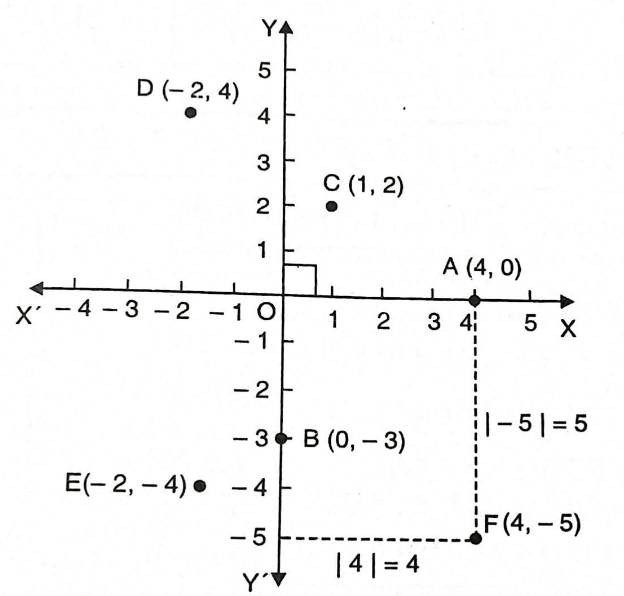
(iii) P is in the I quadrant, then $x > 0$, $y > 0$.

(iv) P is in the II quadrant, then $x < 0$, $y > 0$.

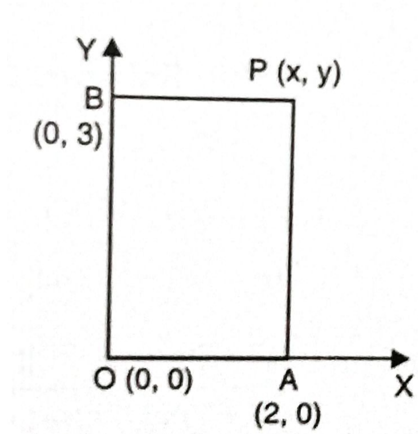
(v) P is in the III quadrant, then $x < 0$, $y < 0$.

(vi) P is in the IV quadrant, then $x > 0$, $y < 0$.

For example, the points $A(4, 0)$, $B(0, -3)$ are on the x – axis and y-axis, respectively. Also, the points $C(1, 2)$, $D(-2, 4)$, $E(-2, -4)$, $F(4, -5)$ are in the I, II, III, and IV quadrants, respectively.



Example 1. *If three vertices of a rectangle are (0, 0), (2, 0) and (0, 3), find the coordinates of the fourth vertex.*



Sol. Let *OAPB* be the rectangle with vertices *O*(0, 0), *A*(2, 0), *P*(*x*, *y*) and *B*(0, 3).

Now, $x = BP = OA = 2$ and $y = AP = OB = 3$

$\therefore$ Coordinates of fourth vertex = **(2, 3)**.

EXERCISE 17.1

SHORT ANSWER TYPE QUESTIONS

1. Plot the following points on a cartesian plane :

   (i) (3, 4) (ii) (3, - 7) (iii) (- 5, - 8)

   (iv) (- 6, 2) (v) (0, 4) (vi) (0, - 6)

2. In which quadrant the following points lie:

   (i) (5, 9) (ii) (- 6, 8) (iii) (15, -7)

   (iv) (- 3, - 4) (v) (9, 2) (vi) (-6, 8) ?

3. Draw the quadrilateral whose vertices are (- 4, 5), (0, 7), (5, - 5) and (- 4, - 2).

4. If three vertices of a rectangle are (0, 0), (- 4, 0), (0, 5), find the coordinates of the fourth vertex.

LONG ANSWER TYPE QUESTIONS

5. The base of an equilateral triangle with side 7 cm lie along the y-axis such that the mid-point of the base is at the origin. Find the vertices of the triangle.

Answers

2. (i) First (ii) second (iii) fourth (iv) third

   (v) first (vi) second.

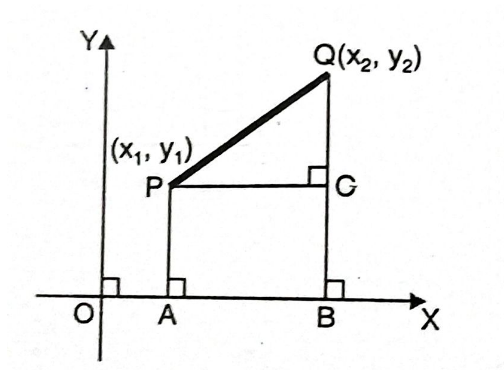
4. (- 4, 5)

5. $\left(0,\frac{7}{2}\right),\left(0,-\frac{7}{2}\right),\left(\frac{7\sqrt{3}}{2},0\right);\ \left(0,\frac{7}{2}\right),\left(0,-\frac{7}{2}\right),\left(-\frac{7\sqrt{3}}{2},0\right)$

DISTANCE FORMULA

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points in the plane. For the sake of exactness, let us assume that the points P and Q are both in the I quadrant. Draw PA and QB perpendicular to x-axis. Draw $PC \perp QB$.

Since, PCQ is a right-angled triangle, therefore by *Pythagoras theorem,*



$$PQ^2 = PC^2 + CQ^2$$

$$\therefore \quad PQ = \sqrt{PC^2 + CQ^2} \quad \ldots.(1)$$

Now, $PC = AB = OB - OA = x_2 - x_1$

and

$$CQ = BQ - BC = BQ - AP = y_2 - y_1$$

$$\therefore \quad (1) \quad \Rightarrow \quad PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\ .$$

$\therefore$ **The distance PQ between the points P(x_1, y_1) and Q(x_2, y_2) is given by**

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\ .$$

Remark 1. If $P(x, y)$ be any point in the plane, then the distance of P from O

$$= OP = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}\ .$$

Remark 2. The distance between points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{[-(x_1 - x_2)]^2 + [-(y_1 - y_2)]^2}$$

$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\ .$$

Thus, in finding PQ, it does not matter whether we subtract x_1 from x_2 or x_1. In practice, we find it easier to subtract smaller abscissa from the bigger abscissa. Similar arguments also work for ordinates.

Remark 3. (i) If PQ is parallel to x-axis, then $y_1 = y_2$ and so

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x_2 - x_1)^2} = | x_2 - x_1 |\ .$$

(ii) If PQ is parallel to y-axis, then $x_1 = x_2$ and so

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(y_2 - y_1)^2} = | y_2 - y_1 | \ .$$

Note 1. When *three* points are given and it is required to prove that they are **collinear** *i.e.,* they lie on a line, then show that sum of the distances between two point-pairs is equal to the distance between the third point-pair.

Note 2. When *three* points are given and it is required to prove that they form :

i. an **isosceles triangle,** show that two of its sides are equal.
ii. an **equilateral triangle,** show that its all sides are equal.
iii. a **right angled triangle,** show that the sum of the squares of two sides is equal to the square of the third side.

Note 3. When *four* points are given and it is required to prove that they form :

i. a **parallelogram,** show that opposite sides are equal.
ii. a **rectangle,** show that opposite sides are equal and diagonals are also equal.
iii. a **parallelogram but not a rectangle,** show that opposite sides are equal and diagonals are not equal.
iv. a **square,** show that all sides are equal and diagonals are also equal.
v. a **rhombus,** show that all sides are equal and diagonals are not equal.

WORKING RULES FOR SOLVING PROBLEMS

Rule I. *The distance between the points* (x_1, y_1) *and* (x_2, y_2) *is* $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Rule II. *The distance of the point* (x_1, y_1) *from the origin is* $\sqrt{x_1^2 + y_1^2}$.

Rule III. *Three points are* **collinear** *if the sum of two distance is equal to the third distance.*

Rule IV. *The triangle ABC is a/an :*

(i) **equilateral triangle** *if AB = BC = CA.*

(ii) **isosceles triangle** *if two of its sides are equal.*

(iii) **right angled triangle** *if the sum of squares of two sides is equal to the square of the third side.*

Rule V. *The quadrilateral ABCD is a:*

(i) **parallelogram** *if AB = CD and BC = DA.*

(ii) **rectangle** *if AB = CD and BC = DA and AC = BD.*

(iii) ***parallelogram but not a rectangle*** *if AB = CD, BC = DA and AC ≠ BD.*

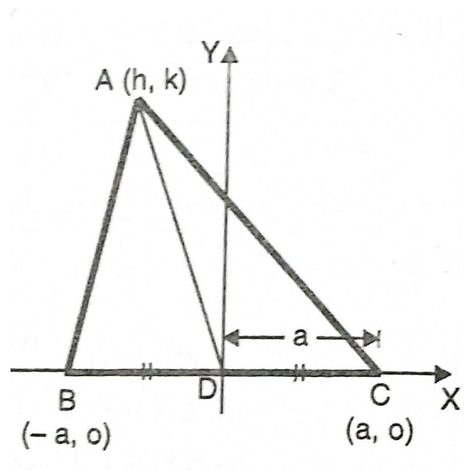
(iv) ***square*** *if AB = BC = CD = DA and AC = BD.*

(v) ***rhombus*** *if AB = BC = CD = DA and AC ≠ BD.*

Example 2. *If D is the mid-point of the side BC of a triangle ABC, prove that*

$$AB^2 + AC^2 = 2(AD^2 + DC^2).$$

Sol. Let *DC* be taken as the *x*-axis and perpendicular to *DC* from *D* as the *y*-axis.



Let $DC = a$.

$\therefore$ $B = (-a, 0)$ and $C = (a, 0)$.

Let $A = (h, k)$

$$\therefore \quad AB^2 + AC^2 = \left[(h+a)^2 + (k-0)^2\right] + \left[(h-a)^2 + (k-0)^2\right]$$

$$= 2h^2 + 2a^2 + 2k^2 = 2\left(h^2 + k^2 + a^2\right)$$

Also, $2\left(AD^2 + DC^2\right) = 2\left(\left[(h-0)^2 + (k-0)^2\right] + a^2\right)$

$\therefore$ $\mathbf{AB^2 + AC^2 = 2(AD^2 + DC^2)}$.

EXERCISE 17.2

SHORT ANSWER TYPE QUESTIONS

1. Find the distance between the following pairs of points :

(i) (0, 0) and (4, 5) (ii) (5, -12) and (9, -9)

(iii) $(x - y, y - x)$ and $(x + y, x + y)$ (iv) $(b + c, c + a)$ and $(c + a, a + b)$

2. Show that each of the following sets of points are the vertices of a right angled triangle:

(i) (4, 4), (3, 5), (-1, -1) (ii) (6, 2), (3, -1), (-2, 4)

3. Show that the following sets of points are collinear :

   (i) (-2, 3), (1, 2) (7, 0) (ii) (4, 3), (2, 0), (-4, -9)

   Also verify the result by drawing the points on a plane.

4. Using distance formula, show that (3, 3) is the centre of the circle passing through the points (6, 2), (0, 4) and (4, 6). Find the radius of the circle.

5. Show that the triangle whose vertices given below are equilateral :

   (i) (-1, -1), (1, 1), $(-\sqrt{3}, \sqrt{3})$ (ii) $(2a, 4a)$, $(2a, 6a)$, $(2a + \sqrt{3}\,a, 5a)$

LONG ANSWER TYPE QUESTIONS

6. Show that the quadrilaterals whose vertices given below are parallelogram :

   (i) (-1, 0), (0, 3), (1, 3), (0, 0) (ii) (-2, -1), (1, 0), (4, 3), (1, 2)

7. Show that the quadrilaterals whose vertices given below are rectangles:

   (i) (0, -1), (-2, 3), (6, 7), (8, 3) (ii) (3, 2), (11, 8), (8, 12), (0, 6).

8. Show that the quadrilaterals whose vertices given below are rhombuses:

   (i) (7, 3), (3, 0), (0, -4), (4, -1) (ii) (3, -4), (4, 2), (5, -4), (4, -10).

Answers

1. (i) $\sqrt{41}$ (ii) 5 (iii) $2\sqrt{x^2 + y^2}$

   (iv) $\sqrt{a^2 + 2b^2 + c^2 - 2ab - 2bc}$

4. $\sqrt{10}$

AREA OF A TRIANGLE

Let ABC be a triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$.

Draw AL, BM, CN perpendiculars on x-axis. Required area ABC

= area of trap. $ALNC$ + area of trap. $CNMB$

– area of trap. $ALMB$

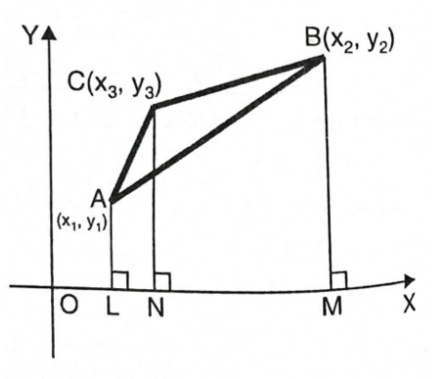
$$= \frac{1}{2}(AL + CN)LN + \frac{1}{2}(CN + BM)NM - \frac{1}{2}(AL + BM)LM$$

Now, $AL = y_1$, $BM = y_2$, $CN = y_3$ and

$LN = ON - OL = x_3 - x_1$

$NM = OM - ON = x_2 - x_3$

$LM = OM - OL = x_2 - x_1$



$\therefore$ Area ABC =

$$\frac{1}{2}(y_1 + y_3)(x_3 - x_1) + \frac{1}{2}(y_3 + y_2)(x_2 - x_3) - \frac{1}{2}(y_1 + y_2)(x_2 - x_1)$$

$$= \frac{1}{2}\left[x_1(-y_1 - y_3 + y_1 + y_2) + x_2(y_3 + y_2 - y_1 - y_2) + x_3(y_1 + y_3 - y_3 - y_2)\right]$$

$$= \frac{1}{2}\left[x_1(y_2 - y_3) + x_3(y_3 - y_1) + x_3(y_1 - y_2)\right].$$

The area of triangle ΔABC will come out to be a positive quantity only when the vertices *A, B, C* are taken in anticlock direction.

Thus, is general $\Delta ABC = \frac{1}{2}\left[x_1(y_2 - y_3) + x_3(y_3 - y_1) + x_3(y_1 - y_2)\right]$.

Remark. The area of a quadrilateral can be found out by dividing the quadrilateral into two triangles.

CONDITION FOR COLLINEARITY OF THREE POINTS

If the points *A, B* and *C* are collinear, then the area of triangle must be *zero.* This also holds *vice-versa*. Thus, to show that given three points are *collinear*, it would be sufficient to show that the area of the triangle with given points as vertices is *zero.*

WORKING RULES FOR SOLVING PROBLEMS

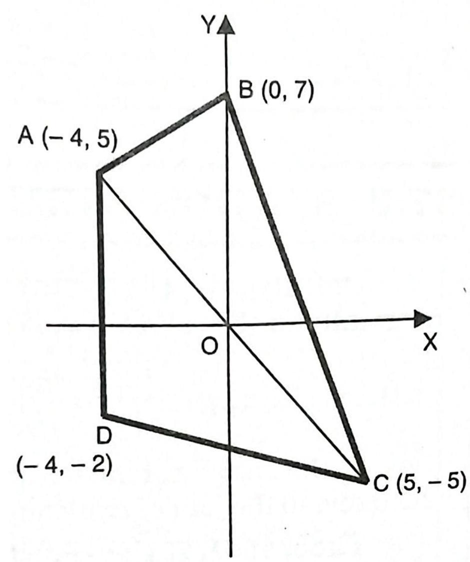
Rule I. *Area of triangle with vertices* (x_1, y_1), (x_2, y_2) *and* (x_3, y_3) *is*

$$= \frac{1}{2}\left[x_1(y_2 - y_3) + x_3(y_3 - y_1) + x_3(y_1 - y_2)\right]$$

Rule II. *Three points are collinear if and only if area of triangle formed by these points is zero.*

Example 3. *Draw a quadrilateral in the Cartesian plane, whose vertices are (-4, 5), (0, 7), (5, -5) and (-4, -2). Also, find its area.*

Sol. Let the vertices of the quadrilateral be $A(-4, 5)$, $B(0, 7)$, $C(5, -5)$ and $D(-4, -2)$. Join AC.



Area of

$$\Delta ABC = \frac{1}{2}\left|-4(7+5)+0(-5-5)+5(5-7)\right|$$

$$= \frac{1}{2}\left|-48+0-10\right|$$

$$= \frac{1}{2}\left|-58\right| = 29 \text{ sq. units.}$$

Area of

$$\Delta ACD = \frac{1}{2}\left|-4-(-5+2)+5(-2-5)+(-4)(5+5)\right|$$

$$= \frac{1}{2}\left|12-35-40\right|$$

$$= \frac{1}{2}\left|-63\right| = 31.5 \text{ sq. units}$$

$\therefore$ Area of quadrilateral $ABCD$

= area of ΔABC + area of ΔACD

= 29 sq. units + 31.5 sq. units = 60.5 sq. units.

EXERCISE 17.3

SHORT ANSWER TYPE QUESTIONS

1. Find the area of the triangle whose vertices are :

   (i) (3, 4), (2, -1), (4, -6) (ii) (1, -1), (-1, -1), ($-\sqrt{3}$, $\sqrt{3}$)

2. Show that the area of the triangle with vertices $(\lambda, \lambda-2)$, $(\lambda+3, \lambda)$ and $(\lambda+2, \lambda+2)$ is independent of λ

3. Show that the following points are collinear :

   (i) (1, 4), (3, -2), (-3, 16) (ii) (-5, 1), (5, 5), (10, 7).

4. Find the value of x so that the points (x, -1), (5, 7), (8, 11) may be collinear.
5. Find y so that the points (a, 0), (0, b), (3a, y) are collinear $(a \neq 0)$.
6. For what value of k do the points (-1, 4), (-3, 8) and (-k + 1, 3k) lie on the straight line?

LONG ANSWER TYPE QUESTIONS

7. Show that the points (p + 1, 1) (2p + 1, 3) and (2p +2, 2p) are collinear if p = - 1/2 or 2.
8. If the vertices of a triangle are $(1,\lambda),(4,-3),(-9,7)$ and its area is 15 sq. units. Find the value of λ.
9. Find the area of the quadrilateral whose vertices, taken in order, are:

   (i) (1, 1), (3, 4), (5, -2), (4, -7) (ii) (4, 3), (-5, 6), (-7, -2), and (0, -7).

Answers

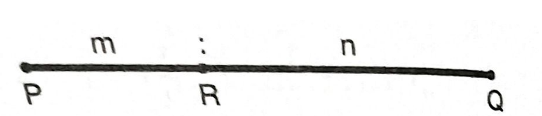
1. (i) $7\frac{1}{2}$ sq. units(ii) $1+\sqrt{3}$ sq. units 4. -1 5. -2b

6. 0 8. -3, or $\frac{21}{13}$ 9.(i) $\frac{41}{2}$ sq. units (ii) 84 sq. units.

SECTION FORMULAE

Definition of internal division. A point R is said to divide the line PQ **internally** in the ratio $m : n$, if R is within PQ and

$$\frac{PR}{RQ}=\frac{m}{n}.$$

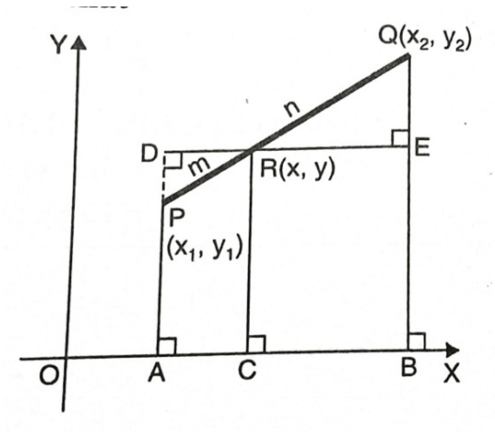


In the next theorem, we shall derive a formula to find the coordinates of a point which divide the join of two points internally in a given ratio.

Theorem 1. If the point R(x, y) divides the join of points P(x_1, y_1) and Q(x_2, y_2) internally in the ratio m : n, then prove that

$$x = \frac{mx_2 + nx_1}{m+n},\ y = \frac{my_2 + ny_1}{m+n}.$$

Proof. The given points are $P(x_1, y_1)$ and $Q(x_2, y_2)$. For the sake of exactness, let us assume that the points P and Q are both in the I quadrant. Let R be such that $PR : RQ = m : n$.



Draw PA, QB and RC $\perp$s on the x-axis. Through R, draw $DE || x$-axis to meet AP produced in D and BQ in E.

Δs PRD and QRE are similar.

$$\therefore \qquad \frac{DR}{RE} = \frac{PD}{EQ} = \frac{PR}{RQ} = \frac{m}{n} \qquad \ldots(1)$$

$$(1) \Rightarrow \quad \frac{m}{n} = \frac{DR}{RE} = \frac{AC}{CB} = \frac{OC - OA}{OB - OC} = \frac{x - x_1}{x_2 - x}$$

$$\Rightarrow \quad mx_2 - mx = nx - nx_1 \quad \Rightarrow \quad mx_2 - nx_1 = (m+n)x$$

$$\Rightarrow \quad x = \frac{mx_2 + nx_1}{m+n}.$$

$$\text{Again } (1) \Rightarrow \quad \frac{m}{n} = \frac{PD}{EQ} = \frac{AD - AP}{BQ - BE} = \frac{y - y_1}{y_2 - y}$$

$$\Rightarrow \quad my_2 - my = ny - ny_1 \quad \Rightarrow \quad my_2 - my_1 = (m+n)y$$

$$\Rightarrow \quad y = \frac{my_2 + ny_1}{m+n}.$$

$\therefore$ **The coordinates of R are** $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$.

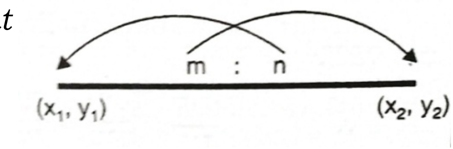
WORKING RULES TO FIND THE POINT OF INTERNAL DIVISION

Step I. *Multiply m by the x-coordinate of the point remote from m and n by the x-coordinate of the point remote from n as shown by the arrows.*

Step II. *Add the products of mx_2 and nx_1.*

Step III. *Divide the sum $mx_2 + nx_1$ by $m + n$. This gives the x-coordinate of the point of internal division.*

Step IV. *Similarly, find y coordinate.*



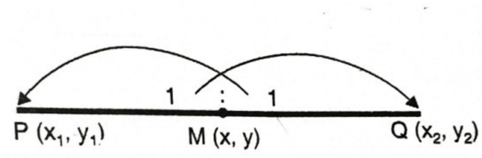
Remark 1. The above section formula also holds good even if either point or both points are not in the I quadrant.

Remark 2. The ratio $m : n$ is equivalent to $\frac{m}{n} : \frac{n}{n}(=1)$ as well as to $\frac{m}{n}(=1) : \frac{n}{m}$. Therefore, in a given ratio, 1 can be kept on either side.

Corollary. Mid-point of a line segment. Let $M(x, y)$ be the mid-point of the line joining $P(x_1, y_1)$ and $Q(x_2, y_2)$.

$\therefore \quad PM = MQ$

$\therefore$ M divides PQ in the ratio 1 : 1 internally.

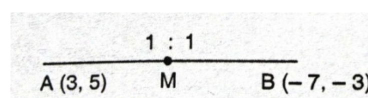


$$\therefore \quad x = \frac{1.x_2 + 1.x_1}{1+1} = \frac{x_1 + x_2}{2}$$

and $$x = \frac{1.y_2 + 1.y_1}{1+1} = \frac{y_1 + y_2}{2}.$$

$\therefore$ **The mid-point is M** $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$**.**

Example 4. *Find the mid-point of the line joining (3, 5) and (-7, -3). Also draw the points on a cartesian plane.*

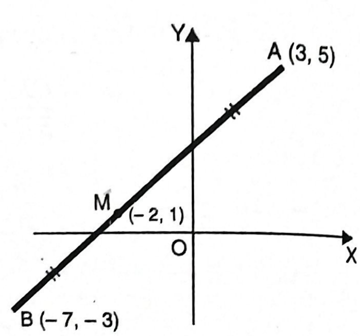


Sol. Let A = (3. 5) and B = (-7, -3).

Let $M(x, y)$ be the mid-point of AB.

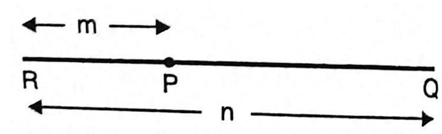
$$\therefore \quad x = \frac{3 + (-7)}{2} = \frac{-4}{2} = -2$$

and $$y = \frac{5 + (-3)}{2} = \frac{2}{2} = 1$$



∴ The mid-point of the given line of **(-2, 1).**

In the adjoining figure, the points are shown on a cartesian plane.

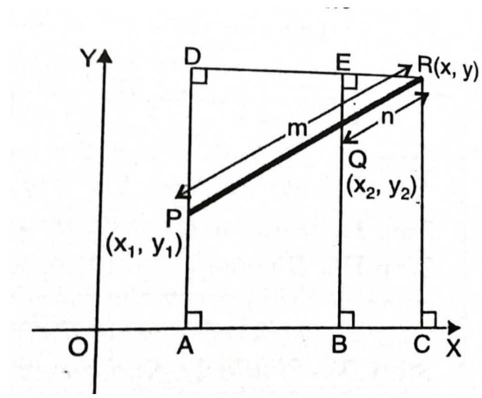


Definition of external division. A point R is said to divide the line PQ **externally** in the ratio $m : n$, where $m \neq n$, if R is outside PQ and

$$\frac{PR}{RQ} = \frac{m}{n}.$$

Theorem II. If the point R(x, y) divides the join of points P(x_1, y_1) and Q(x_2, y_2) externally in the ratio m : n, where m ≠ n, then prove that

$$x = \frac{mx_2 + nx_1}{m + n}, y = \frac{my_2 + ny_1}{m + n}.$$



Proof. Let $m > n$.

∴ The point of division R lies on the right of PQ.

In this case, we have similar triangles PRD are QRE.

$$\therefore \qquad \frac{DR}{RE} = \frac{PD}{EQ} = \frac{PR}{RQ} = \frac{m}{n} \qquad ....(1)$$

$$(1) \Rightarrow \qquad \frac{m}{n} = \frac{DR}{RE} = \frac{AC}{CB} = \frac{OC - OA}{OC - OB} = \frac{x - x_1}{x - x_2}$$

$$\Rightarrow \qquad mx - mx_2 = nx - nx_1 \quad \Rightarrow \quad mx_2 - nx_1 = (m - n)x$$

$$\Rightarrow \qquad x = \frac{mx_2 - nx_1}{m - n}$$ (Division by $m - n$ is justified, because $m \neq n$)

$$\text{Again } (1) \Rightarrow \qquad \frac{m}{n} = \frac{PD}{EQ} = \frac{AD - AP}{BE - BQ} = \frac{y - y_1}{y - y_2}$$

$$\Rightarrow \qquad my - my_2 = ny - ny_1 \quad \Rightarrow \quad my_2 - ny_1 = (m - n)y$$

$$\Rightarrow \qquad y = \frac{my_2 - ny_1}{m - n}$$

$\therefore$ **The coordinates of R are** $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$.

Alternative proof. We observe that $Q(x_2, y_2)$ divides PR in the ratio $m - n : n$ internally.

$$\Rightarrow \quad x_2 = \frac{(m-n)x + nx_1}{(m-n)+n}$$

$$\Rightarrow \quad mx_2 = (m-n)x + nx_1 \quad \Rightarrow \quad x = \frac{mx_2 - nx_1}{m-n}$$

Also, $$y_2 = \frac{(m-n)y + ny_1}{(m-n)+n}$$

or $$my_2 = (m-n)y + ny_1 \text{ or } y = \frac{my_2 - ny_1}{m-n}$$

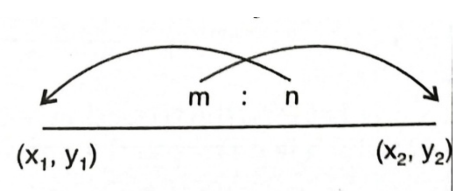
$\therefore$ **The coordinates of R are** $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$.

WORKING RULES TO FIND THE POINT OF EXTERNAL DIVISION

Step I. *Multiply m by the x-coordinate of the point remote from m and n by the x-coordinate of the point remote from n as shown by arrows.*

Step II. *Subtract the product of* nx_1 *and* mx_2.

Step III. *Divide the difference* $mx_2 + nx_1$ *by* $m - n$. *This gives the x-coordinate of the point of external division.*



Step IV. *Similarly, find y- coordinate.*

Remark 1. The above section formula also holds good even if either point or both points are not in the I quadrant.

Remark 2. The ratio $m : n$ is equivalent to $\frac{m}{n} : \frac{n}{n} (= 1)$ as well as to $\frac{m}{m}(=1) : \frac{n}{m}$.

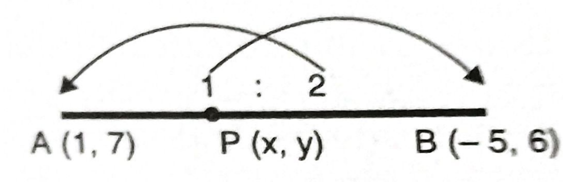
Therefore, in a given ratio, 1 can be kept on either side.

Remark 3. The coordinates $\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$ of the point dividing $P(x_1, y_1)$ and $Q(x_2, y_2)$ externally in the ratio $m : n$, where $m \neq n$ can also be expressed as $\left(\frac{mx_2 + (-n)x_1}{m+(-n)}, \frac{my_2 + (-n)y_1}{m+(-n)}\right)$ and this can be thought of as the coordinates of the point dividing PQ internally in the ratio $m : -n$.

Example 5. *Find the coordinates of the points which divides the join of (1, 7) and (-5, 6) in the ratio (i) 1 : 2 internally (ii) 3 : 2 externally.*

Sol. Let the given points be $A(1,7)$ and $B(-5, 6)$.

(i) Let $P(x, y)$ divides AB in the ratio 1 : 2 internally.



$$\therefore \quad x = \frac{1\times(-5)+2\times 1}{1+2} = \frac{-5+2}{3} = -1$$

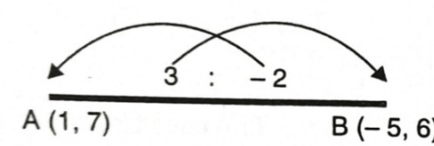
and $$y = \frac{1\times 6 + 2\times 7}{1+2} = \frac{6+14}{3} = \frac{20}{3}$$

$\therefore$ The point of division is **(-1, 20/3).**

(ii) Let $Q(x, y)$ divides AB in the ratio 3 : 2 externally. This is equivalent to say that Q divides AB in the ratio 3 : -2 internally.

$$\therefore \quad x = \frac{3\times(-5)+(-2)\times 1}{3+(-2)} = \frac{-15-2}{1} = -17$$

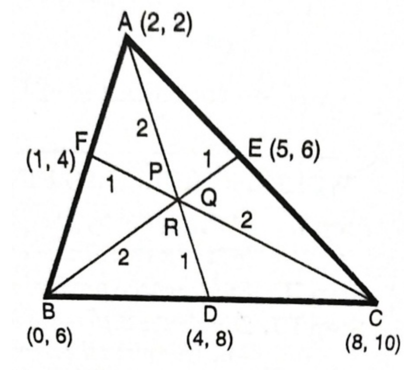
and $$y = \frac{3\times 6 + (-2)\times 7}{3+(-2)} = \frac{18-14}{1} = 4$$



$\therefore$ The point of division is **(-17, 4).**

Example 6. *The vertices of a triangle are at (2, 2), (0, 6) and (8, 10). Find the coordinates of the trisection point of each median which is nearer the opposite side.*

Sol. Let the given points be A (2, 2), $B(0, 6)$ and $C(8, 10)$. Let AD, BE, CF be the medians.



$\therefore \quad D = \left(\frac{0+8}{2}, \frac{6+10}{2}\right) = (4,8)$

$E = \left(\frac{8+2}{2}, \frac{10+2}{2}\right) = (5,6)$

and $F = \left(\frac{2+0}{2}, \frac{2+6}{2}\right) = (1,4)$

Let P be the trisection point of the median AD which is nearer to the opposite side BC.

$\therefore$ P divides DA in the ratio 1 : 2 internally.

$$\therefore \quad P = \left(\frac{1(2)+2(4)}{1+2}, \frac{1(2)+2(8)}{1+2}\right) = \left(\frac{10}{3}, 6\right)$$

Let Q be the trisection point of the median BE which is nearer to the opposite side CA.

$\therefore$ Q divides EB in the ratio 1 : 2 internally.

$$\therefore \quad Q = \left(\frac{1(0)+2(5)}{1+2}, \frac{1(6)+2(6)}{1+2}\right) = \left(\frac{10}{3}, 6\right)$$

Let R be the trisection point of the median CF which is nearer to the opposite side AB.

R divides FC in the ratio 1 : 2 internally.

$$\therefore \quad R = \left(\frac{1(8)+2(1)}{1+2}, \frac{1(10)+2(4)}{1+2}\right) = \left(\frac{10}{3}, 6\right)$$

$\therefore$ Coordinates of required trisection points are **(10/3,6), (10/3,6) and (10/3,6).**

Remark. In the above example, the points P, Q and R are coincident. This point is called the *centroid,* which we shall discuss in detail in the next section.

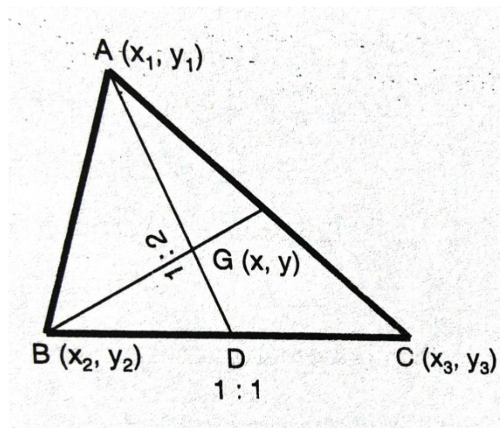
CENTROID OF A TRIANGLE

The point of concurrence of three medians of a triangle is called its **centroid.**

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of a triangle. Let D be the mid-point of BC.

$\therefore$ The coordinates of D are $\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$

Let G divide DA in the ratio 1 : 2 internally.



$$\therefore \quad x = \frac{1(x_1) + 2\left(\frac{x_2 + x_3}{2}\right)}{1+2} = \frac{x_1 + x_2 + x_3}{3}$$

and $$y = \frac{1(y_1) + 2\left(\frac{y_2 + y_3}{2}\right)}{1+2} = \frac{y_1 + y_2 + y_3}{3}$$

$$\therefore \quad G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

The symmetry of coordinates of G shows that it a vertices on the other two medians through B and C. Thus, all medians meet at G *i.e.,* it is the centroid.

$\therefore$ The centroid is $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$.

Example 7. *Find the centroid of the triangle with vertices (1, 7), (-3, -4) and (-6, 4).*

Sol. The vertices are (1, 7), (-3, -4) and (-6, 4).

Let $G(x, y)$ be the centroid of the given triangle.

$\therefore$ By using $x = \frac{x_1 + x_2 + x_3}{3}$, $y = \frac{y_1 + y_2 + y_3}{3}$, we get

$$x = \frac{1 + (-3) + (-6)}{3} = -\frac{8}{3}, \; y = \frac{7 + (-4) + 4}{3} = \frac{7}{3}$$

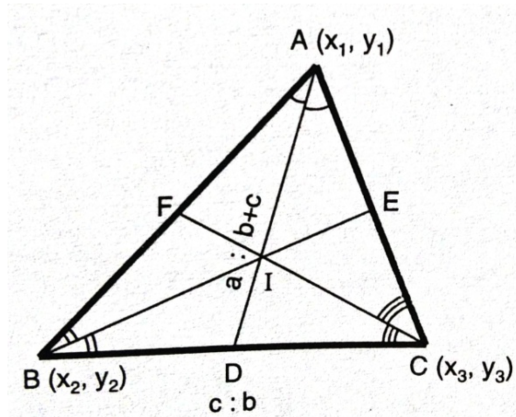
$\therefore$ The centroid is **(-8/3, 7/3)**.

INCENTRE OF A TRIANGLE

The point of concurrence of three internal bisectors of the angles of a triangle is called its **incentre.**

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of a triangle. Let *AD, BE,* and *CF* be the internal bisectors of the angles *A, B, C* respectively.

Let the sides *BC, CA* and *AB* be denoted by *a, b, c* respectively.



By geometry, $\dfrac{BD}{DC} = \dfrac{BA}{CA} = \dfrac{c}{b}$ ....(1)

$\therefore$ *D* divides *BC* in the ratio *c : b* internally.

$\therefore$ The coordinates of *D* are $\left(\dfrac{bx_2 + cx_3}{b+c}, \dfrac{by_2 + cy_3}{b+c}\right)$

(1) $\Rightarrow \dfrac{BD}{DC} + 1 = \dfrac{c}{b} + 1 \Rightarrow \dfrac{BD + DC}{DC} = \dfrac{c+b}{b}$

$\Rightarrow \dfrac{BC}{DC} = \dfrac{c+b}{b} \Rightarrow DC = \dfrac{ab}{b+c}$

Let *AD* and *CF* intersects at *I*.

Since, *CI* is the internal bisector angle *C*, we have

$$\frac{AI}{ID} = \frac{AC}{DC} = \frac{b}{abl\,(a+b)} = \frac{b+c}{a}$$

$\therefore$ *I* divides *AD* internally in the ratio *b + c : a.*

$\therefore$ The coordinates of *I* are $\left(\dfrac{ax_1 + (b+c)\left(\dfrac{bx_2 + cx_3}{b+c}\right)}{a+(b+c)}, \dfrac{ay_1 + (b+c)\left(\dfrac{by_2 + cy_3}{b+c}\right)}{a+(b+c)}\right)$

or $\left(\dfrac{ax_1 + bx_2 + cx_3}{a+b+c}, \dfrac{ay_1 + by_2 + cy_3}{a+b+c}\right)$

The symmetry of the coordinates of I shows that it also lies on the internal bisector through B. Thus, all internal bisectors meet at I *i.e.,* it is the incentre.

$\therefore$ The incentre is $\left(\dfrac{ax_1+bx_2+cx_3}{a+b+c}, \dfrac{ay_1+by_2+cy_3}{a+b+c}\right)$.

Remark. Let ABC be an equilateral triangle.

$$a = b = c$$

$$\therefore \quad \frac{ax_1+bx_2+cx_3}{a+b+c} = \frac{ax_1+ax_2+ax_3}{a+a+a} = \frac{a(x_1+x_2+x_3)}{3a} = \frac{x_1+x_2+x_3}{3}$$

Similarly, $\dfrac{ay_1+by_2+cy_3}{a+b+c} = \dfrac{y_1+y_2+y_3}{3}$

$\therefore$ In an equilateral triangle, incentre and centroid are coincident.

Example 8. *Find the incentre of the triangle whose vertices are (-36, 7), (20, 7) and (0, -8).*

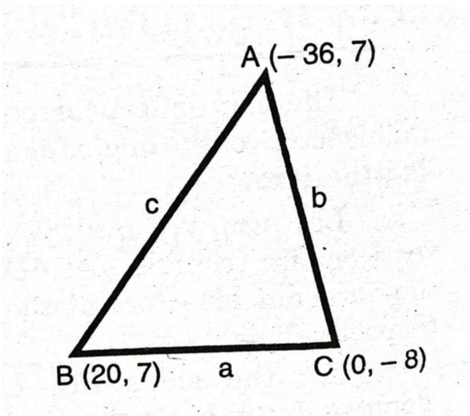
Sol. Let the vertices of the triangle be $A(-36, 7)$, $B(20, 7)$ and $C(0, -8)$.

$$\therefore \quad a = BC = \sqrt{(0-20)^2+(-8-7)^2}$$

$$= \sqrt{400+225} = 25$$

$$b = CA = \sqrt{(-36-0)^2+(7+8)^2}$$

$$= \sqrt{1296+225} = 39$$



$$c = AB = \sqrt{(20+36)^2+(7-7)^2} = \sqrt{3136+0} = 56$$

Let $I(x, y)$ be the incentre of the triangle.

$\therefore$ By using $x = \dfrac{ax_1+bx_2+cx_3}{a+b+c}$, $y = \dfrac{ay_1+by_2+cy_3}{a+b+c}$

We get $x = \dfrac{25(-36)+39(20)+56(0)}{25+39+56} = \dfrac{-120}{120} = -1$,

$$y = \frac{25(7) + 39(7) + 56(-8)}{25 + 39 + 56} = \frac{0}{120} = 0 .$$

$\therefore$ The incentre is **(-1, 0)**.

EXERCISE 17.4

SHORT ANSWER TYPE QUESTIONS

1. (i) Determine the point that bisects the line segment whose end points are (4, -6) and (12, 13).

   (ii) Given *A*(-3, 2) and *B*(5, 4), find the mid-point of *AB*.

2. Find the centroid of the triangle whose vertices are :

   (i) (1, 2), (3, 5), (6, 3) (ii) (0, 9), (-5, 6), (11, -7)

3. One end of diameter of a circle is (4, 1) and the centre is (3, 3), find the coordinates of the other end of the diameter.
4. Show that the quadrilateral with vertices (1, 4), (-2, 1), (0, -1) and (3, 2) is a parallelogram.
5. Prove that the points (1, 1), (4, 4), (4, 8) and (1, 5) form a parallelogram. Is it is a rectangle?
6. If *A*(-1, 3), *B*(1, -1) and *C*(5, 1) are the vertices of a triangle *ABC*, find the length of the median through *A*.
7. Find the third vertex of a triangle if two of its vertices are at (-1, 4) and (5, 2) and the medians meet at (0, -3).

LONG ANSWER TYPE QUESTIONS

8. In what ratio does the point (1/2, 6) divide the line segment joining (3, 5) and (-7, 9) ?
9. If a vertex of a triangle be (1, 1) and the mid-points of the sides through it are (-2, 3) and (5, 2), find the other vertices.
10. If two adjacent vertices of a parallelogram are (3, -2) and (4, 0) and the diagonals intersect at (9/2, -5/2), find the other vertices.
11. If the coordinates of the mid-points of the sides of a triangle are (1, 2), (0, -1) and (2, -1), find the vertices of the triangle.
12. Find the coordinates of the point which is on the line joining the points *A*(5, -4) and *B*(-3, 2) and is twice as far from *A* as from *B*.

13. The line joining the points (2, -2) and (-2, 4) is trisected. Find the points of trisection.

Answers

1. (i) (8, 9/2)	(ii) (1, 3)	2. (i) (10/3, 10/3)	(ii) (2, 8/3)
3. (2, 5)	5. No.	6. 5	7. (-4, -15)
8. 1 : 3 internally	9. (-5, 5), (9, 3)	10. (6, -3), (5, -5)	
11. (1, -4), (3, 2), (-1, 2)		12. (-1/3, 0)	
13. (2/3, 0), (-2/3, 2).			

SUMMARY

1. If $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points in the plane then

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

2. If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle, then area Δ, of the triangle ABC is given by

$$\Delta = \frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right].$$

3. (i) A point R is said to divide PQ **externally** is the ratio $m : n$, where $m \neq n$, if R is outside PQ and

$$\frac{PR}{RQ} = \frac{m}{n}$$

(ii) If $R(x,y)$ divides the join of $P(x_1, y_1)$ and $Q(x_2, y_2)$ externally in the ratio $m : n$, where $m \neq n$, then

$$x = \frac{mx_2 - nx_1}{m - n}, \; y = \frac{my_2 - ny_1}{m - n}$$

4. If $M(x, y)$ is the *mid-point* of the join of $P(x_1, y_1)$ and $Q(x_2, y_2)$, then :

$$x = \frac{x_1 + x_2}{2}, \; y = \frac{y_1 + y_2}{2}.$$

5. If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle, then :

(i) centroid = $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$

(ii) incentre = $\left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c}\right)$, where $a = BC$, $b = CA$, $c = AB$.

TEST YOURSELF

1. Where will the points lie if :

   (i) the ordinate is equal to 2 (ii) the abscissa is equal to – 3?

2. The vertices of a triangle are $(1, 2\sqrt{3})$, (3, 0) and (-1, 0). Is the triangle equilateral, isosceles or scalene ?
3. Find the value of x if the distance between the points (x, -1) and (3, 2) is 5.
4. If the point (2, 1) and (1, -2) are equidistant from the point (x, y), show that $x + 3y = 0$.
5. If the line segment joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ subtend an angle θ at the origin, show that

$$OP.OQ \cos\theta = x_1x_2 + y_1y_2 .$$

6. Show that (1, -3/2), (-3, -7/2) and (-4, -3/2) are the vertices of a right angled triangle.
7. Prove that(2, -2), (-2, 1) and (5, 2) are the vertices of a right angled triangle. Find the area of the triangle and length of its hypotenuse.
8. Find the value (s) of x if the points ($2x$, $2x$), (3, $2x$ + 1) and (1, 0) are collinear.
9. Three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x, y)$ are collinear. Prove that

$$(x - x_1)(y_2 - y_1) = (x_2 - x_1)(y - y_1).$$

Answers

1. (i) On the line parallel to x-axis at a distance 2 units above x-axis.

(ii) On the line parallel to y-axis at a distance 3 units on the left of y-axis.

2. Equilateral 3. -1 7. 12.5 sq. units, $5\sqrt{2}$

8. $\frac{1+\sqrt{2}}{2}$

SECTION – C

18. LOCUS

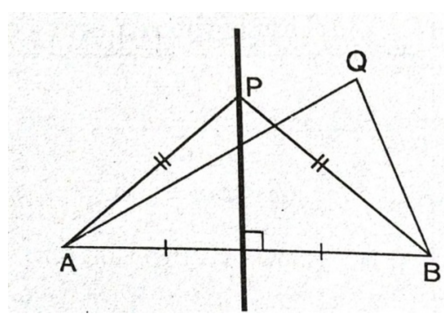
LEARNING OBJECTIVES

- Locus and its Equation

LOCUS AND ITS EQUATION

Let A and B be any two points. Let P be a point which moves so that its distances from A and B are equal. The point P cannot be at Q, because AQ is not equal to BQ. It may be proved that the point P can be any where on the right bisector of AB. We shall say that this line is the *locus* (path) of the point P, under the above mentioned condition. Let us define the term *'locus'* formally.

When a point moves so as always to satisfy a given condition or conditions, the path it traces out is called its **locus** under these conditions. Technically, a *locus* represents the '*set of all points'* which lies on it.



In the above example, the right bisector of AB is the locus under given condition.

If $P(x, y)$ be a general point on the locus, then an equation involving x and y which is satisfied by each point on the locus and such that each point satisfying the equation is on the locus is called the **equation** of the locus.

WORKING RULES FOR FINDING THE EQUATION OF THE LOCUS OF A POINT

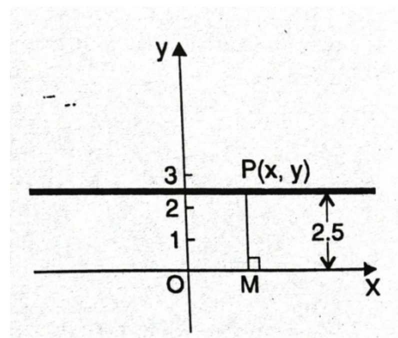
Step I. *Take a general point $P(x, y)$ on the locus.*

Step II.	*Write down the given geometric conditions, under which the point P moves.*
Step III.	*Express the above said conditions in terms of x and y by the help of the formulae and simplify the equation (by squaring both sides if square roots are there and taking L.C.M. to remove the denominators).*
Step IV.	*The simplified equation so obtained is the required equation of the locus.*

Example 1. *Find the equation of the locus of a point which moves so that :*

(i) *it is always 2.5 units above x-axis*

(ii) *it is always 4 units from y-axis.*



Sol. (i) Let $P(x, y)$ be a general point on the locus. Draw $PM \perp x$-axis. By the given conditions, $PM = 2.5$.

$\therefore$ **y = 2.5**.

$\therefore$ This is the required equation of the locus.

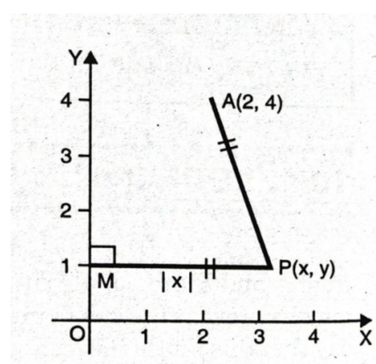
(ii) Let $P(x, y)$ be a general point on the locus.

$\therefore$ By the given conditions, P is either on L_1, or on L_2 which are lines at distance of 4 units from y-axis. If P is on L_1 then we have $x = 4$ and if P is on L_2, then we have ix = -4.

$\therefore$ For any point $P(x, y)$, we have $\mathbf{x = \pm 4}$.

This is the required equation of the locus.

Example 2. *Find the equation of the locus of a point which is equidistant from the point (2, 4) and the y-axis.*



Sol. Let $A(2, 4)$ be the given point. Let $P(x, y)$ be a general point on the locus. Draw $PM \perp y$-axis.

$\therefore$ By the given condition, $PA = PM$.

$$\therefore \quad \sqrt{(x-2)^2 + (y-4)^2} = |x|$$

$\Rightarrow \qquad x^2 + 4 - 4x + y^2 + 16 - 8y = x^2$

$\Rightarrow \qquad$ **$y^2 - 4x - 8y + 20 - 0$.**

This is the required equation of the locus.

EXERCISE 18.1

SHORT ANSWER TYPE QUESTIONS

1. Show that the point (4, 3) lies on the locus of a point whose equation is $x + 3y - 13 = 0$.
2. Show that the points (a, 0), (-a, 0), (0, b), (0, -b) lies on the locus of a point whose equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
3. Which of the following points lie on the locus of $x^2 + y^2 = 16$:

   (i) (4, 1) (ii) (4, 0)

   (iii) (0, 4) (iv) $\left(2\sqrt{2}, -2\sqrt{2}\right)$?
4. Find the equation of the locus of a point which moves so that:

   (i) it is always 2 units above x-axis. (ii) it is always 4 units below x-axis.
5. Find the equation fo the locus of a point which moves so that:

   (i) it is always 6 units on the right of y-axis.
   (ii) it is always 2 units on the left of y-axis.
6. Find the equation of the locus of a point which move so that :

   (i) the sum of the squares of its distances from the coordinates axes is p^2.

   (ii) the sum of the squares of its distances from the points (-a, 0) and (a, 0) is $2k^2$, where k is a constant.
7. Find the equation of the locus of a point which moves so that it is equidistant from the points :

   (i) (-1, -1) and (4, 2) (ii) ($a^2 + b^2$, $a^2 - b^2$) and ($a^2 - b^2$, $a^2 + b^2$).

Answers

3. (ii), (iii), (iv) 4. (i) $y = 2$ (ii) $y = -4$

5. (i) $x = 6$ (ii) $x = -2$

6. (i) $x^2 + y^2 = p^2$ (ii) $x^2 + y^2 = k^2 - a^2$

7. (i) $5x + 3y - 9 = 0$ (ii) $x - y = 0$.

SUMMARY

1. When a point moves so as always to satisfy a given condition or conditions, the path it traces out is called its **locus** under there conditions.
2. To find the equation of locus of a point, we take a general point on the local and find an equation between the coordinates satisfying the given conditions.

TEST YOURSELF

1. Find the equation of the set of all point $P(x, y)$ such that the segment OP has slope 3, where O is the origin.
2. Find the locus of a point which is collinear with the points (x_1, y_1) and (x_2, y_2).
3. Show that the equation of the locus of a point which moves in such a way that its distance from the point $(-g, -f)$ is always equal to a, $x^2 + y^2 + 2gx + 2fy + c = 0$ where $c = g^2 + f^2 - a^2$.
4. Find the equation fo the set of points for which every ordinate is greater than the corresponding abscissa by a given distance.
5. Find the equation of the set of all points $P(x, y)$ such that the line OP is coincident with the line joining P and the point (3, 2).
6. Find the equation of the set of all points which are equidistant from the points $(a+b, a-b)$ and $(a-b, a+b)$.

Answers

1. $y = 3x$ 2. $(y_1 - y_2)x + (x_2 - x_1)y + x_1y_2 - x_2y_1 = 0$

4. $y = x + k$, where k is some constant 5. $2x - 3y = 0$ 6. $x - y = 0$.

SECTION – C

19. STRAIGHT LINES

LEARNING OBJECTIVES

- Introduction
- Inclination of a Line
- Slope of a Line
- Parallel and Perpendicular Lines
- Description of a Line by an Equation
- Equation of a Line Parallel to *x*-axis
- Equation of a Line Parallel to *y*-axis
- Point-Slope form
- Two-Point form
- Intercepts and Axes
- Slope-Intercept form
- Intercept form
- Normal form
- Symmetric form (or Distance form)
- General Equation of a Line
- Reduction of General Equation to Standard form
- Angle between Two Lines
- Condition for Parallelism of Lines
- Condition for Perpendicularity of Lines
- Intersection of Lines
- Condition for concurrency of Three Lines
- Coordinates of Orthocentre and Circumcentre of a Triangle
- Distance of a Point from a Lines
- Distance between Parallel Lines
- Family of Lines
- Equation of Family of Linear Passing Through the Point of Intersection of Two Lines

INTRODUCTION

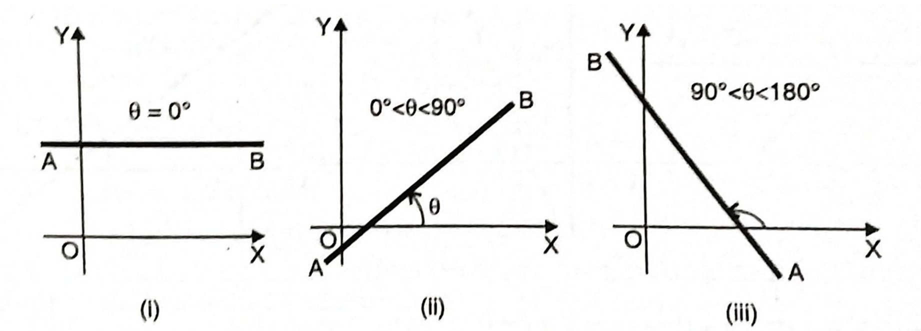
In the present chapter, we shall learn the methods of finding the equations of various types of straight lines. The concept of 'slope' would be used quite often in this chapter. We shall conclude this chapter with the method of finding the equation of family of straight lines passing through the intersection of two straight lines.

INCLINATION OF A LINE

Any line in a coordinate plane may or may not be parallel to *x*-axis. If a line is not parallel to *x*-axis, then the angle which is made by the line in the anti-clockwise direction from the *x*-axis is called the **inclination** of the line. This inclination lies between 0^0 and 180^0. The inclination of a line is generally denoted by θ. The inclination of a line parallel to *x*-axis is defined to be 0^0.

SLOPE OF A LINE

The concept of slope is defined only for lines which are not parallel to the *y*-axis. Let *AB* be a line which is not parallel to *y*-axis and let θ be the inclination of the line. The **slope** of the line *AB* is defined as tan θ.



In other words, the slope of a non-vertical line is the tangent of the inclination of the line.

If $\theta = 0^0$, then	slope of line = $\tan 0^0$ = 0	[Fig. (i)]
If $0^0 < \theta < 90^0$, then	Slope of line = $\tan\theta > 0$	[Fig. (ii)]
If $90^0 < \theta < 180^0$, then	slope of line = $\tan\theta$ < 0	[Fig. (iii)]

The slope of a line is generally denoted by the letter *m*.

$\therefore$ **m = tan** θ.

Remark. If a line is parallel to *y*-axis, then $\theta = 90^0$ and so $\tan\theta = \tan 90^0$ is not defined. That is why, we do not define the slope of a vertical line.

Example 1. (a) *What is the slope of a line whose inclination is :*

(i) *60⁰* (ii) *90⁰* (iii) *120⁰.*

(b) *What is the inclination of a line whose slope as :*

(i) *0* (ii) *1* (iii) *- 1.*

Sol. (a) (i) Here $\theta = 60^0$

Slope $= \tan\theta = \tan 60^0 = \sqrt{3}$

(ii) Here $\theta = 90^0$

$\therefore$ There slope of line is not defined.

(iii) Here $\theta = 120^0$

$\therefore$ Slope $= \tan\theta = \tan 120^0 = \tan(180^0 - 60^0) = -\tan 60^0 = -\sqrt{3}$.

(b) (i) Here $m = 0$. Let θ be the inclination of the line.

$\therefore$ $\tan\theta = 0$ $\Rightarrow$ θ = **0⁰**

(ii) Here $m = 1$. Let θ be the inclination of the line.

$\therefore$ $\tan\theta = 1$ $\Rightarrow$ θ = **45⁰** $[\because\ 0^0 \leq \theta < 180^0]$

(iii) Here $m = -1$. Let θ be the inclination of the line.

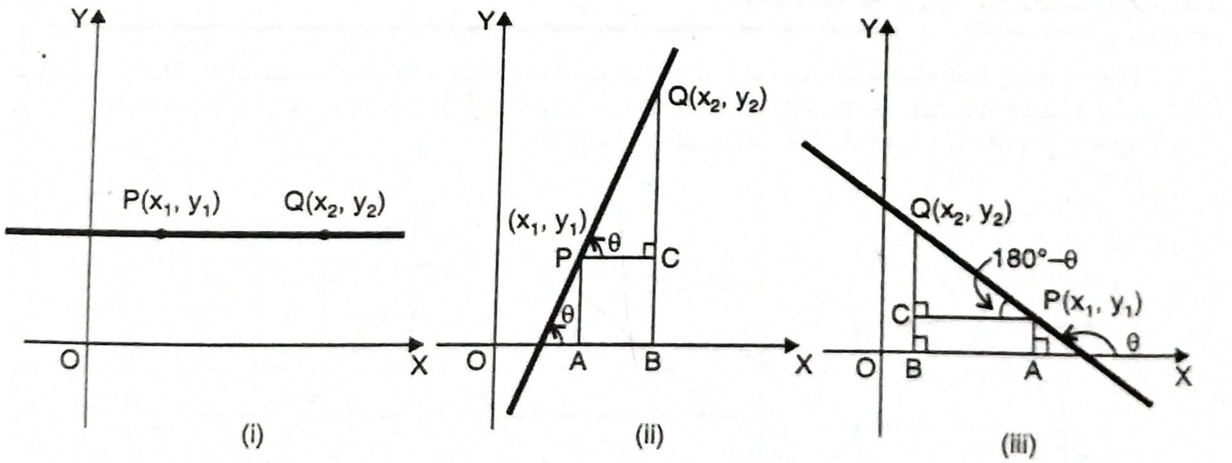
$\therefore$ $\tan\theta = -1$ $\Rightarrow$ θ = **135⁰**

Theorem. If a non-vertical line passes through two distinct points (x_1, y_1), (x_2, y_2), then the slope, m, of the line is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Proof. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the given points on the line. Let θ be the inclination of the line.

Since the line is non-vertical, we have $x_1 \neq x_2$ *i.e.,* $x_2 - x_1 \neq 0$.



Case I. $\theta = 0^0$**.** In this case, $m = 0$

Also, $$\frac{y_2 - y_1}{x_2 - x_1} = \frac{0}{x_2 - x_1} = 0. \quad \therefore \quad m = \frac{y_2 - y_1}{x_2 - x_1}.$$ $[\because y_1 = y_2]$

Case II. $0^0 < \theta < 90^0$. In this case,

$$m = \tan\theta = \frac{QC}{PC} = \frac{QB - BC}{AB} = \frac{QB - AP}{OB - OA} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Case III. $90^0 < \theta < 180^0$. In this case,

$$m = \tan\theta = -\tan(180^0 - \theta) = -\frac{QC}{PC} = -\frac{QB - BC}{AB}$$

$$= -\frac{QB - AP}{OA - OB} = -\frac{y_2 - y_1}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}.$$

$\therefore$ **For any non-vertical line, m =** $\frac{y_2 - y_1}{x_2 - x_1}$ **.**

Example 2. *If the slope of the line joining (2, 5) and (3, λ) is -2, find the value of λ.*

Sol. Let the given points be $A(2, 5)$ and $B(3, \lambda)$.

Slope of AB = -2

$$\therefore \quad -2 = \frac{\lambda - 5}{3 - 2}$$

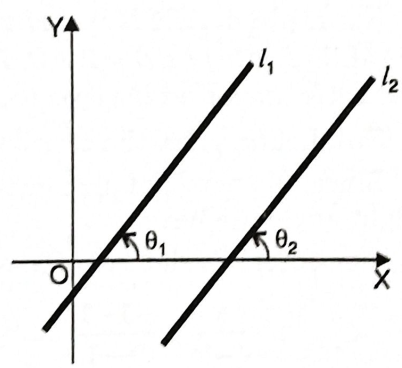
or $\quad -2 = \lambda - 5$ *i.e.*, $\lambda = 3$.

PARALLEL AND PERPENDICULAR LINES

In this section, we shall develop relations between slopes of lines which are either parallel or perpendicular to each other.

Theorem I. Two non-vertical lines are parallel if and only if their slopes are equal.

Proof. Let l_1, l_2 be two non-vertical lines with respective inclinations θ_1, θ_2.



Necessity. Let l_1 and l_2 be parallel.

$\therefore$ Their inclinations are equal.

$\Rightarrow \quad \theta_1 = \theta_2 \quad \therefore \quad \tan\theta_1 = \tan\theta_2$

$\therefore$ Slopes are equal.

Sufficiency. Let the slope of l_1 and l_2 be equal.

$\Rightarrow \quad \tan\theta_1 = \tan\theta_2$

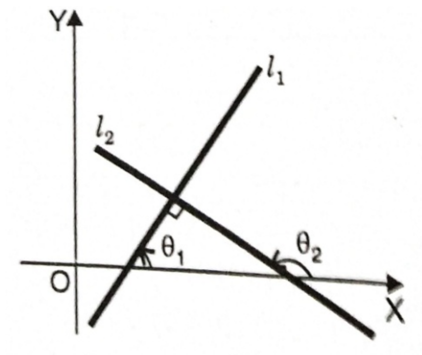
$\Rightarrow \quad \theta_1 = \theta_2$, because θ_1 and θ_2 lie between 0^0 and 180^0, and there exist unique angle $\theta\left(0^0 \leq \theta < 180^0\right)$ for a given value of $\tan\theta$.

$\therefore$ The inclinations of the lines are equal.

Theorem II. Two non-vertical lines are perpendicular if and only. If the product of their slopes is minus one.

Proof. Let l_1, l_2 be two non-vertical lines with respective inclinations θ_1, θ_2.

Necessity. Let l_1 and l_2 be perpendicular.



$$\therefore \quad \theta_2 = \theta_1 + 90^0$$

$$\Rightarrow \quad \tan\theta_2 = \tan(\theta_1 + 90^0) = -\cot\theta_1 = -\frac{1}{\tan\theta_1}.$$

$$\therefore \quad \tan\theta_1 \tan\theta_2 = -1.$$

Product of slopes is -1.

Sufficiency. Let $\tan\theta_1 \tan\theta_2 = -1$.

$$\therefore \quad \tan\theta_1 = \frac{-1}{\tan\theta_2} = -\cot\theta_2 = -\tan(90^0 - \theta_2) = \tan[-(90^0 - \theta_2)]$$

$$\Rightarrow \quad \theta_1 = -(90^0 - \theta_2) \quad \Rightarrow \quad \theta_2 = \theta_1 + 90^0$$

$\therefore$ The lines are perpendicular.

Remark. If the slopes of two non-vertical lines be m_1 and m_2, then

(i) lines are || iff $m_1 = m_2$ (ii) lines are $\perp$ iff $m_1 m_2 = -1$.

WORKING RULES FOR SOLVING PROBLEMS

Rule I. *If m be the slope of a non-vertical line passing through the distinct points (x_1, y_1) and (x_2, y_2), then* $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Rule II. *Two lines are parallel if and only if their slopes are equal.*

Rule III. *Two lines are perpendicular if and only if the product of their slopes is minus one.*

$\therefore$ Slope of $AB = \frac{-9-3}{12-2} = \frac{-12}{10} = -\frac{6}{5}$

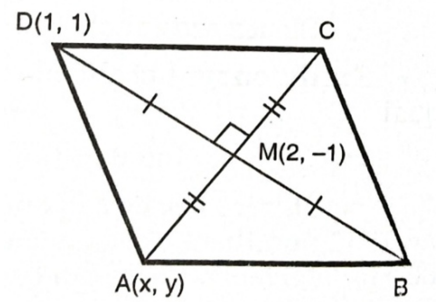
Let the slope of the required line be m.

$$\therefore \qquad m \times \left(-\frac{6}{5}\right) = -1 \quad \therefore \quad m = \frac{5}{6}.$$

Example 4. *ABCD is a rhombus. Its diagonals AC and BD intersect at the point M and satisfy BD = 2 AC. If the coordinates of D and M are (1, 1) and (2, -1) respectively, then find the coordinates of A.*

Sol. Let (x, y) be the coordinates of the vertex A.

Since, diagonals of a rhombus bisects each other at right angle, we have slope of AM x slope of MD = -1



$$\Rightarrow \qquad \frac{y+1}{x-2} \times \frac{-1-1}{2-1} = -1$$

$$\Rightarrow \qquad \frac{y+1}{x-2} \times (-2) = -1$$

$$\Rightarrow \qquad x - 2y = 4 \qquad \ldots(1)$$

Also, $\qquad BD = 2AC \quad \Rightarrow \quad MD = 2AM \quad \Rightarrow \quad (MD)^2 = 4(AM)^2$

$$\Rightarrow \qquad (2-1)^2 + (-1-1)^2 = 4\left[(x-2)^2 + (y+1)^2\right]$$

$$\Rightarrow \qquad 4x^2 + 4y^2 - 16x + 8y + 15 = 0$$

$$\Rightarrow \qquad 4(2y+4)^2 + 4y^2 - 16(2y+4) + 8y + 15 = 0 \qquad \text{[Using (1)]}$$

$$\Rightarrow \qquad 20y^2 + 40y + 15 = 0 \quad \Rightarrow \quad 4y^2 + 8y + 3 = 0 \quad \Rightarrow \quad y = -3/2, -1/2$$

$$\therefore \qquad y = -\frac{1}{3} \quad \Rightarrow \quad x = 2y + 4 = 2\left(-\frac{3}{2}\right) + 4 = 1$$

and $\qquad y = -\frac{1}{2} \quad \Rightarrow \quad x = 2y + 4 = 2\left(-\frac{1}{2}\right) + 4 = 3$

$\therefore$ The coordinates of A are $\left(1, -\frac{3}{2}\right)$ or $\left(3, -\frac{1}{2}\right)$.

EXERCISE 19.1

SHORT ANSWER TYPE QUESTIONS

1. What can be said regarding inclination of a line if its slope is :

   (i) positive (ii) zero

   (iii) negative (iv) not defined ?

2. Find the slope of the line passing through the points (3, -2) and -1, 4).
3. Find the slope of the line, which makes an angle of 30^0 with the positive direction of y-axis measured anticlockwise.
4. Find the angle between the x-axis and the line joining the points (3, -1) and (4, -2).
5. A lien passes through (x_1, y_1) and (h, k). If slope of the line is m, show that

   $$k - y_1 = m(h - x_1).$$

6. Find the value of x, if the slope of the line joining (1, 5) and (x, -7) is 4.
7. Find the value of y, if the slope of the line joining (0, y) and (4, 3y) is -4.

LONG ANSWER TYPE QUESTIONS

8. Show that the line joining (6, -4) and (-3, 2) is :

   (i) parallel to the line joining (1, 3) and (-2, 5).

   (ii) perpendicular to the line joining (0, 4) and (-2, 1).

   Also verify your result graphically in each case.

9. State whether the lines in each part are parallel or perpendicular or neither:

   (i) through (1, -12) and (4, 6); through (10, 5) and (16, 4).

   (ii) through (-3, -4) and (-1, 0); through (6, -3) and (7, -1).

   (iii) through (0, 0) and (6, 7) ; through (0, 0) and (7, 6).

   Also verify your result graphically in each case.

Answers

1. (i) Inclination is acute (ii) Either coincident or parallel to x-axis.
   (iii) Inclination is obtuse (iv) Parallel to y-axis.

2. -3/2 3. $-\sqrt{3}$ 4. 135^0 6. -2

7. -8 9. (i) perpendicular (ii) parallel (iii) neither.

DESCRIPTION OF A LINE BY AN EQUATION

An equation is called the **equation of a straight line** if the coordinates of every point on the straight lien satisfies the equation of the straight line and every point whose coordinates satisfies the equation of the straight line is on the straight line.

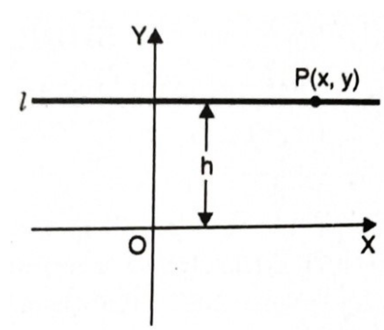
We shall see that every first degree equation line $ax + by + c = 0$ would be the equation of a certain straight line and conversely, the equation of any straight line would always be of the type $ax + by + c = 0$.

Remark. A straight line is briefly written as a 'line'.

EQUATION OF A LINE PARALLEL TO x-AXIS

To find the equation of the straight line parallel to x-axis and at a given directed distance from it.

Let l be a straight line parallel to x-axis and at a directed distance *'h'* from it.



Let $P(x, y)$ be a general point on the line l.

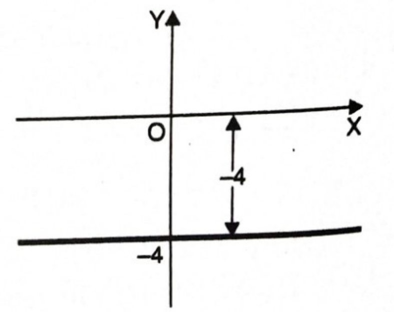
∴ $\mathbf{y = h}$.

This is the required equation of the line.

Remark 1. In particular, the equation of x-axis is $y = 0$.

Remark 2. If $h > 0$, then the line lies above the x-axis and if $h < 0$, then the line lies below the x-axis.

Example 5. *Find the equation of the line which is parallel to x-axis and at a distance of 4 units below the x-axis.*



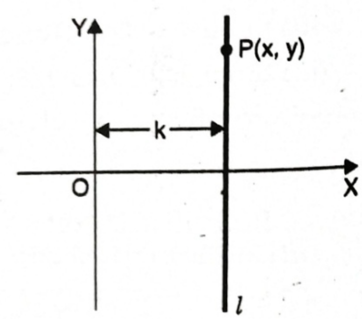
Sol. The gives line is parallel to x-axis and is at a directed distance '- 4' from x-axis.

$\therefore$ Using $y = h$, the equation of the line is $\mathbf{y = -4}$.

EQUATION OF A LINE PARALLEL TO y-AXIS

To find the equation of the straight line parallel to y-axis and at a given directed distance from it.

Let l be a straight line parallel to y-axis and at a directed distance 'k' from it.



Let $P(x, y)$ be a general point on the line.

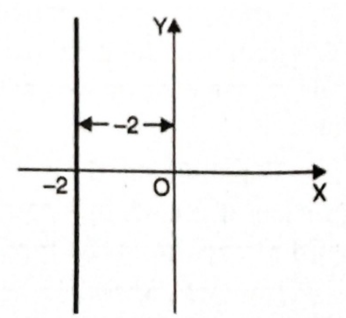
$\therefore$ $\mathbf{x = k}$.

This is the required equation of the line.

Remark 1. In particular, the equation of y-axis is $x = 0$.

Remark 2. If $k > 0$, then the line lies on the right of y-axis and if $k < 0$, then the line lies on the left of y-axis.

Example 6. *Find the equation of the line which is parallel to y-axis and at a distance of 2 units to the left of it.*



Sol. The given line is parallel to y-axis and is at a directed distance '-2' from y-axis.

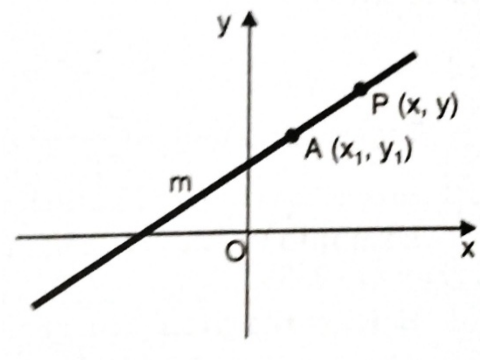
$\therefore$ Using $x = k$, the equation of the line is $\mathbf{x = -2}$.

POINT – SLOPE FORM

To find the equation of the straight line having given one point on the line and its slope.

Let a non-vertical line passes through point $A(x_1, y_1)$ and having slope m.

Let $P(x, y)$ be a general point on the line. Since the line passes through $A(x_1, y_1)$ and $P(x, y)$, so the slope of the line is equal to $\frac{y - y_1}{x - x_1}$*.



Also, the slope of the line is given to be m.

$$\therefore \quad \frac{y - y_1}{x - x_1} = m$$

$$\Rightarrow \quad \mathbf{y - y_1 = m(x - x_1).}$$

This is the required equation of the line. This represents the equation of a line in **point – slope** form.

Remark. If a vertical line passes through the point (x_1, y_1) then its equation is given by $x = x_1$.

Example 7. *The length L (in centimeters) of a copper rod is a linear function of its Celsius temperature C. In an experiment, if L = 124.942 when C = 20 and L = 125.134 when C = 110, express L in terms of C.*

Sol. Let the linear relationship between L and C be $L+ mC + k$.

$C = 20 \quad \Rightarrow \quad L = 124.942 \qquad \therefore \quad 124.942 = 20\,m + k \qquad$ ....(1)

$C = 110 \quad \Rightarrow \quad L = 125.134 \qquad \therefore \quad 125.134 = 110\,m + k \qquad$ ....(2)

(2) – (1) $\quad \Rightarrow \quad 0.192 = 90m \;\Rightarrow\; m = \frac{0.192}{90} = 0.0021333$

$\therefore \quad L = mC + k$ implies $\quad \mathbf{L = 0.0021333\ C + 124.89934.}$

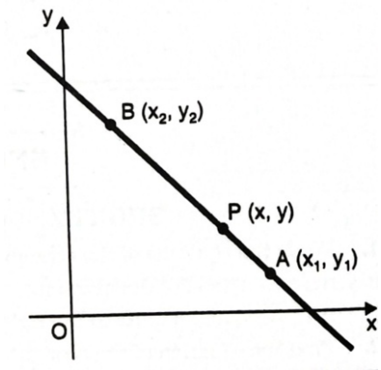
TWO – POINT FORM

To find the equation of the straight line having given two distinct points on the line.

Let a non-vertical line passes through two distinct points $A(x_1, y_1)$ and $B(x_2, y_2)$.

Let $P(x, y)$ be a general point on the line.

Since the line passes through A, B and P, we have slope of AB = slope of AP.



$$\Rightarrow \qquad \frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1}$$

$$\Rightarrow \qquad y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

This is the required equation of the line.

Remark. The equation of the line passing through the points (x_1, y_1) and (x_2, y_2) can also be expressed as

$$y - y_2 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_2).$$

The equations $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ and $y - y_2 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_2)$ appear to be distinct, but these equations become identical, when *expressed* in the form $ax + by + c = 0$.

Example 8. *Find the equation of the straight line passing through the points (4, 2) and (-2, 8).*

Sol. Let the given points be A(4, 2) and B(-2, 8).

The equation of the line passing through (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$\therefore$ The equation of the required line is $y - 2 = \frac{8-2}{-2-4}(x - 4)$

(Here $x_1 = 4, y_1 = 2, x_2 = -2, y_2 = 8$)

or $\qquad y - 2 = (-1)(x - 4)$ or **x + y – 6 = 0.**

Remark. Using $y - y_2 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_2)$, we get $y - 8 = \frac{8-2}{-2-4}(x - (-2))$.

$\Rightarrow \quad y - 8 = (-1)(x + 2) \quad \Rightarrow \quad x + y - 6 = 0$. This is the same equation as obtained above.

WORKING RULES FOR SOLVING PROBLEMS

Rule I. (i) *The equation of the line parallel to x-axis and at a directed distance 'h' from it is* **y = h.**

(ii) *The equation of the line parallel to y-axis and at a directed distance 'k' from it is* **x = k**.

Rule II. *The equation of the line passing through* (x_1, y_1) *and having slope 'm' is* $\mathbf{y - y_1 = m (x - x_1)}$.

Rule III. *The equation of the non-vertical line passing through* (x_1, y_1) *and* (x_2, y_2) *is*

$$\mathbf{y - y_1} = \frac{y_2 - y_1}{x_2 - x_1}\mathbf{(x - x_1)}.$$

EXERCISE 19.2

SHORT ANSWER TYPE QUESTIONS

1. Write the equation of the straight line which is parallel to :

   (i) x-axis and 2 units above it (ii) x-axis and 3 units below it

   (iii) y-axis and 3 units to the right of it

   (iv) y-axis and 2 units to the left to it.

2. Find the equation of the straight line passing through (3, -4) and parallel to:

   (i) x-axis (ii) y-axis.

3. Find the value of k for which the line $(k - 3)x - (4 - k^2)y + k^2 - 7k + 6 = 0$ is

   (i) parallel to the x-axis. (ii) parallel to the y-axis.

   (iii) passing through the origin

4. Find the equation of a line through the origin which makes an angle of 45^0 with the positive direction of x-axis.

5. Find the equation of the straight line :

   (i) passing through $(\sqrt{2}, 2\sqrt{2})$ and having slope 2/3.

(ii) passing through $\left(2, 2\sqrt{3}\right)$ and having inclination of 75^0 with x-axis.

6. Find the equation of the straight line passing through (3, -5) and parallel to the line joining (1, 2) and (-3, 4).
7. Find the equation of the right bisector of the line segment joining (1, 1) and (3, 5).

LONG ANSWER TYPE QUESTIONS

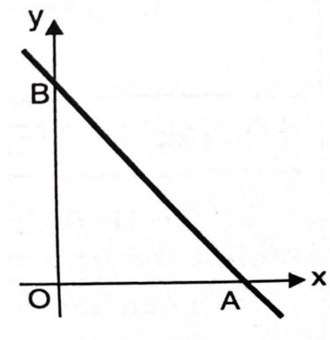
8. Show that the points (1, 4), (3, -2) and (-3, 16) are collinear and find the equation of the line passing through these points.
9. Find the equation of the straight which bisects the distance between the points (a, b), (a', b') and also bisects the distance between the points ($-a$, b), (a', $-b'$).
10. Find the equations of the sides of the triangle whose vertices are (2, 1), (-2, 3), (4, 5).
11. The mid-points of the sides of a triangle are (2, 2), (2, 3), (4, 5). Find the equation of the sides.

Answers

1. (i) $y = 2$ (ii) $y = -3$ (iii) $x = 3$ (iv) $x = -2$
2. (i) $y = -4$ (ii) $x = 3$
3. (i) 3 (ii) ± 2 (iii) 1, 6
4. $x - y = 0$
5. (i) $2x - 3y + 4\sqrt{2} = 0$ (ii) $\left(2+\sqrt{3}\right)x - y - 4 = 0$
6. $x + 2y + 7 = 0$
7. $x + 2y - 8 = 0$
8. $3x + y - 7 = 0$
9. $2b'x - 2ay + ab - a'b' = 0$
10. $x + 2y - 4 = 0$, $x - 3y + 11 = 0$, $2x - y - 3 = 0$
11. $3x - 2y - 2 = 0$, $2x - y - 1 = 0$, $x = 4$.

INTERCEPTS ON AXES

If a line intersect the x-axis at A, then OA (with due regard to sign) is called the **intercept** of the line on x-axis or **x-intercept** of the line.



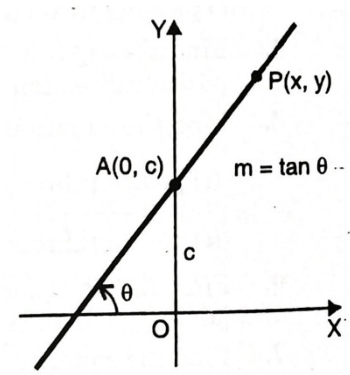
Similarly, if a line intersect the y-axis at B, then OB (with due regard to sign) is called the **intercept** of the line on y-axis or **y-intercept** of the line.

If a line intersect the axes at A and B, then AB is called the **portion of the line intercepted between the axes.**

A non-vertical line is completely determine if its slope and intercept on the y-axis are given.

SLOPE – INTERCEPT FORM

To find the equation of the straight line having given its slope and its intercept on the y-axis.

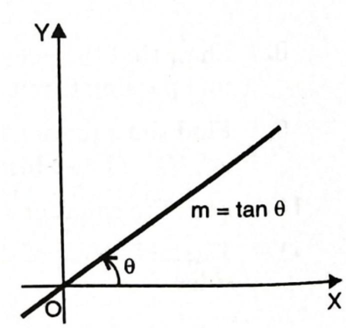


Let a non-vertical line has slope m and intercept on y-axis equal to c.

Let $P(x, y)$ be a general point on the line. Let the line intersect the y-axis at A.

$\therefore$ The coordinates of A are $(0, c)$.

$\therefore$ Slope of line $= \dfrac{y-c}{x-0}$ (Assuming, P is not at A.)



Also, the slope of the line is given to be m.

$$\therefore \quad \frac{y-c}{x} = m$$

$$\Rightarrow \quad \mathbf{y = mx + c}$$

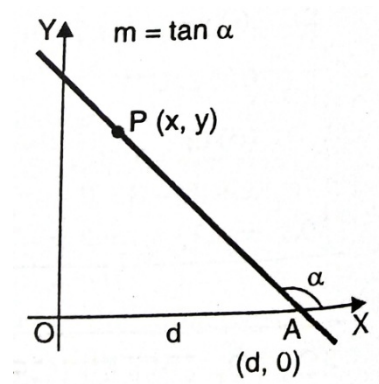
This is required equation of the line.

Corollary. If a line passes through origin and has slope *m*, then its equation is

$$y = mx + 0, \text{ i.e., } \mathbf{y = mx}.$$

Example 9. *Find the equation of a line whose slope in m and the x-intercept is d.*

Sol. Let $P(x, y)$ be a general point on the line. Let the line intersect the *x*-axis at *A*.



∴ The coordinates of *A* are $(d, 0)$.

∴ Slope of line = $\dfrac{y-0}{x-d}$ (Assuming *P* is not at *A*)

∴ $$\frac{y}{x-d} = m$$

⇒ $$\mathbf{y = m(x - d).}$$

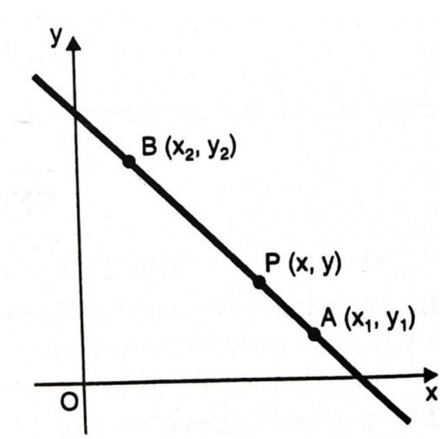
This is the equation of the required line.

INTERCEPT FORM

To find the *equation* of the straight line having given the intercepts which the line makes on the axes.

Let a line make intercepts *a* and *b* on *x*-axis and *y*-axis respectively, where $a \neq 0$.

The line is non-vertical, because *b* is finite. Let $P(x, y)$ be a general point on the line.



Let the line intersects *x*-axis and *y*-axis at *A* and *B* respectively.

∴ The coordinates of *A* and *B* $(a, 0)$ and $(0, b)$ respectively.

Since the line passes through *A*, *B* and *P*, we have

Slope of *AB* = slope of *AP*

⇒ $$\frac{b-0}{0-a} = \frac{y-a}{x-a}$$

$\Rightarrow \qquad bx - ab = -ay$

$\Rightarrow \qquad bx + ab = ab$

$\Rightarrow \qquad \frac{bx}{ab} + \frac{ay}{ab} = \frac{ab}{ab} \quad \Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 1.$

This is the equation of the required line.

Example 10. *Find the equation of the line which makes intercepts – 4 and 5 on the axes.*

Sol. Here $a = -4$ and $b = 5$.

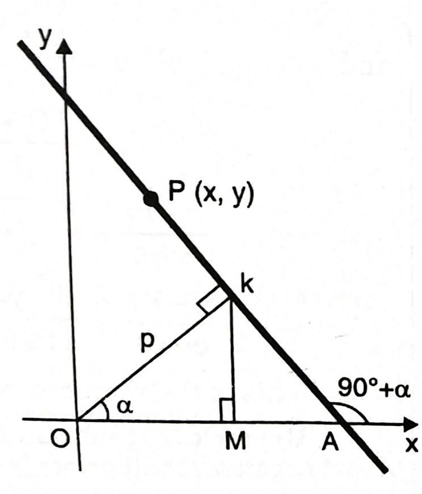
The equation of the line in the **intercept form** is $\frac{x}{a} + \frac{x}{b} = 1$.

$\Rightarrow \qquad \frac{x}{-4} + \frac{y}{5} = 1 \quad \Rightarrow \quad -5x + 4y = 20 \quad \Rightarrow$ **5x – 4y + 20 = 0.**

NORMAL FORM

To find the equation of the straight line on which the length of the perpendicular from the origin and the angle which this perpendicular makes with the x-axis are given.

Let l be a non-vertical straight line on which the length of perpendicular from he origin is p and this perpendicular makes an angle $a(\neq)0$ with the positive direction of x-axis.



$$\angle KAX = 180^0 - \angle OAK = 180^0 - (90^0 - \alpha)$$

$$= 90^0 + \alpha.$$

$\therefore$ Slope of l = tan $(90^0 + \alpha) = -\cot\alpha$

Also $OM = OK.\frac{OM}{OK} = p\cot\alpha$

and $MK = OK.\frac{MK}{OK} = p\sin\alpha$.

$\therefore \qquad K = (p\cos\alpha, p\sin\alpha)$

Let $P(x, y)$ be a general point on the line.

$\therefore$ Slope of l = slope of KP

$$\Rightarrow \quad -\cot\alpha = \frac{y - p\sin\alpha}{x - p\cos\alpha} \quad \Rightarrow \quad -\frac{\cos\alpha}{\sin\alpha} = \frac{y - p\sin\alpha}{x - p\cos\alpha}$$

$$\Rightarrow \quad -x\cos\alpha + p\cos^2\alpha = y\sin\alpha - p\sin^2\alpha$$

$$\Rightarrow \quad x\cos\alpha + y\sin\alpha = p(\sin^2\alpha + \cos^2\alpha)$$

$\Rightarrow$ **x cos** α **+ y sin** α **= p.** This is the required equation of the line.

Remark. In the equation $x\cos\alpha + y\sin\alpha = p$, we observe that:

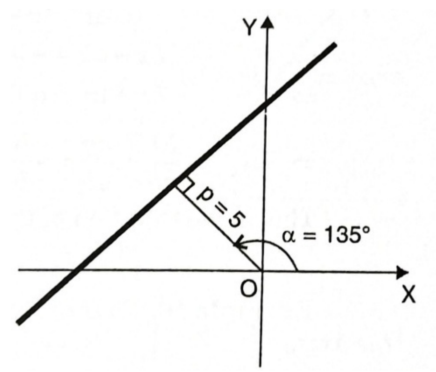
(i) the constant term p on the R.H.S., being the length of perpendicular, is positive.

(ii) (coeff. of x)^2 + (coeff. of y)^2 = $\cos^2\alpha + \sin^2\alpha = 1$.

Example 11. *Find the equation of the line for which p = 5 and $\alpha = 135^0$. Also sketch the line.*

Sol. We have $p = 5$ and $\alpha = 135^0$.

The equation of the line in the **normal form** is



$$x\cos\alpha + y\sin\alpha = p.$$

$$\Rightarrow \quad x\cos 135^0 + y\sin 135^0 = 5$$

$$\Rightarrow \quad x\left(-\frac{1}{\sqrt{2}}\right) + y\left(\frac{1}{\sqrt{2}}\right) = 5$$

$\Rightarrow$ **x – y + 5**$\sqrt{2}$ **= 0.**

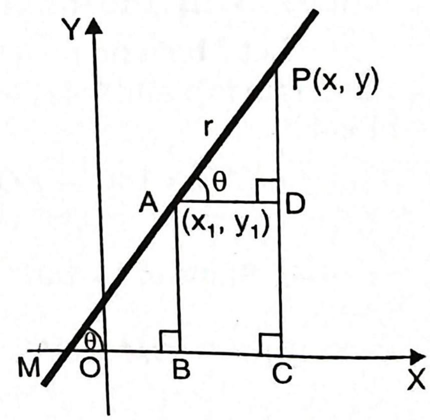
SYMMETRIC FORM (OR DISTANCE FORM)

To find the equation of a straight line having given one point on the line and its inclination.

Let a non-vertical line passes through the point $A(x_1, y_1)$ and having inclination θ *i.e.,* making an angle θ with the positive direction of x – axis.

Let $P(x, y)$ be a general point on the line. Let $AP = r$.

$\therefore \quad \angle PAD = \angle PMX = \theta$



From ΔAPD, we have

$$\cos\theta = \frac{AD}{AP} = \frac{BC}{AP} = \frac{OC - OB}{AP} = \frac{x - x_1}{r}$$

and $$\sin\theta = \frac{PD}{AP} = \frac{PC - CD}{AP}$$

$$= \frac{PC - AB}{AP} = \frac{y - y_1}{r}$$

$$\therefore \quad \frac{x - x_1}{\cos\theta} = r \quad \text{and} \quad \frac{y - y_1}{\sin\theta} = r\ .$$

$$\Rightarrow \quad \frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r\ .$$

This is the required equation of the line.

Remark 1. The distance '*r*' is positive for all points lying on one side of the given point *A* and negative for all points lying on the other side of the given point *A*.

Remark 2. From the equation $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$, we have $x = x_1 + r\cos\theta$ and $y = y_1 + r\sin\theta$.

$\therefore$ The coordinates of any point on this line are $(\mathbf{x_1 + r\cos\theta}, \mathbf{y_1 + r\sin\theta})$.

Here '*r*' is called the *parameter* and represent the distance between the points

$$(x_1 + r\cos\theta, y_1 + r\sin\theta) \text{ and } (x_1, y_1).$$

Example 12. *Find the equation of a line which passes through the point (-2, 3) and makes angle 60^0 with the positive direction of x-axis.*

Sol. Here given point, $(x_1, y_1) = (-2, 3)$ and $\theta = 60^0$.

Using $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$, the equation of the line is $\frac{x - (-2)}{\cos 60^0} = \frac{y - 3}{\sin 60^0} = r$.

$\Rightarrow \quad \dfrac{x+2}{1/2} = \dfrac{y-3}{\sqrt{3}/2} = r,$ **where '*r*' is the distance between (x, y) and (-2, 3).**

Example 13. *Find the equation of the straight line which passes through the point (2, 9) and making an angle of 45^0 with x – axis. Also find the points on the line which are at the distance of (i) 2 units (ii) 5 units from (2, 9).*

Sol. Here given point, (x_1, y_1) = (2, 9) and θ = 45^0.

The equation of the line in the **symmetric form** is $\dfrac{x-x_1}{\cos\theta} = \dfrac{y-y_1}{\sin\theta} = r,$ where '*r*' is the distance between (*x, y*) and (x_1, y_1).

$$\Rightarrow \qquad \frac{x-2}{\cos 45^0} = \frac{y-9}{\sin 45^0} = r \quad \Rightarrow \quad \frac{x-2}{1/\sqrt{2}} = \frac{y-9}{1/\sqrt{2}} = r \qquad \text{....(1)}$$

(1) implies $x - 2 = y - 9 \quad \Rightarrow$ **x – y + 7 = 0**. This is the required equation.

$$(1) \Rightarrow \qquad x = 2 + \frac{1}{\sqrt{2}} r, \quad y = 9 + \frac{1}{\sqrt{2}} r \qquad \text{....(2)}$$

(i) Let $r = 2$. $\therefore$ (2) $\Rightarrow$ $x = 2 + \dfrac{1}{\sqrt{2}}(2) = 2 + \sqrt{2}$

and $$x = 9 + \frac{1}{\sqrt{2}}(2) = 9 + \sqrt{2}\ .$$

$\therefore$ The point $(2+\sqrt{2}, 9+\sqrt{2})$ is on the lien and at a distance of 2 units from (2,9).

(ii) Let $r = 5$. $\therefore$ (2) $\Rightarrow$ $x = 2 + \dfrac{1}{\sqrt{2}}(5) = 2 + \dfrac{5}{\sqrt{2}}$

and $$x = 9 + \frac{1}{\sqrt{2}}(5) = 9 + \frac{5}{\sqrt{2}}$$

$\therefore$ The point $\left(2 + \dfrac{5}{\sqrt{2}}, 9 + \dfrac{5}{\sqrt{2}}\right)$ is on the line and is at a distance of 5 units from (2, 9).

WORKING RULES FOR SOLVING PROBLEMS

Rule I. *The equation of the line having slope 'm' and y-intercept 'c' is* **y =mx + c.**

Rule II. *The equation of the line having intercepts 'a' and 'b' is* $\frac{x}{a}+\frac{y}{b}=1$

Rule III. *The equation of the line for which the length of perpendicular from the origin is 'p' and this perpendicular is inclined at an angle* α *to the x-axis is*

$$\mathbf{x}\cos\alpha + \mathbf{y}\sin\alpha = \mathbf{p}.$$

Rule IV. *The equation of the line passing through* (x_1, y_1) *and having inclination '*θ*' is*

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r,$$

'r' is the distance between the points (x, y) and (x_1, y_1).

EXERCISE 19.3

SHORT ANSWER TYPE QUESTIONS

1. Find the equation of the straight line which makes :

   (i) an angle 30^0 with x-axis and cuts off intercept 4 from the positive direction of y-axis.

   (ii) an angle $\tan^{-1} 2$ with the x-axis and cuts off intercepts 5 from the negative side of y-axis.

2. Find the equation of the straight line cutting off intercepts a and b from the axes where :

   (i) $a = 1, b = 4$ (ii) $a = 5,\ b = -10$.

3. Find the equation of the straight line for which :

   (i) $p = 1,\ \alpha = 30^0$ (ii) $p = 2,\ \alpha = 135^0$.

4. Find the equation of the straight line in symmetric form which passes through the point (x_1, y_1) and having inclination θ, where :

   (i) $(x_1, y_1) = (-2, 1),\ \theta = 45^0$ (ii) $(x_1, y_1) = (-1, 0),\ \theta = 120^0$.

5. Find the equation of the line which passes through (2, 5) and cuts off equal intercepts on the axes.
6. Find the equation of the line which passes through (3, -5) and cuts off intercepts on the axes which are equal in magnitude but opposite in sign.
7. A line passes through (1, 3) and its y-intercept is three times its intercept on x-axis. Find the equation of this line.
8. Find the equations of the liens which cut off intercepts on the axes whose sum and product are 1 and -6 respectively.
9. Find the equation of the line which passes through (4, 1) and is such that the portion of the line intercepted between the axes is divided by this point internally in the ratio 1 : 2.
10. Find the equation of the line which passes through (2, 2) and the sum of whose intercepts on coordinate axes is 9.
11. Find the equation of the line which passes through the point (22, -6) and is such that the intercept on x-axis exceeds the intercept on y-axis by 5.

LONG ANSWER TYPE QUESTIONS

12. A straight line is such that the segment of the line intercepted between the axes is bisected at the point $P(a, b)$. Show that its equation is $\frac{x}{a} + \frac{y}{b} = 2$.
13. If the straight line $\frac{x}{a} + \frac{y}{b} = 1$ passes through the points (8, -9) and (12, -15), find the values of a and b.
14. Find the equation of the straight line at a distance of 3 units from the origin such that the perpendicular from the origin to the line makes an angle α, given by $\tan\alpha = 5/2$, with the positive direction of x-axis.
15. Find the equation of the line in the normal form for which $p = 2$, $\sin\alpha = 4/5$.

Answers

1. (i) $x-\sqrt{3}y+4\sqrt{3}=0$ (ii) $2x-y-5=0$

2. (i) $4x+y-4=0$ (ii) $2x-y-10=0$

3. (i) $\sqrt{3}x+y-2=0$ (ii) $x-y+2\sqrt{2}=0$

4. (i) $\frac{x+2}{1/\sqrt{2}}=\frac{y-1}{1/\sqrt{2}}=r$ (iii) $\frac{x+1}{-1/2}=\frac{y}{\sqrt{3}/2}=r$

5. $x+y-7=0$

6. $x-y-8=0$

7. $3x+y-6=0$

8. $2x-3y-6=0,\quad 3x-2y+6=0$

9. $x+2y-6=0$

10. $2x+y=6,\quad x+2y=6$

11. $x+2y-10=0$, $6x+11y-66=0$

13. 2, 3

14. $12x+5y-39=0$, $12x+5y+39=0$

15. $3x+4y-10=0$, $3x-4y+10=0$.

GENERAL EQUATION OF A LINE

We shall prove that the equation of any line can be expressed in the form *Ax + By + C = 0*. This will also hold conversely, *i.e.,* the points whose coordinates satisfies the above equation would all lie on a straight line.

Theorem. Prove that every straight line has equation of the form Ax + By + C = 0 and conversely an equation of the type Ax + By + C = 0 (A, B are not both zero) always represents a straight line.

Proof. Let l be any straight line. The line is either perpendicular to x-axis or not.

Case I. 1 is perpendicular to x-axis.

Let the line be at a distance k from the y-axis.

$\therefore$ The equation of the line $x = k$. This equation can be expressed as $1(x) + 0(y) + (-k) = 0$.

This equation is of the type $Ax + By + C = 0$, where $A = 1$, $B = 0$, $C = -k$.

Case II. 1 is not perpendicular to x-axis.

Let m and c be the slope and y-intercept of the line respectively.

The equation of the line in **slope-intercept** form is $y = mx + c$.

This equation can be expressed as $m.x + (-1)y + c = 0$.

This equation is of the type $Ax + By + C = 0$, where $A = m$, $B = -1$, $C = c$.

$\therefore$ Every line can be represented by a linear equation of the type $Ax + By + C = 0$

Converse. Let $Ax + By + C = 0$ be a linear equation in x and y with A, B not both zero.

Case I. B = 0. In this case, the equation reduces to $Ax + 0.\ y + C = 0$.

$\Rightarrow \qquad x = -C / A \qquad (\because\ B = 0 \Rightarrow A \neq 0)$

This represents the straight line parallel to y-axis and at a distance $-C/A$ from it.

Case II. B$\neq$0. In this case, the equation reduces to $By = -Ax - C$.

$$\Rightarrow \qquad y = \left(-\frac{A}{B}\right)x + \left(-\frac{C}{B}\right)$$

This represents the straight line whose slope and y-intercept are $-\frac{A}{B}$ and $-\frac{C}{B}$ respectively.

Here, the result holds.

Example 14. *Find the equation of the line passing through (3, 5) and (1, -2), assuming the equation of the line to be Ax + By + C = 0.*

Sol. The equation of the required line is $Ax + By + C = 0$. ...(1)

(3, 5) and (1, -2) are on the line.

$\therefore \qquad \mathbf{3A + 5B + C = 0}$ ...(2)

$\qquad \mathbf{A - 2B + C = 0}$ ...(3)

$$\therefore \qquad \frac{A}{5+2} = \frac{B}{1-3} = \frac{C}{-6-5} = k \text{ (say)}$$

$\therefore$ (1) $\Rightarrow$ $7kx - 2ky - 11k = 0$ or $\mathbf{7x - 2y - 11 = 0.}$

This is the required equation.

REDUCTION OF GENERAL EQUATION TO STANDARD FORM

Let $Ax + By + C = 0$ be the general equation of a straight line.
$\therefore$ A and B are not both zero.

(i) **Reduction of 'slope-intercept' form.** Give equation is $Ax + By + C = 0$.

$\Rightarrow$ $$By = -Ax - C \Rightarrow y = \left(-\frac{A}{B}\right)x + \left(-\frac{C}{B}\right).$$ (Assuming $B \neq 0$).

Comparing this with $y = mx + c$, we get slope = $-\frac{A}{B}$ and y – intercept = $-\frac{C}{B}$.

WORKING RULES FOR REDUCING Ax + By + C = 0
TO 'SLOPE-INTERCEPT' FORM

Step I. *Keep only 'By' on the L.H.S. and get By = - Ax – C.*

Step II. *Divide both sides by 'B' and get* $y = \left(-\frac{A}{B}\right)x + \left(-\frac{C}{B}\right)$.

Step III. *This expresses the given equation in the 'slope-intercept' form. Here*

m = -A/B and c = - C/B.

(ii) **Reduction to 'Intercept' form.** Given equation is $Ax + By + C = 0$.

$\Rightarrow$ $$Ax + By = -C \Rightarrow \frac{A}{-C}x + \frac{B}{-C}y = 1.$$ (Assuming $C \neq 0$).

$\Rightarrow$ $$\frac{x}{(-C/A)} + \frac{y}{(-C/B)} = 1$$ (Assuming $A \neq 0$, $B \neq 0$)

Comparing this with $\frac{x}{a} + \frac{x}{b} = 1$, we get x – intercept = $-\frac{C}{A}$ and y-intercept = $-\frac{C}{B}$

WORKING RULES FOR REDUCING Ax + By + C = 0

TO 'INTERCEPT' FORM

Step I. *Shift constant 'C' to the R.H.S. and get Ax + By = - C.*

Step II. *Divide both sides by '-C' and get* $\frac{Ax}{-C}+\frac{By}{-C}=1$.

Step III. *Make coefficients of x and y occur as their denominator and get*

$$\frac{x}{(-C/A)}+\frac{y}{(-C/B)}=1.$$

Step IV. *This expresses the given equation in the 'intercept' form. Here*

$a=-C/A$ and $b=-C/B$.

Remark. The *x*-intercept can be obtained by putting y = 0 in the equation and then solving for *x*. The value of *x* gives *x*- intercept.

Similarly, *y*-intercept can also be obtained by putting *x* = 0 in the equation and then solving for y. The value of y gives y-intercept.

(iii) **Reduction to 'Normal' form.** Given equation is *Ax* + *By* + *C* = 0.

Let its **normal form** be $x\cos\alpha+y\sin\alpha=p$.

$$\Rightarrow \quad \frac{A}{\cos\alpha}=\frac{B}{\sin\alpha}=\frac{C}{-p} \Rightarrow \cos\alpha=\frac{-Ap}{C} \text{ and } \sin\alpha=\frac{-Bp}{C} \quad (\text{Assuming } C\neq 0)$$

$$\therefore \quad 1=\cos^2\alpha+\sin^2\alpha=\left(\frac{-Ap}{C}\right)^2+\left(\frac{-Bp}{C}\right)^2=\frac{p^2}{C^2}\left(A^2+B^2\right)$$

$$\therefore \quad p=\pm\frac{C}{\sqrt{A^2+B^2}}.$$

Case I. C is positive. $\therefore \quad p=\frac{C}{\sqrt{A^2+B^2}}$. ($\because$ *p* is to be positive)

$$\therefore \quad \cos\alpha=-\frac{A}{C}\left(\frac{C}{\sqrt{A^2+B^2}}\right)=-\frac{A}{\sqrt{A^2+B^2}}$$

and $$\cos\alpha = -\frac{B}{C}\left(\frac{C}{\sqrt{A^2+B^2}}\right) = -\frac{B}{\sqrt{A^2+B^2}}$$

$\therefore$ $x\cos\alpha + y\sin\alpha = p$ becomes $\left(-\frac{A}{\sqrt{A^2+B^2}}\right)x + \left(-\frac{B}{\sqrt{A^2+B^2}}\right)y = \frac{A}{\sqrt{A^2+B^2}}$.

This is the required normal form of the given equation.

Case II. C is negative. $\therefore$ $p = -\frac{C}{\sqrt{A^2+B^2}}$ ($\because$ p is to be positive)

$$\therefore \quad \cos\alpha = -\frac{A}{C}\left(\frac{C}{\sqrt{A^2+B^2}}\right) = \frac{A}{\sqrt{A^2+B^2}}$$

and $$\sin\alpha = -\frac{B}{C}\left(\frac{C}{\sqrt{A^2+B^2}}\right) = \frac{B}{\sqrt{A^2+B^2}}$$

$\therefore$ $x\cos\alpha + y\sin\alpha = p$ becomes $\left(\frac{A}{\sqrt{A^2+B^2}}\right)x + \left(\frac{B}{\sqrt{A^2+B^2}}\right)y = -\frac{C}{\sqrt{A^2+B^2}}$.

This is the required normal form of the given equation.

WORKING RULES FOR REDUCING Ax + By + C = 0
TO 'NORMAL' FORM

Step I. *Shift constant 'C' to the R.H.S. and get Ax + By = - C.*

Step II. *If the R.H.S. is not positive, then make it so by multiplying the whole equation by '- 1'.*

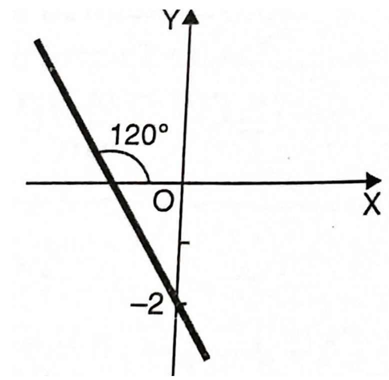
Step III. *Divide the whole equation by* $\sqrt{A^2+B^2}$.

Step IV. *This express the given equation in the 'normal' form.*

Example 15. *Reduce* $\sqrt{3}x + y + 2 = 0$ *to the 'slope-intercept form' and hence find its slope, inclination and y-intercept. Also sketch the line on the coordinate plane.*

Sol. We have $\sqrt{3}x + y + 2 = 0$. $\therefore$ $y = -\sqrt{3}x - 2$.

Comparing it with $y = mx + c$, we get



$m = -\sqrt{3}$ and $c = -2$.

$\therefore$ Slope = $-\sqrt{3}$ and y-intercept = -2.

Let inclination of the line be θ.

$\therefore$ $\tan\theta = -\sqrt{3} = -\tan 60^0$

$= \tan\left(180^0 - 60^0\right) = \tan 120^0$

$\therefore$ $\theta = 120^0$.

$\therefore$ Inclination = **120⁰.**

The sketch of the line is shown in the figure.

EXERCISE 19.4

SHORT ANSWER TYPE QUESTIONS

1. Reduce the following equations to the slope – intercept form and find the values of slope and y-intercept:

   (i) $x + 3y = 10$ (ii) $3x + 3y = 5$.

2. Reduce the following equations to the intercept form and find its intercepts on the axes.

   (i) $2x + y - 6 = 0$ (ii) $5x - y - 15 = 0$.

3. Reduce the following equations to the normal form and find the values of p and α.

   (i) $x + y - 2 = 0$ (ii) $\sqrt{3}x + y + 2 = 0$.

LONG ANSWER TYPE QUESTIONS

4. Reduce the equation $x\sec\alpha - y\operatorname{cosec}\alpha = a$ to the slope – intercept form and hence find the slope and the coordinates of the point where it meets the y-axis.
5. Which of the lines $3x + 4y - 9 = 0$ and $2x + 6y + 19 = 0$ is farther from the origin?

6. Find the intercepts of the line $x\sin\alpha + y\cos\alpha = \sin 2\alpha$ on the axes. Also find the mid-point of the line segment intercepted between the axes.
7. Find the equation of the line passing through (- 1, - 2) and (-5, -3), assuming the equation of the line to be $Ax+By+C=0$. Also reduce this equation to the intercept form.
8. Show that the origin is equidistant from the straight lines $4x+3y+10=0$, $5x-12y+26=0$ and $7x+24y=50$.

Answers

1. (i) $y = -\frac{1}{3}x + \frac{10}{3}$, Slope = $-\frac{1}{3}$, y-intercept = $\frac{10}{3}$

   (ii) $y = -x + \frac{5}{3}$, Slope = - 1, y-intercept = $\frac{5}{3}$

2. (i) $\frac{x}{3} + \frac{y}{6} = 1$, x- intercept = 3, y-intercept = 6.

   (ii) $\frac{x}{3} + \frac{y}{-15} = 1$, x-intercept = 3, y-intercept = - 15.

3. (i) $x\cos 45^0 + y\sin 45^0 = \sqrt{2}, p = \sqrt{2}, a = 45^0$

   (ii) $x\cos 210^0 + y\sin 210^0 = 1, p = 1, \alpha = 210^0$

4. $y = x\tan\alpha - a\sin\alpha, \tan\alpha, (0, -a\sin\alpha)$

5. $2x+6y+19=0$

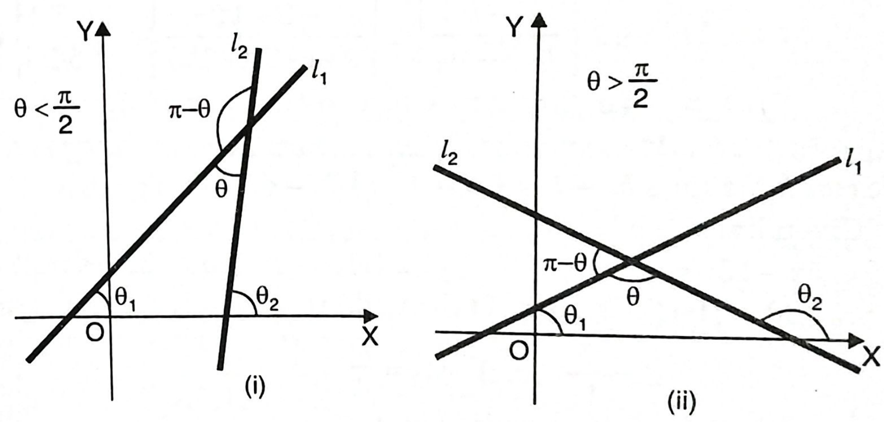
6. $a = 2\cos\alpha, b = 2\sin\alpha, (\cos\alpha, \sin\alpha)$

7. $x - 4y - 7 = 0, \frac{x}{7} + \frac{y}{-7/4} = 1$.

ANGLE BETWEEN TWO LINES

Two intersecting lines intersect at two angles, which are supplementary to each other. For example, if angle between two intersecting lines is 60^0, then the other angle would be $180^0 - 60^0 = 120^0$. Generally, the angle which is not greater than 90^0 is taken as the angle between the intersecting lines.

Now, we shall find the angle between two lines with given slopes.



Let l_1 and l_2 be two non-vertical lines with slopes m_1 and m_2 respectively. Let θ_1, θ_2 be the inclinations of these lines.

$\therefore$ $$m_1 = \tan\theta_1 \text{ and } m_2 = \tan\theta_2$$

Let θ and $\pi - \theta$ be the angles between the lines $(\theta \neq \pi/2)$.

$\therefore$ Exterior angle, $\theta_2 = \theta_1 + \theta$ implies $\theta = \theta_2 - \theta_1$

$\therefore$ $$\tan\theta = \tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 - \tan\theta_1}$$

$\therefore$ $$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}.$$

Remark. In Fig. (i), angle θ is acute, so $\tan\theta > 0$.

$\therefore$ $$\frac{m_2 - m_1}{1 + m_2 m_1} > 0 \text{ and we have } \tan\theta = \left|\frac{m_2 - m_1}{1 + m_2 m_1}\right|.$$

In Fig. (ii), angle θ is obtuse, so $\tan\theta < 0$.

$\therefore$ $$\frac{m_2 - m_1}{1 + m_2 m_1} < 0 .$$

In this case, the other angle $\pi - \theta$ would be acute.

Also, $$\tan(\pi - \theta) = -\tan\theta = -\frac{m_2 - m_1}{1 + m_2 m_1} = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| \qquad \left(\because -\frac{m_2 - m_1}{1 + m_2 m_1} > 0\right)$$

$\therefore$ **The tangent of the acute angle between the lines with slopes m_1 and m_2 is equal to** $\left|\frac{m_2 - m_1}{1 + m_2 m_1}\right|$, **which is also equal to** $\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$ **.**

WORKING RULES TO FIND THE ANGLE BETWEEN GIVEN LINES

Step I. *Find the slopes of the given lines. Let these slopes be m_1 and m_2,*

Step II. *Assume θ to be the acute angle between the given lines.*

Step III. *Put* $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$ *and simplify the R.H.S. to get the value of θ.*

Example 16. *Find the angle between the lines $3x + y - 7 = 0$ and $x + 2y + 9 = 0$.*

Sol. Let m_1 and m_2 be the slopes of the given lines $3x + y - 7 = 0$ and $x + 2y + 9 = 0$ respectively.

$\therefore \quad m_1 = -3$ and $m_2 = -1/2$

[$\because$ $3x + y - 7 = 0 \Rightarrow y = (-3)x + 7$ and $x + 2y + 9 = 0$

$\Rightarrow y = (-1/2)x - (9/2)$.]

Let θ be the acute angle between the given lines.

$$\therefore \quad \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-3 + 1/2}{1 + (-3)(-1/2)}\right| = \left|\frac{-5/2}{5/2}\right| = |-1| = 1 .$$

$\therefore \quad \theta$ = **45^0.**

Example 17. *Find the equations of the lines which pass through (4, 5) and make equal angles with the lines $5x - 12y + 6 = 0$ and $3x = 4y + 7$.*

Sol. Given lines are

$5x - 12y + 6 = 0$ ....(1) and $3x = 4y + 7$ ...(2)

Let m_1 and m_2 be the slopes of (1) and (2) respectively.

$$\therefore \quad m_1 = \frac{5}{12} \text{ and } m_2 = \frac{3}{4}$$

Let m be the slope of the required line.

Let θ be the acute angle which the required line make with given lines.

$$\therefore \qquad \tan\theta = \left|\frac{m-m_1}{1+mn_1}\right| = \left|\frac{m-\frac{5}{12}}{1+m.\frac{5}{12}}\right| = \left|\frac{12m-5}{12+5m}\right|$$

Also $$\tan\theta = \left|\frac{m-m_2}{1+mn_2}\right| = \left|\frac{m-\frac{3}{4}}{1+m.\frac{3}{4}}\right| = \left|\frac{4m-3}{4+3m}\right|$$

$$\Rightarrow \qquad \left|\frac{12m-5}{12+5m}\right| = \left|\frac{4m-3}{4+3m}\right| \quad \Rightarrow \quad \frac{12m-5}{12+5m} = \pm\frac{4m-3}{4+3,}$$

$$\frac{12m-5}{12+5m} = +\frac{4m-3}{4+3m} \quad \Rightarrow \quad m^2 = -1, \text{ which is impossible.}$$

$$\frac{12m-5}{12+5m} = -\frac{4m-3}{4+3m} \quad \Rightarrow \quad 28m^2 + 33m - 28 = 0 \quad \Rightarrow \quad m = \frac{4}{7}, -\frac{7}{4}$$

If $m = \frac{4}{7}$, the equation of the required line is

$$y - 5 = \frac{4}{7}(x-4) \quad \textbf{or} \quad \mathbf{4x - 7y + 19 = 0.}$$

If $m = -\frac{4}{7}$, the equation of the required line is

$$y - 5 = -\frac{4}{7}(x-4) \quad \textbf{or} \quad \mathbf{7x + 4y - 48 = 0.}$$

EXERCISE 19.5

SHORT ANSWER TYPE QUESTIONS

1. Find the acute angle between the lines whose slopes are:

   (i) 3 and 1/2 (ii) $\sqrt{3}$ and $1/\sqrt{3}$.

2. Find the obtuse angle between the lines whose slopes are :

   (i) $\sqrt{3}$ and $1/\sqrt{3}$ (ii) $2-\sqrt{3}$ and $2+\sqrt{3}$.

3. Find the angle between the lines :;

   (i) $3x + y - 8 = 0$ and $x + 2y + 2 = 0$

   (ii) $2x - y + 1 = 0$ and $x + y + 8 = 0$

   (iii) $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{a} - \frac{y}{b} = 1$

   (iv) $y - \sqrt{3}x - 5 = 0$ and $\sqrt{3}y - x - 6 = 0$

   (v) $x\cos\alpha_1 + y\sin\alpha_1 = p_1$ and $x\cos\alpha_2 + y\sin\alpha_2 = p_2$, where $\alpha_1 > \alpha_2$.

4. Find the tangent of the angle between the lines whose intercepts on the axes are respectively, *p, -q* and *q, -p*.
5. The angle between two lines is $\pi/4$ and the slope of one of them is 1. Find the inclination of the other line.
6. Two lines passing through the point (2, 3) make an angle of 45^0. If the slope of one of the lines is 2, find the slope of the other.

LONG ANSWER TYPE QUESTIONS

7. (i) Find the equation of the straight line which passes through (4, 5) and making angle 45^0 with the straight lien $2x + y + 1 = 0$.

   (ii) Find the equations of the lines through the point (3, 2) and making angle of 45^0 with the line $x - 2y = 3$.

8. The line through (4, 3) and (-6, 0) intersects the line $5x + y = 0$. Find the angle of intersection.
9. Find the angles of the triangle whose vertices are (3, 4), (4, 4) and (4, 5). It is given that the triangle is not an obtuse angled.
10. Find the angles of the acute angled triangle whose vertices are (1, 2), (3, -2).

Answers

1. (i) 45^0	(ii) 30^0	2. (i) 150^0	(ii) 120^0
3. (i) 45^0	(ii) $\tan^{-1} 3$	(iii) 90^0	(iv) 30^0

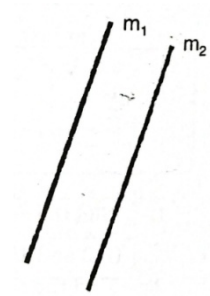
(v) $\alpha_1 - \alpha_2$ 4. $\left|\frac{p^2 - q^2}{2pq}\right|$ 5. 0^0, 90^0 6. $\frac{1}{3}, -3$

7. (i) $3x - y - 7 = 0, x + 3y - 19 = 0$ (ii) $3x - y - 7 = 0, x + 3y - 9 = 0$

8. $\tan^{-1}\frac{\sqrt{3}}{5}$ 9. 45^0, 45^0, 90^0 10. $\tan^{-1}\frac{4}{7}, \tan^{-1}\frac{2}{3}, \tan^{-1} 2$

CONDITION FOR PARALLELISM OF LINES

Let two lines with slopes m_1 and m_2 be parallel.



$\therefore$ Angle between these lines is 0^0.

$$\therefore \quad \tan 0^0 = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$\Rightarrow \quad \frac{m_1 - m_2}{1 + m_1 m_2} = 0$ *i.e.*, $m_1 - m_2 = 0$ or $m_1 \neq m_2$.

$\therefore$ **If two lines are parallel then their slopes are equal.**

Corollary. The slope of the line $Ax + By + C = 0$ is $-A/B$.

$\therefore$ Any line parallel to this line will have slope $-A/B$.

$\therefore$ The equation of a line parallel to $Ax + By + C = 0$ is of the form

$$y = -\frac{A}{B}x + C' \qquad \text{....(1)}$$

where C' is the y-intercept of the line (1)

$\Rightarrow \quad Ax + By + BC' = 0 \quad \Rightarrow \quad Ax + By + k = 0,$

where $k = -BC'$ is some constant.

$\therefore$ **The equation of a line parallel to Ax + By + C = 0 is of the type Ax + By + k = 0, where k is some constant to be determined by using other given conditions.**

Equivalently, $Ax + By + k = 0$ represents the family of lines parallel to the line

$$Ax + By + C = 0.$$

Example 18. *Find the equation of the straight line that has y-intercept 4 and is parallel to the straight line 2x – 3y = 7.*

Sol. Given line is $2x - 3y = 7$. ...(1)

(1) $\Rightarrow$ $3y = 2x - 7 \Rightarrow y = \frac{2}{3}x - \frac{7}{3}$ $\therefore$ Slope of (1) is 2/3.

The required line is parallel to (1), so its slope is also 2/3. y-intercept of required line = 4.

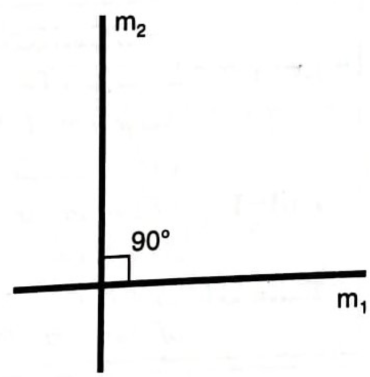
$\therefore$ By using 'y = $mx + c$' form, the equation of the required line is

$$y = \frac{2}{3}x + 4 \quad \text{or} \quad \mathbf{2x - 3y + 12 = 0.}$$

CONDITION FOR PERPENDICULARITY OF LINES

Let two lines with slopes m_1 and m_2 be perpendicular.

$\therefore$ Angle between these lines is 90^0.



$\therefore$ $\tan 90^0 = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$ is not defined

$\Rightarrow$ $1 + m_1 m_2 = 0$ *i.e.,* $\mathbf{m_1 m_2 = -1.}$

$\therefore$ **If two lines are perpendicular then product of their slopes is '-1'.**

Corollary. The slope of the line $Ax + By + C = 0$ is $-A/B$.

$\therefore$ Any line perpendicular to this line will have slope equal to negative reciprocal of $-A/B$, which is B/A.

$\therefore$ The equation of a line perpendicular to $Ax + By + C = 0$ is of the form

$$y = \frac{B}{A}x + C' \qquad \text{....(1)}$$

where C' is the y-intercept of the line.

(1) $\Rightarrow$ $Bx - Ay + AC' = 0$ $\Rightarrow$ $Bx - Ay + k = 0$,

where $k = AC'$ is some constant.

$\therefore$ **The equation of a line perpendicular to Ax + By + C = 0 is of the type Bx – Ay + k = 0, where k is some constant to be determined by using other given conditions.**

Equivalently, $Bx + Ay + k = 0$ represents the family of lines perpendicular to the line

$$Ax + By + C = 0.$$

Example 19. *Find the equation of the straight line passing through (2, 3) perpendicular to 4x – 3y = 10.*

Sol. Given line is $4x - 3y = 10$. ...(1)

(1) $\Rightarrow$ $3y = 4x - 10$ $\Rightarrow$ $y = \frac{4}{3}x - \frac{10}{3}$ $\therefore$ Slope of (1) is $\frac{4}{3}$.

The required line is perpendicular to (1), so its slope is negative reciprocal of 4/3, which is $-\frac{1}{4/3} = -\frac{3}{4}$. The required line is also to pass through (2, 3).

$\therefore$ By using $y - y_1 = m(x - x_1)$ form, the equation of the required line is

$$y - 3 = -\frac{3}{4}(x - 2) \text{ i.e., } 4y - 12 = -3x + 6 \text{ or } \mathbf{3x + 4y - 18 = 0.}$$

Example 20. *The line 7x – 9y – 19 = 0 is perpendicular to the line through the points (x, 3) and (4, 1). Find the value of x.*

Sol. Given lien is $7x - 9y - 19 = 0$.

$\Rightarrow$ $9y = 7x - 19$ $\Rightarrow$ $y = \frac{7}{9}x - \frac{19}{9}$ $\therefore$ Slope of = $\frac{7}{9}$.

Slope of line joining $(x, 3)$ and $(4, 1)$ = $\frac{3-1}{x-4} = \frac{2}{x-4}$

These lines are perpendicular.

$$\Rightarrow \qquad \left(\frac{7}{9}\right)\left(\frac{2}{x-4}\right) = -1 \Rightarrow 14 = -9x + 36 \Rightarrow x = \frac{22}{9}.$$

WORKING RULES FOR SOLVING PROBLEMS

Rule I. *Lines with slopes m_1 and m_2 are :*

(i) *parallel if and only if $m_1 = m_2$.*

(ii) *perpendicular if and only if $m_1 m_2 = -1$.*

Rule II. *The equation of a straight ling parallel to Ax + By + C = 0 is of the form Ax + By + k = 0.*

Rule III. *The equation of a straight line perpendicular to Ax + By + C = 0 is of the form Bx – Ay + k = 0.*

EXERCISE 19.6

SHORT ANSWER TYPE QUESTIONS

1. Find the slope of a line which is parallel to the line :

   (i) $2x - y + 8 = 0$ (ii) $\frac{x}{a} + \frac{y}{b} = 1$

2. Find the slope of a line which is perpendicular to the line :

   (i) $x + y - 9 = 0$ (ii) $x \cos \alpha + y \sin \alpha = p$.

3. (i) Find the equation of the straight line parallel to $2x + 5y = 7$ and passing through (-1, 4).

   (ii) Find the equation of the straight line parallel to $2x - y + 8 = 0$ and having y-intercept 4.

4. (i) Find the equation of the straight line perpendicular to $2x + 4y - 7 = 0$ and passing through (9, 2).

   (ii) Find the equation of the straight lien perpendicular to $2x + 4y - 7 = 0$ and having y-intercept 5.

5. The perpendicular from the origin to a line meets it at the point (-2, 9), find the equation of the line.
6. Find the equation of the straight line which is (i) parallel (ii) perpendicular to the line $4x - y + 8 = 0$ and passing through the mid-point of the line segment joining (1, 5) and (3, 11).
7. Find the equation of the straight line which is (i) parallel (ii) perpendicular to the line $Ax + By + C = 0$ and passing through the point (x_1, y_1).
8. Find the equation of the right bisector of the line segment joining the points (α, β) and (β, α).
9. For the triangle *ABC* whose vertices *A* (-2, 3), *B* (4, -3) and *C*(4, 5), find the equation of the :

   (i) right bisector of *BC* (ii) altitude from *A*.

   (iii) straight line parallel to *BC* and passing through *A*.

Answers

1. (i) 2 (ii) $-b/a$ 2. (i) 1 (ii) $\tan\alpha$

3. (i) $2x + 5y - 18 = 0$ (ii) $2x - y + 4 = 0$

4. (i) $3x - 2y - 23 = 0$ (ii) $2x - y + 5 = 0$

5. $2x - 9y + 85 = 0$ 6. (i) $4x - y = 0$ (ii) $x + 4y - 34 = 0$

7. (i) $Ax + By - (Ax_1 + By_1) = 0$ (ii) $Bx - Ay - (Bx_1 - Ay_1) = 0$

8. $x - y = 0$ 9. (i) $y - 1 = 0$ (ii) $y - 3 = 0$ (iii) $x + 2 = 0$.

INTERSECTION OF LINES

Two lines are said to be **intersecting** if there is exactly one point which is common to both lines.

Let $A_1 x + B_1 y + C_1 = 0$ ....(1)

and $A_2 x + B_2 y + C_2 = 0$ ....(2) be the equations of two lines.

The point of intersection of (1) and (2) will on both lines. So, the coordinates of the point of intersection will satisfy both equations.

Solving (1) and (2) by cross – multiplying, we get

$$\frac{x}{B_1C_2 - B_2C_1} = \frac{y}{C_1A_2 - C_2A_1} = \frac{1}{A_1B_2 - A_2B_1}.$$

$\Rightarrow$ $$x = \frac{B_1C_2 - B_2C_1}{A_1B_2 - A_2B_1}$$

and $$y = \frac{C_1A_2 - C_2A_1}{A_1B_2 - A_2B_1}$$, provided $A_1B_2 - A_2B_1 \neq 0$.

$\therefore$ The lines intersect at the point $\left(\frac{B_1C_2 - B_2C_1}{A_1B_2 - A_2B_1}, \frac{C_1A_2 - C_2A_1}{A_1B_2 - A_2B_1}\right)$, provided

$A_1B_2 - A_2B_1 \neq 0$ *i.e.*, $\frac{A_1}{A_2} \neq \frac{B_1}{B_2}$.

In case $\frac{A_1}{A_2} = \frac{B_1}{B_2}$, two possibilities arises :

If $\frac{A_1}{A_2} = \frac{B_1}{B_2} \neq \frac{C_1}{C_2}$, then the lines are *coincident* and there are *infinitely many points* which are on both lines.

Example 21. *Find which of the following pairs of lines are intersecting, parallel or coincident :*

(i) $2x - y + 7 = 0$ *and* $2x + y - 9 = 0$ (ii) $x + 6y + 11 = 0$ *and* $2x + 12y = -22$.

(iii) $3x - y + 6 = 0$ *and* $2y - 6x + 11 = 0$

Sol. We know that the lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ are :

(a) **intersecting** if $\frac{A_1}{A_2} \neq \frac{B_1}{B_2}$ (c) **parallel** if $\frac{A_1}{A_2} = \frac{B_1}{B_2} \neq \frac{C_1}{C_2}$

(b) **coincident** if $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$.

(i) Given lines are $2x - y + 7 = 0$ and $2x + y - 9 = 0$.

Here $\frac{A_1}{A_2} = \frac{2}{2} = 1$ and $\frac{B_1}{B_2} = \frac{-1}{1} = -1$.

$\therefore \quad \frac{A_1}{A_2} \neq \frac{B_1}{B_2}$, so the lines are **intersecting.**

(ii) Given lines are $x + 6y + 11 = 0$ and $2x + 12y + 22 = 0$.

Here $\quad \frac{A_1}{A_2} = \frac{1}{2}, \ \frac{B_1}{B_2} = \frac{6}{12} = \frac{1}{2}$ and $\frac{C_1}{C_2} = \frac{11}{22} = \frac{1}{2}$.

$\therefore \quad \frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$, so the lines are **coincident.**

(iii) Given lines are $3x - y + 6 = 0$ and $-6x + 2y + 11 = 0$.

Here $\quad \frac{A_1}{A_2} = \frac{3}{-6} = \frac{1}{-2}, \ \frac{B_1}{B_2} = \frac{-1}{2} = -\frac{1}{2}$ and $\frac{C_1}{C_2} = \frac{6}{11}$.

$\therefore \quad \frac{A_1}{A_2} = \frac{B_1}{B_2} \neq \frac{C_1}{C_2}$, so the lines are **parallel.**

CONDITION FOR CONCURRENCY OF THREE LINES

Three lines are said to be **concurrent** if all the three lines passes through a common point. The common point of three lines is called the **point of concurrence** of the lines.

Let $\quad A_1x + B_1y + C_1 = 0 \quad \ldots(1)$

$A_2x + B_2y + C_2 = 0 \quad \ldots(2)$

and $\quad A_3x + B_3y + C_3 = 0 \quad \ldots(3)$

be any three lines.

Solving (1) and (2), we get $\frac{x}{B_1C_2 - B_2C_1} = \frac{y}{C_1A_2 - C_2A_1} = \frac{1}{A_1B_2 - A_2B_1}$.

$\therefore \quad x = \frac{B_1C_2 - B_2C_1}{A_1B_2 - A_2B_1}$ and $y = \frac{C_1A_2 - C_2A_1}{A_1B_2 - A_2B_1}$, assuming $A_1B_2 - A_2B_1 \neq 0$.

$\therefore$ (1) and (2) intersect at the point $\left(\frac{B_1C_2 - B_2C_1}{A_1B_2 - A_2B_1}, \frac{C_1A_2 - C_2A_1}{A_1B_2 - A_2B_1}\right)$.

The given lines are concurrent if the point of intersection of (1) and (2) lies on (3),

i.e., if
$$A_3\left(\frac{B_1C_2 - B_2C_1}{A_1B_2 - A_2B_1}\right) + B_3\left(\frac{C_1A_2 - C_2A_1}{A_1B_2 - A_2B_1}\right) + C_3 = 0$$

or if
$$\mathbf{A_3(B_1C_2 - B_2C_1) + B_3(C_1A_2 - C_2A_1) + C_3(A_1B_2 - A_2B_1) = 0.}$$

This is the required condition.

Remark. The three lines are concurrent, them the lines must be mutually non-parallel.

WORKING RULES FOR SOLVING PROBLEMS

Step I. *Find the point of intersection of any two lines.*

Step II. *Check wheather this point lie on the third line or not.*

Step III. *If this point lie on the third line then the lines are concurrent and the point obtained in Step I, is the point of concurrence of the given lines.*

Example 22. *For what value of k, are the three lines:*

$x - 2y + 1 = 0$, $2x - 5y + 3 = 0$ and $5x - 9y + k = 0$ are concurrent ?

Sol. Given lines are $x - 2y + 1 = 0$ ...(1)

$2x - 5y + 3 = 0$ ...(2)

and $5x - 9y + k = 0$ ...(3)

Solving (1) and (2), we get $\dfrac{x}{-6+5} = \dfrac{y}{2-3} = \dfrac{1}{-5+4}$.

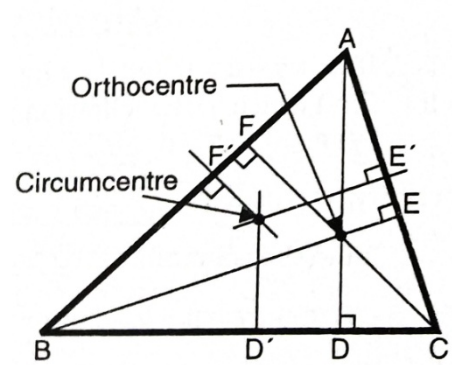
$\therefore$ $x = \dfrac{-1}{-1} = 1$ and $y = \dfrac{-1}{-1} = 1$.

$\therefore$ The lines (1) and (2) intersects at the point (1, 1).

Let the given lines be concurrent. $\therefore$ (1, 1) must lie on the line (3).

$\Rightarrow$ $5(1) - 9(1) + k = 0$ $\Rightarrow$ $k = 9 - 5 = 4$.

COORDINATES OF ORTHOCENTRE AND CIRCUMCENTRE OF A TRIANGLE

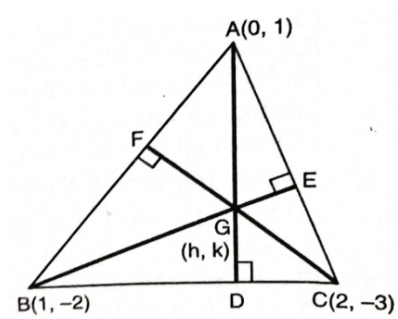


The **orthocenter** of a triangle is the point of concurrence of the altitudes drawn from the vertices to the opposite sides of the triangle.

The **circumcentre** of a triangle is the point of concurrence of the right bisectors of the sides of the triangle.

Remark. We have already proved that the altitudes of a triangle are always concurrent, so the definition of orthocenter is justified. Similar argument also work for the circumcentre.

Example 23. *Find the coordinates of the orthocenter of the triangle whose vertices are (0, 1), (1, -2) and (2, -3).*



Sol. Let the vertices of the triangle be *A*(0, 1), *B*(1, -2) and *C*(2, -3). The orthocenter of the triangle is the point of concurrence of the altitude from the vertices.

Let *AD, BE* and *CF* be the altitudes, and *G*(*h, k*) be the orthocenter of the triangle.

$AG \perp BC \quad \Rightarrow \quad$ slope of *AG* X slope of *BC* = - 1.

$$\Rightarrow \quad \frac{k-1}{h-0} \times \frac{-3+2}{2-1} = -1$$

$$\Rightarrow \quad -k+1 = -h \quad \Rightarrow \quad h-k+1 = 0 \qquad \text{.....(1)}$$

Also $BG \perp AC \quad \Rightarrow \quad \frac{k+2}{h-1} \times \frac{-3-1}{2-0} = -1$

$$\Rightarrow \quad -4k-8 = -2h+2 \quad \Rightarrow \quad h-2k-5 = 0 \qquad \text{...(2)}$$

$$(1)-(2) \quad \Rightarrow \quad k+6 = 0 \quad \Rightarrow \quad k = -6$$

$$\therefore \quad (1) \quad \Rightarrow \quad k-(-6)+1=0 \quad \Rightarrow \quad h = -7$$

$\therefore$ The orthocenter is **(-7, -6).**

WORKING RULES FOR SOLVING PROBLEMS

Rule I. *The lines* $A_1x + B_1y + C_1 = 0$ *and* $A_2x + B_2y + C_2 = 0$ *are :*

(i) *intersecting if* $\frac{A_1}{A_2} \neq \frac{B_1}{B_2}$ (ii) *parallel if* $\frac{A_1}{A_2} = \frac{B_1}{B_2} \neq \frac{C_1}{C_2}$

(iii) *coincident if* $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$

Rule II. *If the point of intersection of two lines also lie on the third line, then the lines are concurrent.*

Rule III. *The altitudes of a triangle are concurrent and their point of concurrence is called the* **orthocenter.**

Rule IV. *The right bisectors of a triangle are concurrent and their point of concurrence is called the* **circumcentre.**

EXERCISE 19.7

SHORT ANSWER TYPE QUESTIONS

1. If $2x + y + a = 0$ and $4x + by + 3 = 0$ represent the same line, find the values of *a* and *b*.
2. Find which of the following pairs of lines are intersecting, parallel, coincident :

   (i) $6x - y + 7 = 0$ and $y - 6x = 8$ (ii) $4x + y + 9 = 0$ and $8x + 2y = -18$

   (iii) $x - y = 0$ and $x + y = 0$ (iv) $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$, $a \neq b$.
3. Find the point intersection of the lines :

   (i) $3x + 2y - 9 = 0$ and $x - y + 2 = 0$ (ii) $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$.

LONG ANSWER TYPE QUESTIONS

4. Find the foot of perpendicular from the point (-1, 2) on the straight line $x - y + 5 = 0$.

5. Obtain the coordinates of the foot of perpendiculars drawn from the origin upon the lines $3x-5y+2=0$ and $4x-3y+5=0$ show that the equation of the straight line joining these feet is $26x+53y=11$.

6. The line $2x-3y-4=0$ is the perpendicular bisector of the line *AB* and the coordinates of *A* are (-3, 1). Find the coordinates of *B*.

7. (i) Find the area of the triangle formed by the lines $y-x=0, x+y=0$ and $x = k$.

   (ii) Find the area of the triangle formed by the lines $x+y-6=0, x-3y-2=0$ and $5x-3y+2=0$.

8. Show that the diagonals of the parallelogram formed by the four lines $3x+y=0$, $3y+x=0, 3x+y=4$ and $3x+y=4$ are perpendicular.

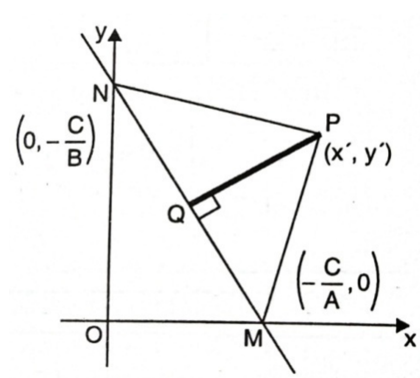
Answers

1. $a = 3/2$, $b = 2$	2. (i) Parallel	(ii) Coincident
(iii) Intersecting	(iv) Intersecting	3. (i) (1, 3)
(ii) $\left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$	4. (-2, 3)	5. (-3/17, 5/17), (-4/5,3/5)
6. (1, -5)	7. (i) k^2 sq. units	(ii) 12 sq. units.

DISTANCE OF A POINT FROM A LINE

The perpendicular distance of the point P(x', y') from the line Ax + By + C = 0 is equal to $\left|\frac{Ax'+By'+C}{\sqrt{A^2+B^2}}\right|$.

Proof. Given line is $Ax + By + C = 0$.



$$\Rightarrow \quad \frac{x}{(-C/A)} + \frac{y}{(-C/B)} = 1$$

Let the given line intersects the axes at the point *M* and *N*.

$\therefore$ The coordinates of *M* and *N* are respectively (-

C/A, 0) and (0, -C/B).

Let us assume that P is not on the given line.

$\therefore$ Area of $\Delta MNP = \frac{1}{2}\left|x'\left(0+\frac{C}{B}\right)+\left(-\frac{C}{A}\right)\left(-\frac{C}{B}=y'\right)+0.(y'-0)\right|$

$$= \frac{1}{2}\left|\frac{Cx'}{B}+\frac{C^2}{AB}+\frac{Cy'}{A}\right|$$

$$= \frac{1}{2}\left|\frac{C}{AB}\left(Ax'+By'+C\right)\right| = \frac{1}{2}\left|\frac{C}{AB}\right| \quad |Ax'+By'+C|.$$

Also, area of $\Delta MNP = \frac{1}{2}MN \times PQ = \frac{1}{2}\sqrt{\left(0+\frac{C}{A}\right)^2+\left(-\frac{C}{B}-0\right)^2} \times PQ$

$$= \frac{1}{2}\sqrt{\frac{C^2\left(A^2+B^2\right)}{A^2B^2}} \times PQ = \frac{1}{2}\left|\frac{C}{AB}\right|\sqrt{A^2+B^2} \times PQ\ .$$

Equating the values of area of ΔMNP, we get

$$\frac{1}{2}\left|\frac{C}{AB}\right|\sqrt{A^2+B^2} \times PQ = \frac{1}{2}\left|\frac{C}{AB}\right| \quad |Ax'+By'+C|$$

$\Rightarrow$ $$PQ = \frac{|Ax'+By'+C|}{\sqrt{A^2+B^2}} = \left|\frac{Ax'+By'+C}{\sqrt{A^2+B^2}}\right|$$

$\therefore$ The distance of $P(x', y')$ from the line $Ax + By + C = 0$ is equal to $\left|\frac{Ax'+By'+C}{\sqrt{A^2+B^2}}\right|$.

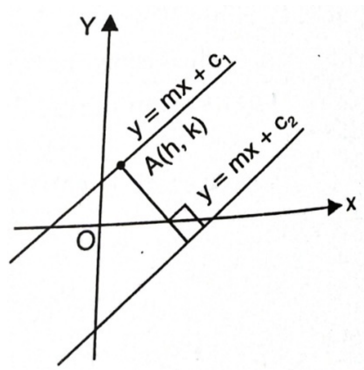
If $P(x', y')$ is on the $Ax + By + C = 0$, then $Ax' + By' + C = 0$ and its distance from the line is '0'.

Also, $$\left|\frac{Ax'+By'+C}{\sqrt{A^2+B^2}}\right| = \left|\frac{0}{\sqrt{A^2+B^2}}=0\right|.$$

$\therefore$ **The distance of P(x', y') from the line Ax + By + C = 0 is equal to** $\left|\frac{Ax'+By'+C}{\sqrt{A^2+B^2}}\right|$.

Remark. The length of perpendicular from the origin to the line $Ax + By + C = 0$ is $\left|\frac{A.0+B.0+C}{\sqrt{A^2+B^2}}\right|$ *i.e.,* $\left|\frac{C}{\sqrt{A^2+B^2}}\right|$.

DISTANCE BETWEEN PARALLEL LINES



Let $y = mx + c_1$ ...(1)

and $y = mx + c_2$ ....(2)

be two parallel lines.

(1) $\Rightarrow$ $mx - y + c_1 = 0$ ....(3)

(2) $\Rightarrow$ $mx - y + c_2 = 0$ ....(4)

Let A(h, k) be any point on the line (3).

$\therefore$ $mh - k + c_1 = 0$ ...(5)

Distance between lines (1) and (2)

= perpendicular distance of $A(h, k)$ from line (4)

$$= \left|\frac{mh-k+c_2}{\sqrt{m^2+(-1)^2}}\right| = \left|\frac{-c_1+-c_2}{\sqrt{m^2+1}}\right| = \left|\frac{c_1-c_2}{\sqrt{m^2+1}}\right|.$$

Similarly, we can show that the distance between parallel lines $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ is equal to $\left|\frac{C_1-C_2}{\sqrt{A^2+B^2}}\right|$.

Example 24. *Find the length of perpendicular from (4, 2) to the line $5x - 12y - 9 = 0$.*

Sol. The line is $5x - 12y - 9 = 0$.

Length of $\perp$ from (4, 2) = $\left|\frac{5(4)-12(2)-9}{\sqrt{(5)^2+(-12)^2}}\right| = \left|\frac{20-24-9}{13}\right|$

$$= \left|\frac{-13}{13}\right| = |-1| = \mathbf{1.}$$

Example 25. *Find the distance between the lines*

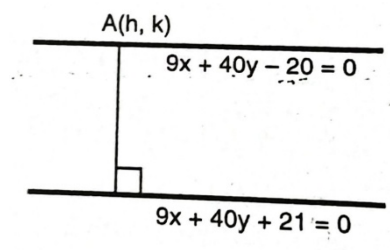
$$9x + 40y - 20 = 0 \text{ and } 9x + 40y + 21 = 0.$$

Sol. The lines are $9x + 40y - 20 = 0$ ....(1)

and $9x + 40y + 21 = 0$ ...(2)

These lines are parallel because their slopes are same.

Distance between given lines is equal to perpendicular distance from any point on (1) to the line (2) and *vice – versa.*



Let $A(h, k)$ be any point on (1).

$\therefore \quad 9h + 40k - 20 = 0$ ....(3)

Distance between lines

= length of $\perp$ from $A(h, k)$ to line (2)

$$= \left|\frac{(9h+40k)+21}{\sqrt{(9)^2+(40)^2}}\right| = \left|\frac{20+21}{\sqrt{81+1600}}\right| = \left|\frac{41}{41}\right| = |1| = \mathbf{1}. \quad \text{[Using (3)]}$$

WORKING RULES FOR SOLVING PROBLEMS

Rule I. *The distance of (x', y') from the line Ax + By + C = 0 is equal to*

$$\left|\frac{Ax'+By'+C}{\sqrt{A^2+B^2}}\right|$$

Rule II. *To find the distance between two parallel line, take any arbitrary point on one line and find its distance from the second line. This gives the required distance.*

SHORT ANSWER TYPE QUESTIONS

1. Find the distance of the point from the line in the following cases :

(i) $(-2, -1)$; $4x + 3y - 5\ 0$ (ii) (a, b); $\frac{x}{a}+\frac{y}{b}=1$, $a > 0$, $b > 0$.

2. If p be the length of the perpendicular from the origin to the line whose intercepts on the axes are a and b, then show that $\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}$.

3. Find the distance between the parallel lines $5x-12y+2=0$ and $5x-12y-7=0$

4. Show that the origin is equidistant from the lines $4x+3y+10=0$, $5x-12y+26=0$ and $7x+24y-50=0$.

LONG ANSWER TYPE QUESTIONS

5. Which of the lines $x + 6y - 9 = 0$ and $2x - 5y + 8 = 0$ is farther from the point (1, 5)?

6. If $5x - 12y - 65 = 0$ and $5x - 12y + 26 = 0$ are the equations of a pair of opposite sides of a square, find the area of the square.

7. Find the length of the perpendicular from the origin to the line joining two points whose coordinates are ($\cos\theta$, $\sin\theta$) and ($\cos\phi$, $\sin\phi$).

8. Find the length of the altitude of the triangle with vertices (-2, 3), (2, 1) and (-10, -13).

9. (i) What are the points on the axis of x whose perpendicular distance from the line $\frac{x}{3}+\frac{y}{4}=1$ is 4?

10. (ii) What are the points on the axis of y whose perpendicular distance from the line $\frac{x}{3}+\frac{y}{4}=1$ is 4?

11. Show that the product of distance of the line $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$ from the points $\left(*\sqrt{a^2-b^2},0\right)$ is b^2.

Answers

1. (i) $\frac{16}{5}$ (ii) $\frac{ab}{\sqrt{a^2+b^2}}$ 3. $\frac{9}{13}$ 5. $X + 6y - 9 = 0$

6. 49 sq. units 7. $\cos\frac{\theta-\phi}{2}$ 8. $\frac{8\sqrt{85}}{17},2\sqrt{5},8\sqrt{5}$

9. (i) (8, 0), (-2, 0) (ii) (0, 32/3), (0, -8/3).

FAMILY OF LINES

We have already observed that two independent conditions are necessary and sufficient to identify a straight line in a plane. Just one condition is not sufficient to identify a line.

For example, if we say that a particular line is parallel to x-axis, then the line cannot be identified, because there are infinitely many lines which are parallel to x-axis. Similarly, if it is known that a particular line passes through a point, say, (a, b), then also the line cannot be identified, because there are infinitely many lines passing through (a, b).

A set of lines satisfying a given condition is called a **family of lines.** A family of liens can be represented by a linear equation in x and y and involving one arbitrary constant, which is called the **parameter** of the family of lines, under consideration.

For example, $3x + ky + 9 = 0$, $y = mx + 9$, $x = k$ etc., represent family of lines

EQUATION OF FAMILY OF LINES PASSING THROUGH THE POINT OF INTERSECTION OF TWO LINES

Let $a_1a + b_1y + c_1 = 0$ ....(1) and $a_2a + b_2y + c_2 = 0$ ...(2)

be two intersecting lines. Let $P(x', y')$ be their point of intersection.

Consider the equation $a_1x + b_1y + c_1 + k(a_2x + b_2y + c_2) = 0$ . ...(3)

where k is any constant.

Now $a_1x' + b_1y' + c_1 + k(a_2x' + b_2y' + c_2) = +k.0 = 0.$

[$\because$ $P(x', y')$ lies on the lines (1) and (2)]

$\therefore$ $P(x', y')$ lies on the locus of equation (3).

The equation (3) can also be expressed as $(a_1 + ka_2)x + (b_1 + kb_2)y + (c_1 + kc_2) = 0.$
This equation is linear in x and y.

$\therefore$ Equation (3) represents a straight line for all values of k.

$\therefore$ For any k, (3) is a straight line passing through the point of intersection of the given lines.

$\therefore$ **The equation $a_1 x + b_1 y + c_1 + k(a_2 x + b_2 y + c_2) = 0$ is a family of straight lines passing through the point of intersection of the lines $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$.**

Example 26. *Find the equation of the line passes through the point of intersection of the lines 4x + 7y – 3 = 0 and 2x – 3y + 1 = 0, that has equal intercepts on the axes.*

Sol. Given lines are $4x + 7y - 3 = 0$ and $2x - 3y + 1 = 0$.

Let the equation of the required line be

$$4x + 7y - 3 + \lambda(2x - 3y + 1) = 0 \qquad \text{....(1)}$$

$$\Rightarrow \quad (4 + 2\lambda)x + (7 - 3\lambda)y = 3 - \lambda$$

$$\Rightarrow \quad \frac{x}{\frac{3-\lambda}{4+2\lambda}} + \frac{x}{\frac{3-\lambda}{7-3\lambda}} = 1$$

$\therefore$ Intercepts on axes are $\frac{3-\lambda}{4+2\lambda}$ and $\frac{3-\lambda}{7-3\lambda}$.

This line has equal intercepts. $\therefore \frac{3-\lambda}{4+2\lambda} = \frac{3-\lambda}{7-3\lambda}$

$$\Rightarrow \quad (3-\lambda)\left[\frac{1}{4+2\lambda} - \frac{1}{7-3\lambda}\right] = 0 \quad \Rightarrow \quad (3-\lambda)\left[\frac{7-3\lambda-4-2\lambda}{(4+2\lambda)(7-3\lambda)}\right] = 0$$

$$\Rightarrow \quad (3-\lambda)(3-5\lambda) = 0 \quad \Rightarrow \quad \lambda = 3, 3/5$$

Case I. $\lambda = 3$

(1) $\Rightarrow \quad 4x + 7y - 3 + 3(2x - 3y + 1) = 0 \quad \Rightarrow \quad 5x - y = 0$.

Case II. $\lambda = 3/5$

(1) $\Rightarrow \quad 4x + 7y - 3 + \frac{3}{5}(2x - 3y + 1) = 0$

$\Rightarrow \quad 20x+35y-15+6x-9y+3=0 \quad \Rightarrow \quad 13x+13y-6=0$

$\therefore$ The required lines are **5x – y = 0** and **13x + 13y – 6 = 0.**

EXERCISE 19.9

SHORT ANSWER TYPE QUESTIONS

1. Find the equation of the line passing through the point (-4, 5) and the point of intersection of the lines $4x-3+7=0$ and $2x+3y+5=0$.
2. Find the equation of the lines which passes through the point of intersection of the lines $x+2y-3=0$ and $4x-y+7=0$ is parallel to the line y – x + 10 = 0.
3. Find the equation of the straight line which passes through the point intersection of the lines $3x-4y+1=0$ and $5x+y-1=0$ cuts off equal intercepts on the axes.
4. Find the equation of the line passing through the point of intersection of the lines $2x-5y+3=0$ and $x-3y-7=0$ perpendicular to the line whose equation is $4x+y-1=0$.
5. Find the equation of the line passing through the point of intersection of the lines $2x-3y+1=0$ and $x+y-2=0$ is parallel to y-axis.
6. Find the equation of the line passing through the point of intersection of the line $x-3y+1=0$ and $2x+5y-9=0$ whose distance from the origin is $\sqrt{5}$.
7. Find the equation of the straight line passing through the point of intersection of $5x-3y=1$ and $2x+3y=23$ perpendicular to the line x = 0.
8. Find the equation of the straight line drawn through the point of intersection of the lines x+ y = 4 and 2x – 3y = 1 and perpendicular to the line cutting off intercepts 5, 6 on the axes.
9. Find the equation of the line parallel to y-axis and drawn through the point of intersection of the lines x- 7y + 5 = 0 and 3x + y = 0.
10. Find the equation of a line drawn perpendicular to the line $\frac{x}{4}+\frac{y}{6}=1$ through the point, where it meets the y-axis.

Answers

1. $8x + 3y + 17 = 0$ 2. $3x - 3y + 10 = 0$ 3. $23x + 23y = 11$

4. $x - 4y - 24 = 0$ 5. $x - 1 = 0$ 6. $2x + y - 5 = 0$ 7. $21y - 113 = 0$

8. $25x - 30y - 23 = 0$ 9. $22x + 5 = 0$ 10. $2x - 3y + 18 = 0$.

SUMMARY

1. (i) The equation of a straight line parallel to x- axis and at a distance h from it is given by $y = h$.

   (ii) The equation of the straight line parallel to y-axis and at a distance k from it is given $x = k$.

   (iii) The equation of the straight line having slope m and intercept on y- axis as c is given by $y = mx + c$. **(Slope-intercept form)**

   (iv) The equation of the straight line having intercept a and b on x-axis and y-axis respectively is given by $\frac{x}{a}+\frac{y}{b}=1$. **(Intercept form)**

   (v) The equation of the straight line passing through (x_1, y_1) and having slope m is given by $y - y_1 = m(x - x_1)$. **(Point- slope form)**

   (vi) The equation of the straight passing through the points (x_1, y_1) and (x_2, y_2) is given by $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$. **(Two – point form)**

   Here we assume that $x_1 \neq x_2$.

   In case $x_1 \neq x_2$, then the line is vertical and its equation is $x = x_1$ (or x_2).

   (vii) The equation of the straight line passing through (x_1, y_1) and making angle θ with the positive direction of x – axis is given by $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$,

   (Distance form)

   were r is the distance between the point (x, y) and (x_1, y_1).

   (viii) The equation of a straight line for which the perpendicular from the origin makes an angle α and is of length p, is given by $x\cos\alpha + y\sin\alpha = p$.

   (Normal form)

2. Three lines are said to be **concurrent** if all the three lines passes through a point. The common point of concurrent lines is called the **point of concurrence.**

3. (i) The **orthocenter** of a triangle is the point of concurrence of altitudes draw from the vertices to the opposite sides of the triangle.

   (ii) The **circumcentre** of a triangle is the point of concurrence of right bisectors of the straight line $ax + by + c = 0$ is equal to

4. The **length of perpendicular** of the point (x_1, y_1) from the straight line $ax+by+c=0$ is equal to

$$\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|.$$

5. A set of lines satisfying a given condition is called a **family of lines.** A family of lines can be represented by linear equation in x and y involving one arbitrary constant, which is called the **parameter** of the family of lines under consideration.

TEST YOURSELF

1. The mid-points of the sides of a triangle are (2, 1) (-5, 7), (-5 7), (-5, -5). Find the equations of the sides.
2. Find the equation of the straight line whose inclination is 135^0 and is at a distance of 2 units from the origin. Also sketch the line.
3. Find the equation of the straight line whose x-intercept is 12 and is at a distance of 12 units from the origin. Also sketch the line.
4. Find the equation of the straight line whose nearest point to the origin is (-3, -2).
5. Find the acute angle between the lines $y-\sqrt{3}x-6=0$ and $\sqrt{3}y-x+1=0$.
6. Find the orthocenter of the triangle whose vertices are (-1, -1), (2, 4) and (5, 1).
7. Find the circumcentre of the triangle whose vertices are (-2, 2), (2, -1) and (4, 0).
8. Find the point on the line x + y = 4 which is at a unit distance from the line $4x+3y-10=0$4.

Answers

1. $6x-7y+79=0$, $6x+7+65=0$, x- 2 = 0.

2. $x\cos 45^0+y\sin 45^0=2$ *i.e.,* $x+y-2\sqrt{2}=0$, $x\cos 225^0+y\sin 225^0=2$ *i.e.,* $x+y+2\sqrt{2}=0$

3. $x\left(\frac{12}{13}\right)+y\left(\frac{5}{13}\right)=12$ *i.e.,* $12x+5y-156=0,\ x\left(\frac{12}{13}\right)+y\left(-\frac{5}{13}\right)=12$

i.e., 12*x*- 5y – 156 = 0.

4. $x\left(-\frac{3}{\sqrt{13}}\right)+y\left(-\frac{2}{\sqrt{13}}\right)=\sqrt{13}$ *i.e.,* $3x+2y+13=0$.

6. $\left(\frac{5}{2},\frac{5}{2}\right)$ 7. $\left(\frac{3}{2},\frac{5}{2}\right)$ 8. (-7, 11), (3, 1).

SECTION – C

20. CIRCLES

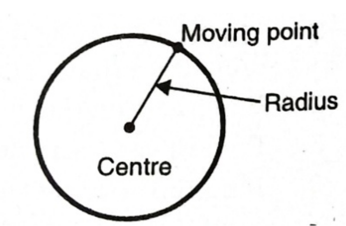
LEARNING OBJECTIVES

- Definition of a circle
- Standard form of the Equation of a Circle.
- General form of the Equation of a Circle, its Radius and Centre.
- Equation of a Circle when the Coordinates of End Points of a Diameter are Given.

DEFINITION OF A CIRCLE

A **circle** is the locus of a point which moves so that its distance from a fixed point is constant.

The fixed point is called the **centre** of the circle and the constant distance is called the **radius** of the circle.



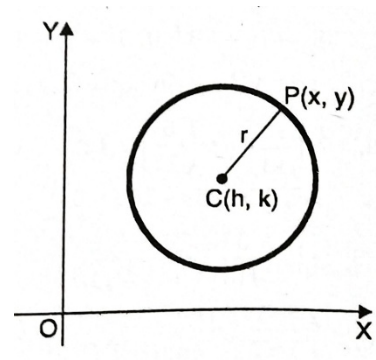
STANDARD FORM OF THE EQUATION OF A CIRCLE

Let $C(h, k)$ and r be the centre and radius of a circle respectively.

Let $P(x, y)$ be a general point on the circle.

$\therefore$ By definition, $PC = r$.

$\therefore$ $$\sqrt{(x-h)^2+(y-k)^2} = r$$



$\Rightarrow$ $\quad$ $\mathbf{(x - h)^2 + (y - k)^2 = r^2}$ $\quad$ …(1)

This is the required equation of the given circle and is called the **standard form** of the equation of a circle.

If in the equation $(x-h)^2+(y-k)^2=r^2$,

(i) $r^2 > 0$, then there do exists points which satisfies the equation. In this case, the circle is called a **real circle.**

(ii) $r^2 = 0$, then there exists just one point, namely $C(h, k)$ which can satisfy the equation. In this case, the circle is called a **point circle.**

(iii) $r^2 < 0$, then there does not exist any point which may satisfy the equation. In this case, the circle is called an **imaginary circle.**

Remark 1. If centre of a circle is at the origin, then the equation of circle is in the form $(x-0)^2+(y-0)^2=r^2$ or $\mathbf{x^2 + y^2 = r^2}$. (r is the radius of circle)

Remark 2. The equation of the circle with centre (h, k) and radius 'r' is

$$(x-h)^2+(y-k)^2=r^2.$$

This can be written as $(x^2-2hx+h^2)+(y^2-2ky+k^2)=r^2$.

$\Rightarrow$ $\quad$ $x^2+y^2-2hx-2ky+(h^2+k^2-r^2)=0$

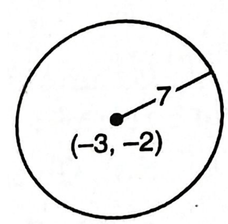
$\Rightarrow$ $\quad$ $x^2+y^2+2gr+2gy+c=0$, where $g = -h$, $f = -k$ and $c = h^2 + k^2 - r^2$.

$\therefore$ The equation of any circle can be expressed in the form

$$\mathbf{x^2 + y^2 + 2gx + 2fy + c = 0.}$$

Example 1. *Find the equation of the circle whose centre is at (-3, -2) and radius equal to 7.*

Sol. Here $(h, k) = (-3, -2)$ and $r = 7$.



Using $(x - h)^2 + (y - k)^2 = r^2$, the equation of the circle is $(x+3)^2+(y+2)^2=(7)^2$.

$\Rightarrow$ $\quad$ $\mathbf{x^2 + y^2 + 6x + 4y - 36 = 0.}$

GENERAL FORM OF THE EQUATION OF A CIRCLE, ITS RADIUS AND CENTRE

Let us consider the equation $x^2 + y^2 + 2gx + 2fy + c = 0$ ...(1)

This implies $\left(x^2 + 2gx\right) + \left(y^2 + 2fy\right) = -c$.

$\Rightarrow \quad \left(x^2 + 2gx + g^2\right) + (y^2 + 2fy + f^2) = g^2 + f^2 - c$

$\Rightarrow \quad (x+g)^2 + (y+f)^2 = g^2 + f^2 - c$

$\Rightarrow \quad (x - (-g))^2 + (y - (-f))^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$ ....(2)

Equation (2) represents a circle in the standard form whose centre is at (-g, -*f*) and radius equal to $\sqrt{g^2 + f^2 - c}$.

$g^2 + f^2 - c > 0 \quad \Rightarrow \quad$ radius = $\sqrt{g^2 + f^2 - c} > 0$

$\therefore$ Equation (2) (*i.e.,* (1)) represents a *real circle*.

$g^2 + f^2 - c = 0 \quad \Rightarrow \quad$ radius = $\sqrt{0} = 0$

$\therefore$ Equation (1) represents a *point circle*.

$g^2 + f^2 - c < 0 \quad \Rightarrow \quad$ radius is imaginary

$\therefore$ Equation (1) represents an *imaginary circle* with real centre and imaginary radius.

Thus, we see that the equation (1) represents a circle. This is called the **general form** of the equation of a circle.

$\therefore$ **The general second degree equation ax² + 2hxy + by² + 2gx + 2fy+c = 0 represents a circle if** (i) **a = b and** (ii) **h = 0.**

Remark. The general equation of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ contains three constants *g, f* and *c*. Thus is order to fix the position of a circle, three independent conditions are required.

Example 2. *Find the centre and radius of the circle* $x^2 + y^2 - 4x + 6y = 5$.

Sol. The given equation is $x^2 + y^2 - 4x + 6y = 5$ ....(1)

$\Rightarrow \quad (x^2 - 4x) + (y^2 + 6y) = 5 \Rightarrow (x^2 - 4x + 4) + (y^2 + 6y + 9) = 4 + 9 + 5$

$\Rightarrow \quad (x-2)^2 + (y+3)^2 = 18 \Rightarrow (x-2)^2 + (y-(-3)^2 = (3\sqrt{2})^2$ ....(2)

Equation (2) represent a circle, in the standard form, whose centre is at (2, -3) and radius equal to $3\sqrt{2}$.

Alternative method. The given equation is $x^2 + y^2 - 4x + 6y = 5$ ....(1)

$\Rightarrow \quad x^2 + y^2 - 4x + 6y - 5 = 0$.

Comparing this equation with equation of circle in general form

$x^2 + y^2 + 2gx + 2fy + c = 0$, we get g = -2, *f* = 3, *c* = -5.

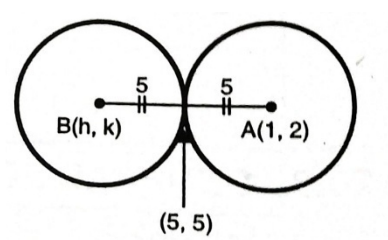
$\therefore$ Centre = (-g, -*f*) = (-(-2), -3) = **(2, -3)**

Radius = $\sqrt{g^2 + f^2 + c} = \sqrt{(-2)^2 + (3)^2 - (-5)}$

$= \sqrt{4 + 9 + 5} = \sqrt{18} = 3\sqrt{2}$.

Example 3. *Find the equation of the circle whose radius is 5 and which touches the circle* $x^2 + y^2 - 2x - 4y - 20 = 0$ *externally at the point (5, 5).*

Sol. Given circle is $x^2 + y^2 - 2x - 4y - 20 = 0$.



Here g = - 1, *f* = - 2, *c* = - 20

$\therefore$ Center = (-g, -*f*) = (1, 2)

and radius = $\sqrt{g^2 + f^2 - c} = \sqrt{1 + 4 + 20} = 5$

Let *B(h, k)* be the centre of the required circle.

$\therefore$ (5, 5) is the mid-point of *BA*.

$\therefore \quad 5 = \frac{h+1}{2}$ and $5 = \frac{h+2}{2}$

$\therefore$ Equation of required circle is

$$(x-9)^2+(y-8)^2=(5)^2.$$

$\Rightarrow$ **$x^2 + y^2 - 18x - 16y + 120 = 0$.**

EXERCISE 20.1

SHORT ANSWER TYPE QUESTIONS

1. Find the equation of the circle having :

   (i) centre at (1/2, 1/4) and radius 1/12

   (ii) centre at (-*a*, -*b*) and radius $\sqrt{a^2-b^2}$ (iii) centre at (*a*, *a*) and radius $\sqrt{2}a$

   (iv) centre at $(a\cos\theta, a\sin\theta)$ and radius *a*.

2. Find the centre and radius of the circle given by :

   (i) $x^2+y^2-6x+12y-75=0$ (ii) $2x^2+2y^2-x=0$

   (iii) $4x^2+4y^2+12ax-6ay-a^2=0$ (iv) $x^2+y^2-2ax\cos\theta-2ay\sin\theta=0$.

3. Show that $Ax^2+Ay^2+Dx+Ey+F=0$ represents a circle. Find its centre and radius.

4. Determine whether the following equations represents a circle, a point circle or no circle :

   (i) $x^2+y^2+x-y=0$ (ii) $x^2+y^2-3x+3y+10=0$

   (iii) $x^2+y^2+2x+10y+26=0$.

5. Find the equation of the circle whose centre is (*h*, *k*) and which passes through the point (*p*, *q*).

6. Find the equation of the circle whose centre is at (4, 5) and which passes through the centre of the circle $x^2+y^2+4x-6y-12=0$.

7. Find the equation of the circle whose centre is (2, 3) and which passes through the point of intersection of the lines $3x-2y-1=0$ and $x+y-27=0$.

8. Show that the radii of the circle $x^2, y^2=1$, $x^2+y^2-2x-6y-6=0$ and $x^2+y^2-4x-12y-9=0$ are in A.P.

LONG ANSWER TYPE QUESTIONS

9. Find the equation of the circle passing through the points :

   (i) (1, 0), (-1, 0) and (0, 1) (ii) (1, -2), (5, 4) and (10, 5).

10. Find the equation of the circle which passes through the origin (0, 0) and cuts off chords of length 4 and 6 on the positive sides of the x-axis and y-axis respectively.

11. Find the equation of the circle which passes through the origin and cuts off intercepts 3 and 4 from the positive parts of x-axis and y-axis respectively.

12. Find the equation of the circle which passes through the origin and the points where the line $3x + 4y = 12$ meets the coordinates axes.

ANSWERS

1. (i) $36x^2 + 36y^2 - 36x - 18y + 11 = 0$ (ii) $x^2 + y^2 + 2ax + 2by + 2b^2 = 0$

   (iii) $x^2 + y^2 - 2ax - 2ay = 0$ (iv) $x^2 + y^2 - 2ax\cos\theta - 2ay\sin\theta = 0$

2. (i) (3, 6) ; $2\sqrt{30}$ (ii) (1/4,0). 1/4 (iii) (-3a/2. 3a/4); 7a/4

   (iv) $(a\cos\theta, a\sin\theta); a$

3. $-\frac{D}{2A}, -\frac{E}{2A}; \frac{\sqrt{D^2 + E^2 - 4AF}}{2A}$

4. (i) Real (ii) Imaginary (iii) Point circle

5. $x^2 + y^2 - 2hx - 2ky - p^2 - q^2 + 2ph + 2qk = 0$

6. $x^2 + y^2 - 8x - 10y + 1 = 0$

7. $x^2 + y^2 - 4x - 6y - 237 = 0$

9. (i) $x^2 + y^2 = 1$

   (ii) $x^2 + y^2 - 18x + 6y + 25 = 0$

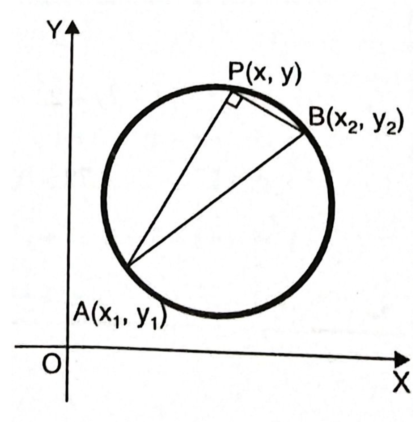
10. $x^2 + y^2 - 4x - 6y = 0$

11. $x^2 + y^2 - 3x - 4y = 0$

12. $x^2 + y^2 - 4x - 3y = 0$.

EQUATION OF A CIRCLE WHEN THE COORDINATES OF END POINTS OF A DIAMETER AREA GIVEN

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the end points of a circle.



Let $P(x, y)$ be a general point on the circle

$\therefore \quad PA \perp PB$

$\Rightarrow \quad$ (slope of PA) x (slope of PB) = -1

$$\Rightarrow \quad \frac{y-y_1}{x-x_1} \times \frac{y-y_2}{x-x_2} = -1$$

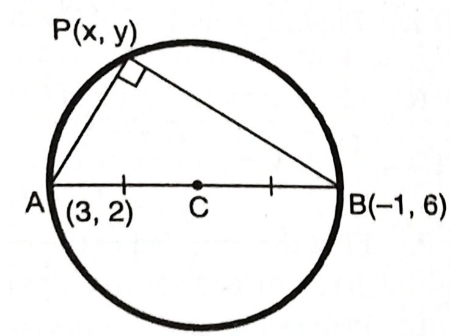
$$\Rightarrow \quad (y-y_1)(y-y_2) = -(x-x_2)$$

$\Rightarrow$ $\mathbf{(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.}$

This is the required equation of the circle.

Example 4. *Determine the equation of the circle if (3, 2) and (-1, 6) are the end points of a diameter of the circle.*

Sol. Let $P(x, y)$ be a general point on the circle.



$\therefore \quad PA \perp PB$

$\Rightarrow$ Slope of PA x Slope of PB = -1

$$\Rightarrow \quad \frac{y-2}{x-3} \times \frac{y-6}{x+1} = -1$$

$$\Rightarrow \quad y^2 - 2y - 6y + 12 = -\left(x^3 - 3x + x - 3\right)$$

$\Rightarrow$ $\mathbf{x^2 + y^2 - 2x - 8y + 9 = 0}$

This is the required equation of the circle.

EXERCISE 20.2

SHORT ANSWER TYPE QUESTIONS

1. Find the equation of the circle when the end points of a diameter are :

   (i) (5, -3) and (2, -4) (ii) (p, q) and (r, s).

2. The line $2x - y + 6 = 0$ meets the circle $x^2 + y^2 - 2y - 9 = 0$ at A and B. Find the equation of the circle on AB as diameter.

3. Find the equation of the circle drawn on the intercept made by the line $2x + 3y = 6$ between the coordinates axes as diameter.

4. If one end of a diameter of the circle $x^2 + y^2 - 4x - 6y + 11 = 0$ is (8, 4), show that the coordinates of the other end are (-4, 2).

LONG ANSWER TYPE QUESTIONS

5. The sides of a square are $x = 6$, $x = 9$, $y = 3$ and $y = 6$. Find the equation of the circle drawn on the diagonal of this square as a diameter.

6. Find the equation of the circle drawn on a diagonal of the rectangle as its diameter whose sides are given by $x = 5$, $x = 8$, $y = 4$, $y = 7$.

7. On the line joining (1, 0) and (3, 0) an equilateral triangle is drawn, having its vertex in the first quadrant. Find the equations of the circles described on its sides as diameter.

8. Find the equations of the circles which pass through the origin and cuts off equal chords of length 'a' from the straight lines $y = x$ and $y = -x$.

Answers

1. (i) $x^2 + y^2 - 7x + 7y + 22 = 0$ (ii) $x^2 + y^2 - (p+r)x - (q+s)y + pr + qs = 0$

2. $x^2 + y^2 + 4x - 4y + 3 = 0$ 3. $x^2 + y^2 - 3x - 2y = 0$

5. $x^2 + y^2 - 15x - 9y + 72 = 0$ 6. $x^2 + y^2 - 13x - 11y + 68 = 0$

7. $x^2 + y^2 - 4x + 3 = 0,\ x^2 + y^2 - 3x - \sqrt{3}y + 2 = 0, x^2 + y^2 - 5x - \sqrt{3}y + 6 = 0$

8. $x^2 + y^2 \pm \sqrt{2}ax = 0, x^2 + y^2 \pm \sqrt{2}ay = 0$.

SUMMARY

1. A **circle** is the locus of a point which moves so that its distance from a fixed point is constant.
   The fixed point is called the **centre** of the circle and the constant distance is called the **radius** of the circle.
2. If (h,k) and r be respectively the centre and radius of a circle, then the equation of the circle is $(x-h)^2+(y-k)^2=r^2$.
3. The equation $x^2+y^2+2gx+2fy+c=0$ represents the circle whose centre and radius are $(-g,-f)$ and $\sqrt{g^2+f^2-c}$ respectively.
4. If (x_1, y_1) and (x_2, y_2) are the coordinates of the end points of a diameter of a circle, then the equation of the circle is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$.

TEST YOURSELF

1. Find the equation the circle passing through the point of intersection of the lines $x + 3y = 0$ and $2x - 7y = 0$ whose centre is at the point of intersection of the lines $x + y + 1 = 0$ and $x - 2y + 4 = 0$.
2. Find the equation of the circle which touches both axes at a distance of 6 units from the origin.
3. Show that the centres of the circles $x^2+y^2-4x-6y-12=0, x^2+y^2+2x+4y-10=0$ and $x^2+y^2-10x-16y-1=0$ are collinear. Find the equation of the line on which the centres lie.
4. Find the equation of the circle with the line joining the centre of the circles $x^2+y^2+6x-14y-1=0$ and $x^2+y^2-4x+10y-2=0$ as a diameter.
5. Find the equation of the circle circumscribing the rectangle whose sides are $x - 3y = 4$, $3x+y=22$, $x-3y=14$ and $3x+y=62$.
6. Find the point of intersection of the circle $x^2+y^2=25$ and the lines $x + y = 5$.
7. For what value of k will the line $4x+3y+k=0$ touches the circle $2x^2+2y^2=5x$?

8. Show that the line $y = x + k\sqrt{2}$ touches the circle $x^2 + y^2 = k^2$.

Answers

1. $x^2 + y^2 + 4x - 2y = 0$
2. $x^2 + y^2 - 12x - 12y + 36 = 0$
3. $5x - 3y - 1 = 0$
4. $x^2 + y^2 + x - 2y - 41 = 0$
5. $x^2 + y^2 - 27x - 3y + 142 = 0$
6. (5, 0), (0, 5)
7. $-\frac{45}{4}, \frac{5}{4}$.

SECTION – D

21. PLOTTING OF CURVES

LEARNING OBJECTIVES

- Introduction
- Plotting of Curve of $y = f(x)$, where f(x) is a Linear Function of x.
- Plotting of Curve of $y = f(x)$, where $f(x)$ is a Quadratic Function of x.

INTRODUCTION

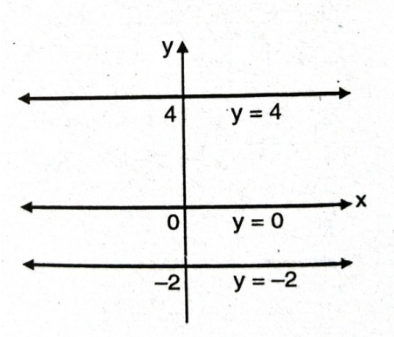
When a point $P(x, y)$ moves under a given set of conditions then the path traced by the point P is called the **curve** (or **graph**) of P. In the present chapter, we shall confine only to the plotting of curves when they y-coordinate of the point P is of the form $f(x)$, where $f(x)$ is a linear (or quadratic) function of the x- coordinate of P.

PLOTTING OF CURVE OF y = f(x), WHERE f(x) IS A LINEAR FUNCTION OF x

Let $y = f(x)$, where $f(x) = ax + b$. The curve of this function is always a straight line. The constant a may or may not be zero.

Case I. a = 0

$\therefore \quad y = ax + b$ reduces to $y = b$.



The curve of this function is a straight line parallel to x-axis and at a distance of b (with due regard to sign) from it. If a is +ve, the line is above x-axis. If a is zero, the line coincides with x – axis. If a is –ve, the line is below x – axis.

In the adjoining figure the *lines* y = 4, y = 0 and y = -2 are shown.

Case II. a$\neq$0

We have $y = ax + b$. The curve of this function is a straight line. We known that only two points are sufficient to fix the position of a line. We gives three convenient values to x and find the corresponding values of y. We plot these three points and join them to get the required straight line represent by the function $y = ax + b$. It is advisable to find three points on the line instead of two points. If these three points do not lie on the line then it is confirmed that some mistake has occurred in finding the points on the line.

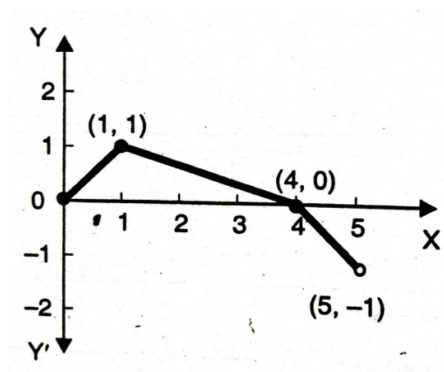
Example 1. *Draw the graph of the function f given by*

$$f(x) = \begin{cases} x & for\ \ 0 \leq x \leq 1 \\ \frac{4-x}{3} & for\ \ 1 \leq x \leq 4 \\ -x+4 & for\ \ 4 \leq x \leq 5 \end{cases}$$

Sol. The given function is $f(x) = \begin{cases} x & for\ \ 0 \leq x \leq 1 \\ \frac{4-x}{3} & for\ \ 1 \leq x \leq 4 \\ -x+4 & for\ \ 4 \leq x \leq 5 \end{cases}$

$0 \leq x \leq 1$. In this interval, $f(x) = x$.

$\therefore$ The graph will be straight line.



For $x = 0$, y = 0. For $x = 1$, y = 1.

$\therefore$ (0, 0), (1, 1) are on the graph.

We take these points on the graph and join them.

$1 \leq x \leq 4$. In this interval $f(x) = \frac{4-x}{3}$.

$\therefore$ The graph will be straight line.

For $x = 1$, $f(x) = \frac{4-1}{3} = 1$. For $x = 4$, $f(x) = \frac{4-4}{3} = 0$.

$\therefore$ (1, 1) and (4, 0) are on the graph.

We take these points on the graph and join them.

$4 \leq x < 5$. In this interval, $f(x) = -x + 4$.

$\therefore$ The graph will be a straight line.

For $x = 4$, $f(x) = -4 + 4 = 0$. For $x = 5$, $f(x) = -5 + 4 = -1$.

The point (5, -1) will be excluded from the graph, because $4 \leq x \leq 5$. We take these points on the graph and join them.

The graph of the given function is shown in the figure.

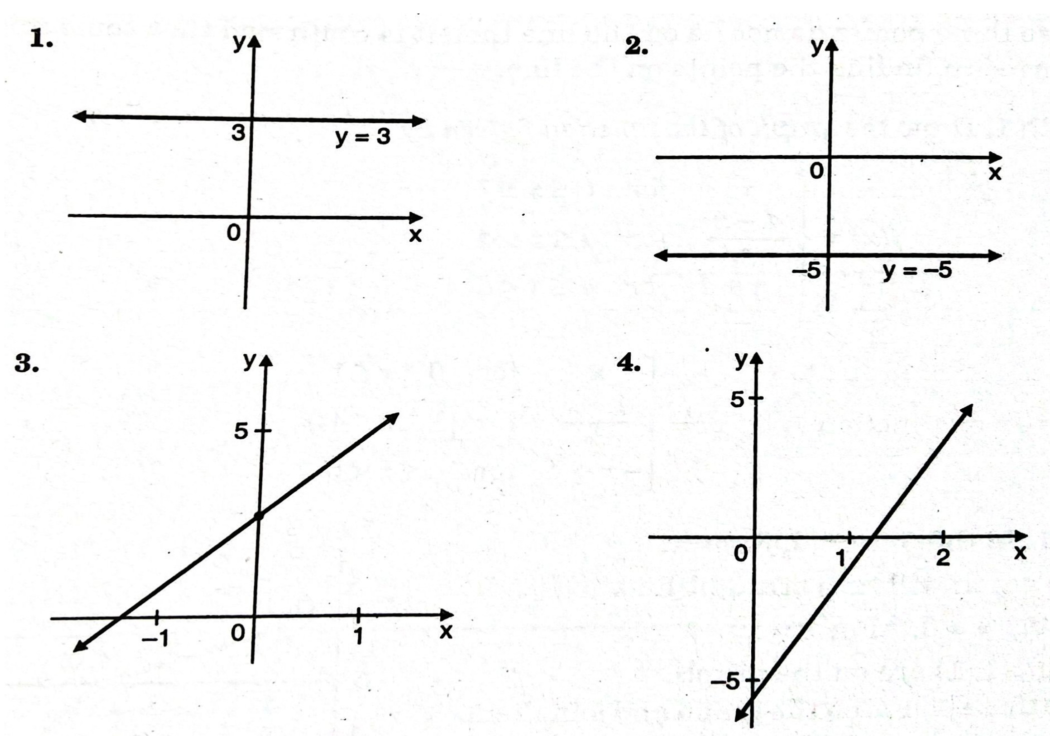
EXERCISE 21.1

SHORT ANSWER TYPE QUESTIONS

Draw the graph of the following functions :

1. $y = 3$
2. $y = -5$
3. $y = 2x + 3$
4. $y = 5x - 6$
5. $y = -x + 5$
6. $y = -3x + 9$
7. $y = \begin{cases} 1 - x, & x < 0 \\ 1 \quad , & x = 0. \\ x + 1, & x > 0 \end{cases}$

Answers



PLOTTING OF CURVE OF y = f(x), WHERE f(x) IS A QUADRATIC FUNCTION OF x

Let $y = f(x)$, where $f(x) = ax^2 + bx + c,\ \ a \neq 0$. This function is also called **parabolie function.** The graph of this function is always a parabola opening either upward or downward. We have $y = ax^2 + bx + c$.

$$\therefore \qquad y = a\left(x^2 + \frac{b}{a}x\right) + c = a\left(x^2 + 2\frac{b}{2a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

$$\therefore \qquad y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

The least value of $\left(x + \frac{b}{2a}\right)^2$ is zero and that will be so when $x = -\frac{b}{2a}$.

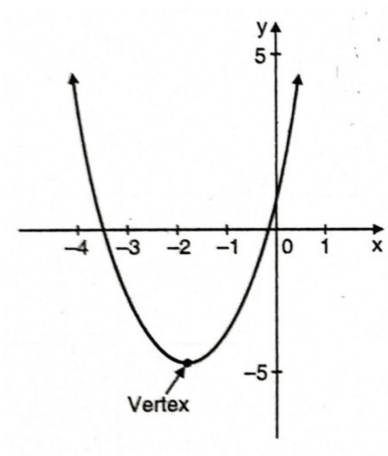
When $x = -\frac{b}{2a}$, $\ y = a(0)^2 + \frac{4ac - b^2}{4a} = \frac{4ac - b^2}{4a}$.

The point $\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$ is the **vertex** of the parabola.

If $a > 0$, then by (1), the value of y will be greater than $\frac{4ac - b^2}{4a}$ for every value of x other than $-\frac{b}{2a}$.

$\therefore \quad a > 0 \qquad \Rightarrow \qquad y \geq \frac{4ac - b^2}{4a} \qquad \therefore$ The parabola will open **downward.**

Example 2. *Draw the graph of the function* $y = 2x^2 + 8x + 3$.



Sol. The given function is

$$y = 2x^2 + 8x + 3 \qquad \ldots(1)$$

The graph of this function is a parabola.

Here, coefficient of $x^2 = 2$ (> 0).

$\therefore$ The parabola will open upward.

$$(1) \Rightarrow \qquad y = 2(x^2 + 4x) + 3$$

$$= 2(x^2 + 4x + 4) - 8 + 3$$

$$= 2(x+2)^2 - 5$$

$\therefore \qquad y = 2(x+2)^2 - 5$

The least value of y is -5 and this is so when

$X + 2 = 0$ *i.e.,* $x = -2$.

$\therefore$ The vertex is (-2, -5).

Now we take some points on the graph :

x	-5	-4	-3	-2	-1	0	1
y	13	3	-3	-5	-3	3	13

The graph of the given function is shown in the figure.

EXERCISE 21.2

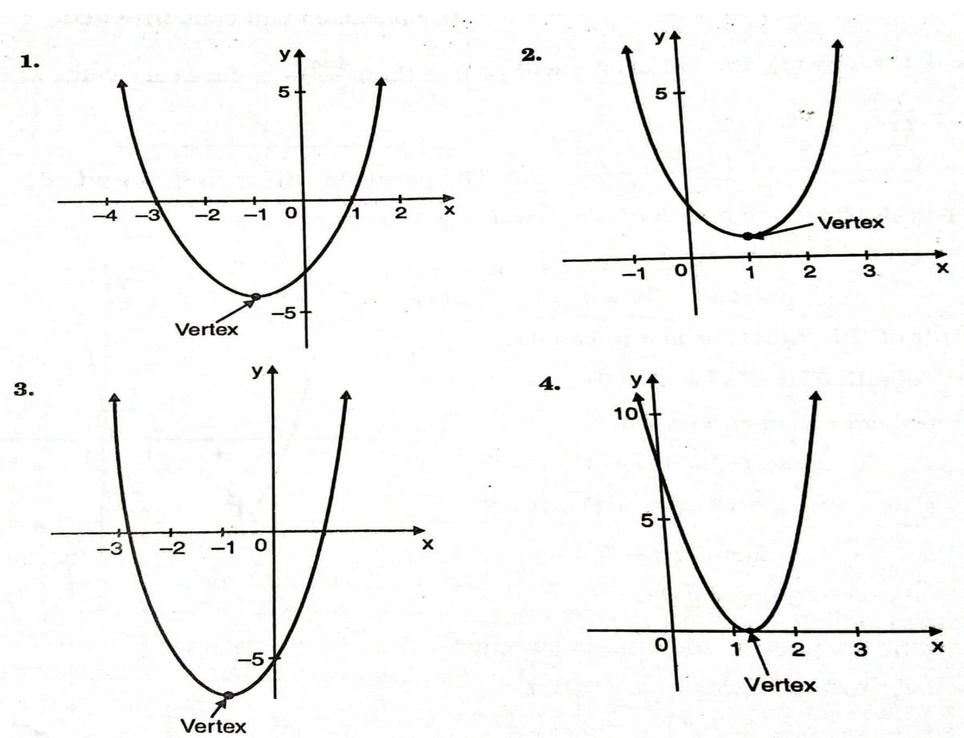
SHORT ANSWER TYPE QUESTIONS

Draw the graph of the following functions :

1. $y = x^2 + 2x - 3$ 2. $y = x^2 - 2x + 3$ 3. $y = 2x^2 + 3x - 5$

4. $y = 4x^2 - 12x + 9$.

Answers



SECTION – D

22. TRANSLATION OF AXES

LEARNING OBJECTIVES

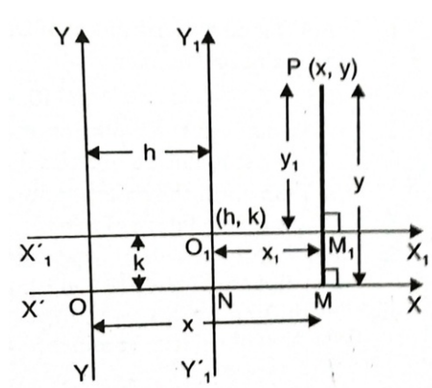
- Introduction
- Translation of Axes

INTRODUCTION

It is sometime convenient to solve a problem by changing the position of coordinate axes. The change may involve a *change of origin* or may involve *rotation of axes* or both type of changes. A change of origin without changing the directions of coordinate axes is called **translation of axes.** In this chapter we shall consider only the method of translation of axes. This technique is also used for finding the foci, vertices, directrices etc., of a conic when its equation is given.

TRANSLATION OF AXES

Let XOX' and YOY' be a system of rectangular coordinate axes with origin O. let $P(x, y)$ be a general point in the plane.



Let $X_1O_1X_1'$ and $Y_1O_1Y_1'$ be a system of another coordinate axes such that:

(i) $X_1O_1X_1'$ is parallel to XOX'

(ii) $Y_1O_1Y_1'$ is parallel to YOY'.

Let (h, k) be the coordinates of the point O_1 w.r.t. new axes.

Now $\quad x = OM = ON + NM \quad = ON + O_1M_1 = h + x_1$

Also $y = PM = M_1M + M_1P = O_1N + M_1P = k + y_1$

$\therefore$ $\mathbf{x = x_1 + h}$ and $\mathbf{y = y_1 + k.}$

Remark. The above transformation equations also holds good even if h or k or both h and k are not positive.

Aid to memory. If the new origin is (h, k), then we have

(i) old x-coordinate = new x-coordinate + h.

(ii) old y-coordinate = new y-coordinate + k.

WORKING RULES FOR SOLVING PROBLEMS

Step I. *Let the axes be translated so that the new origin is the point (h, k).*

Step II. *Let (x_1, y_1) be the coordinates of the point (x, y), under the translation.*

Step III. *Put $x = x_1 + h$ and $y = y_1 + k$.*

Step IV. *Simplify the given equation. The gives the required equation.*

Example 1. *Find the new coordinates of the following points if the origin is shifted to (-3, -2) under a translation :*

(i) *(1, 1)* (ii) *(-2, 1)* (iii) *(5, 0)* (iv) *(-1, -2).*

Sol. New origin is (-3, -2). Let (x, y) be the coordinates of a general point referred to original axes and let (x_1, y_1) be the coordinates of the same w.r.t. new axes

$\therefore$ $x = x_1 + (-3) = x_1 - 3$ and $y = y_1 + (-2) = y_1 - 2$.

$\therefore$ $x_1 = x + 3$ and $y_1 + y + 2$

(i) $x = 1, y = 1 \Rightarrow x_1 = 1 + 3 = 4$ and $y_1 = 1 + 2 = 3$

$\therefore$ New coordinates of (1, 1) are **(4, 3).**

(ii) $x = -2, y = 1 \Rightarrow x_1 = -2 + 3 = 1$ and $y_1 = 1 + 2 = 3$

$\therefore$ New coordinates of (-2, 1) are **(1, 3).**

(iii) $x = 5, y = 0 \Rightarrow x_1 = 5 + 3 = 8$ and $y_1 = 2 + 0 = 2$

$\therefore$ New coordinates of (5, 0) are **(8, 2).**

(iv) $x = -1, y = -2 \Rightarrow x_1 = (-1) + 3 = 2$ and $y_1 = (-2) + 2 = 0$

$\therefore$ New coordinates of (-1, -2) are **(2, 0).**

EXERCISE 22.1

SHORT ANSWER TYPE QUESTIONS

1. Find the new coordinates of the following points if the origin is shifted to (2, -7) under a translations of axes :

   (i) (1, 4) (ii) (0, -3) (iii) (-3, -5) (iv) (-8, 0).

2. (i) Transform the equation $x + y + 2 = 0$ when the origin is shifted to the point (1, 2), after translation of axes.

   (ii) transform the equation $2x - 3y + 5 = 0$ when the origin is shifted to the point (3, -1) after translation of axes.

3. What does the equation $x^2 + y^1 - 4x - 6y + 11 = 0$ become, when the origin is shifted to the point (1, 1) after translation of axes ?

4. On shifting the origin to (4, -5), the axes remaining parallel to the original axes, the equation of a curve becomes $x - 6y + 9 = 0$. Find the original equation of the curve.

SUMMARY

1. A Change of origin without changing the directions of coordinate axes is called a **translation of axes.**
2. If the new origin is (h, k), then
   (i) old x – coordinate = new x – coordinate + h.
   (ii) old y – coordinate = new y – coordinate + k.

TEST YOURSELF

1. Transform the equations :
   (i) $xy - x - y + 1 = 0$ (ii) $x^2 + xy - 3x + 2 = 0$
   when the origin is shifted to (1, 1).
2. Verify that the area of the triangle with vertices (2, 3), (5, 7) and (-3, -1) remains invariant under the translation of axes when the origin is shifted to the point (-1, 3).

Answers

1. (i) $xy = 0$ (ii) $x^2 + xy = 0$.

Hint

2. Vertices under the new system are (2 – (- 1), 3 – 3), (5 – (- 1), 7 – 3) and

$(-3-(-1), -1-3)$.

SECTION – D

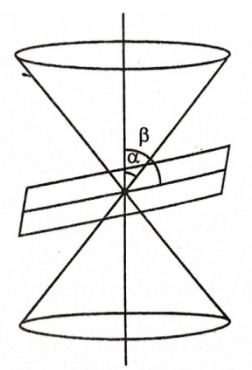
23. PARABOLAS

LEARNING OBJECTIVES

- Conic Section
- Definition of a Parabola
- Equation of a Parabola in the General Form
- Equation of a Parabola in the Standard Form
- Some Definitions Related to a Parabola
- Four Standard Forms of Parabola
- Position of a Point with Respect to a Parabola
- Problems Based on Translation of Axes

COIN SECTION

When a double-napped right circular hollow cone extending infinitely far in both direction is intersected by a plane, the curve so obtained is called a **conic section.** The shape of the conic section depends upon the position of the intersecting plane. Let α be the semi-vertical angle of the cone and let β be the angle made by the intersecting plane with the axis of the cone.



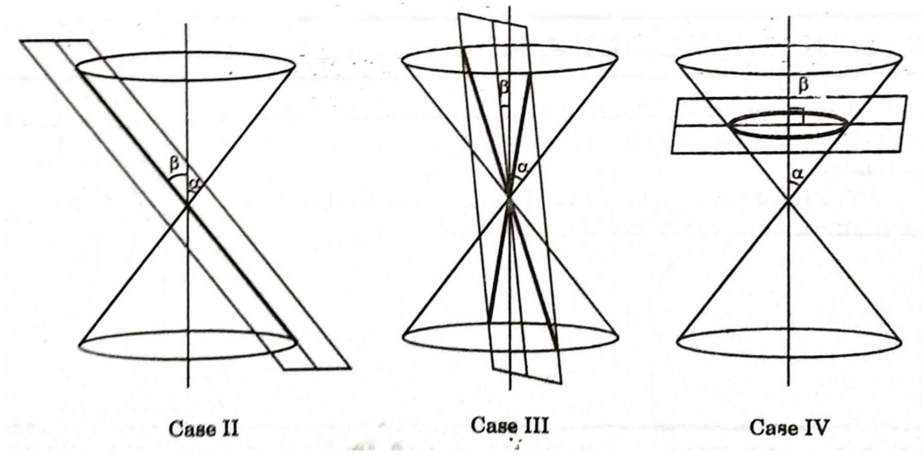
Case I

Case I. Plane passing through the vertex and $\alpha < \beta \leq 90^0$

In this case, the section of the come is a *point.*

Case II. Plane passing through the vertex and $\beta = \alpha$.

In this case, the section of the cone is a straight line. We have already learnt that the equation of a straight line is of the form $ax+by+c-0$.



Case III. Plane passing through the vertex and 0 $\leq \beta < \alpha$.

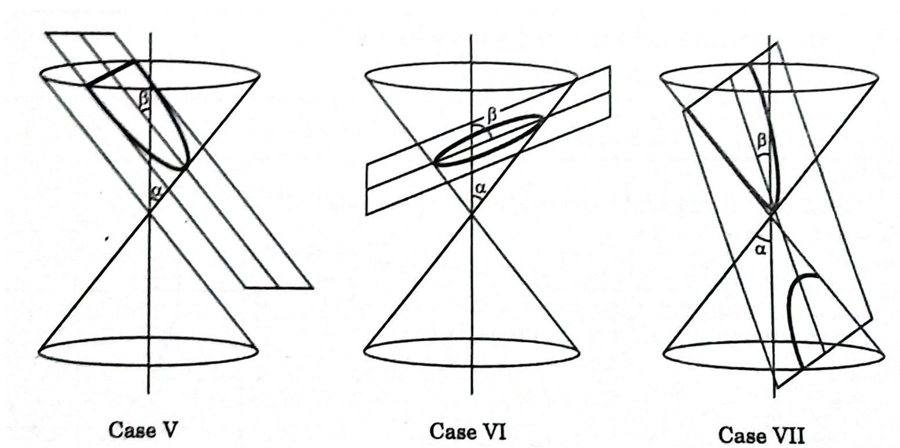
In this case, the section of the cone is a *pair of intersecting straight lines.*

Case IV. Plane not passing through the vertex, cutting only one nappe and $\beta =$ **90⁰.**

In this case, the section of the cone is a *circle*.

Case V. Plane not passing through the vertex, cutting only one nappe and $\beta = \alpha$**.**

In this case, the section of the cone is a *parabola.*



Case VI. Plane not passing through the vertex, cutting only one nappe and $\alpha < \beta < 90^0$.

In this case, the section of the cone is an *ellipse*.

Case VII. Plane not passing through the vertex, cutting both nappes and $0 \leq \beta < \alpha$.

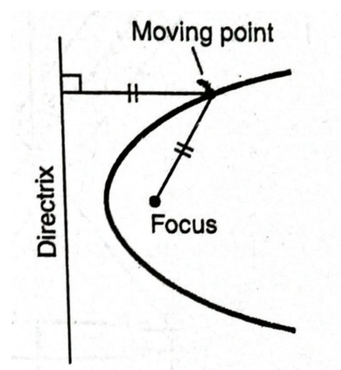
In this case, the section of the cone is a *hyperbola*.

In the following chapters we shall study parabola, ellipse and hyperbola in detail. We shall consider these conic sections as plane curves and define these conic sections alternatively in terms of some specific points and lines lying in the plane containing the conic section.

DEFINITION OF A PARABOLA

A **parabola** is the locus of a plant which moves so that its distance from a fixed point is equal to its distance from a fixed line.

The fixed point and the fixed lien are respectively called the **focus** and the **directrix** of the parabola.



EQUATION OF A PARABOLA IN THE GENERAL FORM

Let $S(h, k)$ and $ax+by+c=0$ be the focus and the directrix of a parabola respectively.

Let $P(x, y)$ be a general point on the parabola.

$\therefore$ By definition, PS = length of $\perp$ from P to $ax+by+c=0$

$$\therefore \quad \sqrt{(x-h)^2+(y-k)^2} = \left|\frac{ax+by+c}{\sqrt{a^2+b^2}}\right|$$

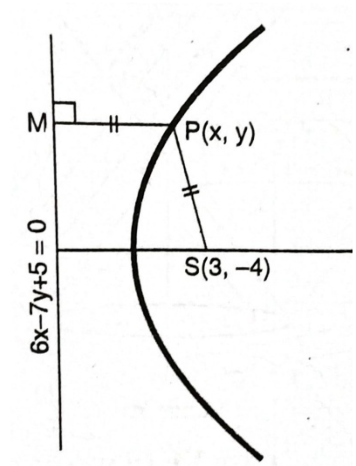
or **$(a^2 + b^2) [(x - h)^2 + (y - k)^2] = (ax + by + c)^2$.**

This is the equation of the required parabola.

Example 1. *Find the equation of the parabola whose focus and directrix are respectively (3, - 4) and 6x – 7y + 5 = 0.*

Sol. Let $P(x, y)$ be a general point on the parabola.

$\therefore$ By definition, distance of P from (3, - 4) is equal to the distance of P from the directrix $6x-7y+5=0$.



$$\Rightarrow \quad PS = PM$$

$$\Rightarrow \quad \sqrt{(x-3)^2+(y+4)^2} = \left|\frac{6x-7y+5}{\sqrt{36+49}}\right|$$

$$\Rightarrow \quad 85(x^2+9-6x+y^2+16+8y)$$

$$= 36x^2+49y^2+25-84xy-70y+60x$$

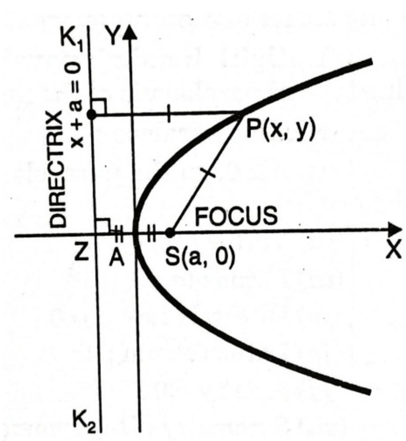
Or $\quad 49x^2+36y^2+84xy-570x+570y+2100=0$

Or $\quad$ **$(7x + 6y)^2 – 570x – 750y + 2100 = 0$.**

This is the equation of the required parabola.

EQUATION OF A PARABOLA IN THE STANDARD FORM

Let S be the focus and K_1K_2, the directrix of a parabola. Draw SZ perpendicular to K_1K_2. Let A be the middle point to SZ.



By definition, A lies on the parabola. Let A be the origin and AX and AY as coordinate axes.

Let $\quad AS = a$.

$\therefore$ The equation of the directrix is $x + a = 0$ and the focus is $S(a, 0)$.

Let $P(x, y)$ be a general point on the parabola.

$\therefore$ By definition, $\quad PS = PM$

$\therefore \quad \sqrt{(x-a)^2+(y-0)^2} = \left|\frac{x+a}{\sqrt{1^2+0^2}}\right| = |x+a|$

$\Rightarrow \quad x^2+a^2-2ax+y^2 = x^2+a^2+2ax$

$\Rightarrow \quad$ **$y^2 = 4ax$.**

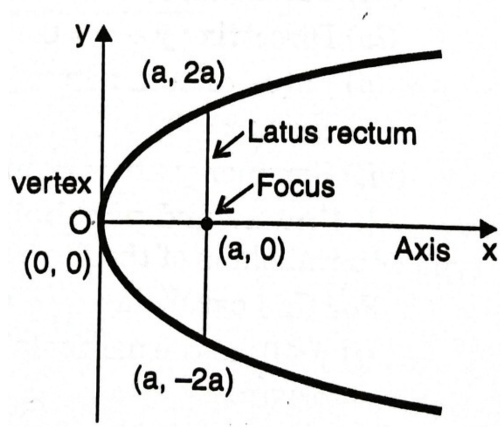
This is the required equation of the parabola in the standard form.

Remark. The *parametric equations* of the parabola $y^2=4ax$ are $x=at^2, y=2at$, where t is the parameter.

SOME DEFINITIONS RELATED TO A PARABOLA

The equation of a parabola in the standard form is $y^2=4ax$, where a is some positive constant.

(i) The line through the focus and perpendicular to the directrix is called the **axis** of the parabola. For the parabola $y^2=4ax, OX$ is the axis. A parabols is always symmetric about its axis, because if (x, y) is on the parabola, then $(x, -y)$ is also on the parabola.



(ii) The point of intersection of the parabola and its axis is called the **vertex** of the parabola. For the parabola $y^2 = 4ax$, the point O is the vertex.

(iii) The double ordinate at the focus is called the **latus rectum** of the parabola.

For the parabola $y^2 = 4ax$, the focus is $(a, 0)$.

Putting $x = a$ in $y^2 = 4ax$, we get

$$y^2 = 4a(a) = 4a^2$$

i.e., $\quad y = \pm 2a$.

$\therefore$ The coordinates of the double ordinate at the focus are $(a, 2a)$ and $(a, -2a)$.

∴ The latus rectum of the parabola is equal to the distance between the points (a, $2a$) and (a, - $2a$) and this is equal to $4a$.

∴ Latus rectum = **4a**.

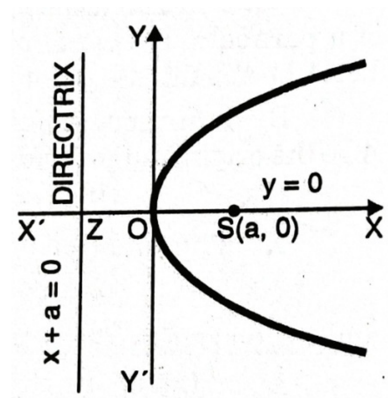
FOUR STANDARD FORMS OF PARABOLA

There are four standard forms of parabola with vertex at the origin and axis along either of coordinate axes.

1. **Right handed parabola.** The equation of this type of parabola is of the form $y^2 = 4ax, a > 0$.

   For this parabola :

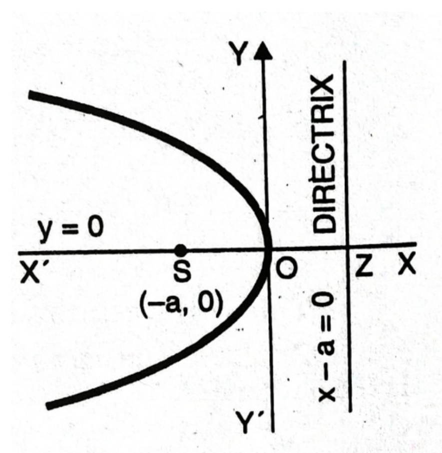
   i. $x \geq 0$, so that parabola opens to the *right* of the origin.
   ii. Vertex : (0, 0)
   iii. Focus : (a, 0)
   iv. Directrix : $x + a = 0$
   v. Latus rectum : $4a$
   vi. Axis : $y = 0$
   vii. Symmetry : It is symmetric about x- axis.



2. **Left handed parabola.** The equation of this type of parabola is of the form $y^2 = -4ax, a > 0$.

   For this parabola :

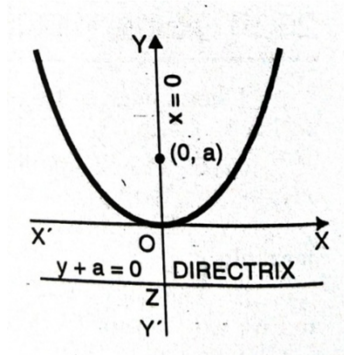
   i. $x \leq 0$, so the parabola opens to the *left* of the origin.
   ii. Vertex : (0, 0)
   iii. Focus : (- a, 0)
   iv. Directrix : $x - a = 0$
   v. Latus rectum : $4a$
   vi. Axis : $y = 0$
   vii. Symmetry : It is symmetric about x – axis.



3. **Upward parabola.** The equation of this type of parabola is of the form $x^2 = -4ax, a > 0$.

   For this parabola :

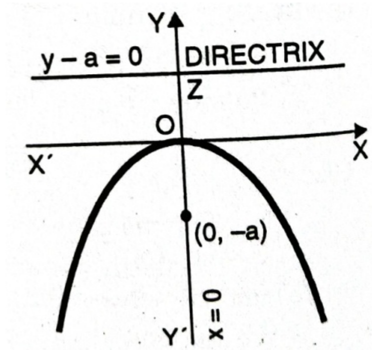
   i. y $\geq$ 0 so the parabola opens *upward* of the origin.
   ii. Vertex : (0, 0)



iii. Focus : (0, a)
iv. Directrix : $y + a = 0$
v. Latus rectum : $4a$
vi. Axis : $x = 0$
vii. Symmetry : It is symmetric about y-axis.

4. **Downward parabola.** The equation of this type of parabola is of form $x^2 = -4ax, a > 0$.

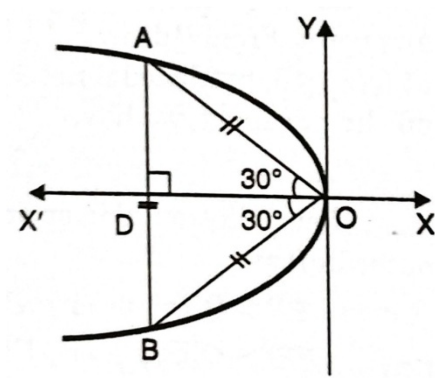
For this parabola :



i. y $\leq$ 0, so the parabola opens downward of the origin.
ii. Vertex : (0, 0)
iii. Focus : (0, - a)
iv. Directrix : y – a = 0
v. Latus rectum : $4a$
vi. Axis : $x = 0$
vii. Symmetry: It is symmetric about y-axis.

Example 2. *An equilateral triangle is inscribed in the parabola $y^2 = -8x$, where one vertex is at the vertex of the parabola. Find the length of the side of the triangle.*

Sol. The given equation is $y^2 = -8x$.



$\Rightarrow \quad y^2 = -4(2)\,x \Rightarrow \quad y^2 = -4ax$, where $a = 2 > 0$.

This represents a parabola with vertex at (0, 0) and axis along OX'.

Let AOB be the equilateral triangle. Let $AB = 2k$.

$$\therefore \qquad OD = AO\cos 30^0 = 2k \,.\, \frac{\sqrt{3}}{2} = \sqrt{3}k$$

and $\qquad AD = AO\sin 30^0 = 2k \,.\, \frac{1}{2} = k$.

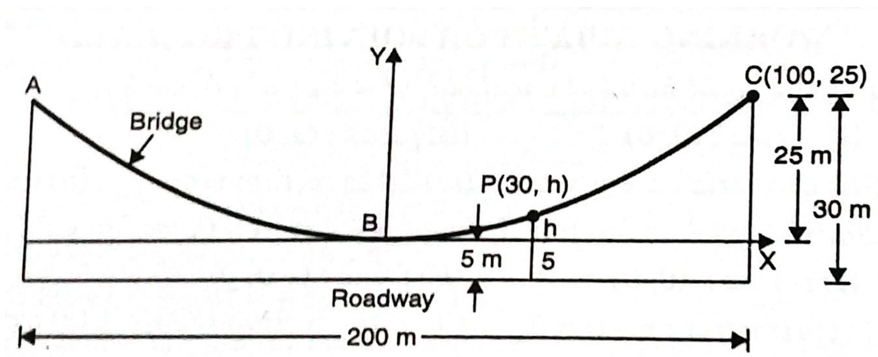
$\therefore$ Coordinates of A are $\left(-\sqrt{3}k, k\right)$.

Since A lies on $y^2 = -8x$, we have $k^2 = -8\left(-\sqrt{3}k\right)$ or $k = 8\sqrt{3}$.

$\therefore$ Side of triangle = $2k = 2\left(8\sqrt{3}\right)$ = $16\sqrt{3}$ units.

Example 3. *The towers of a bridge, hung in the form of a parabola, have their tops 30 metres above the road – way and are 200 metres apart. If the cable is 5 metres above the road-way at the centre of the bridge, find the length of the vertical supporting cable 30 metres from the centre.*

Sol. Let *ABC* be the bridge with *B* as the lowest point. Let the horizontal through *B* and in the plane of the bridge be taken as the *x*-axis. Let vertical through *B* be the *y*-axis.



$\therefore$ *ABC* is a parabola with axis along y-axis and opening upward.

Let its equation be $x^2 = 4ay$, where a is some + ve constant.

The coordinates of *C* are (100, 25) and it lies on the parabola.

$$\Rightarrow \quad (100)^2 = 4a(25) \quad \Rightarrow \quad a = \frac{(100)^2}{100} = 100$$

$\therefore$ The parabola is $x^2 = 4(100)y$ *i.e.,* $x^2 = 400y$.

Let *P*(30, *h*) be the point on the parabola 30 metres from the centre.

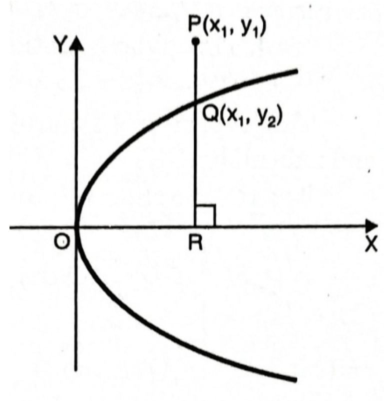
$$\therefore \quad (30)^2 = 400h \quad \text{or} \quad h = \frac{900}{400} = \frac{9}{4}$$

$\therefore$ Length of vertical supporting cable 30 metres from the centre

$$= 5 + \frac{9}{4} = 7\frac{1}{4}\,m.$$

POSITION OF A POINT WITH RESPECT TO A PARABOLA

Let $y^2 = 4ax$ be a parabola and let $P(x_1, y_1)$ be any point. From P draw $PR \perp OX$ meeting the parabola at $Q(x_1, y_2)$, produce if necessary. Since $Q(x_1, y_2)$ lies on the parabola, we have



$$y_2^2 = 4ax_1$$

Now P lies outside or on or inside the parabola according as

	$PR > QR$	or	$PR = QR$	or	$PR < QR$
i.e,	$PR^2 > QR^2$	or	$PR^2 = QR^2$	or	$PR^2 < QR^2$
i.e.,	$y_1^2 > y_2^2$	or	$y_1^2 > y_2^2$	or	$y_1^2 > y_2^2$
i.e.,	$\mathbf{y_1^2 > 4ax_1}$	or	$\mathbf{y_1^2 = 4\,ax_1}$	or	$\mathbf{y_1^2 < 4ax_1}$.

Example 4. *Find the position of the points (2, 3), (2, -4), (3, 7) w.r.t. the parabola* $y^2 = 8x$.

Sol. We have $y^2 = 8x$.

At (2, 3), $y^2 - 8x = (3)^2 - 8(2) = -7 < 0$ *i.e.,* $y^2 < 8x$

$\therefore$ (2, 3) is inside the parabola.

At (2, - 4), $y^2 - 8x = (-4)^2 - 8(2) = 0$ *i.e.,* $y^2 = 8x$

$\therefore$ (3, 7), $y^2 - 8x = (7)^2 - 8(3) = 25 > 0$ *i.e.,* $y^2 > 8x$

$\therefore$ (3, 7) is outside the parabola.

WORKING RULES FOR SOLVING PROBLEMS

Rule I. *For the right handed parabola,* $y^2 = 4ax, a > 0$, *we have :*

(i) *vertex : (0, 0)* (ii) *focus : (a, 0)* (iii) *directrix : x + a = 0*
(iv) *latus rectum = 4a* (v) *axis : y = 0.*

Rule II. *For the left handed parabola,* $y^2 = -4ax, a > 0$, *we have :*

(i) *vertex : (0, 0)* (ii) *focus : (-a, 0)* (iii) *directrix : x - a = 0*
(iv) *latus rectum = 4a* (v) *axis : y = 0.*

Rule III. *For the upward parabola,* $x^2 = 4ay, a > 0$*, we have :*

(i) *vertex : (0, 0)* (ii) *focus : (0, a)* (iii) *directrix : y + a = 0*

(iv) *latus rectum = 4a* (v) *axis : x = 0.*

Rule IV. *For the downward parabola,* $x^2 = -4ay, a > 0$*, we have :*

(i) *vertex : (0, 0)* (ii) *focus : (0, -a)* (iii) *directrix : y - a = 0*

(iv) *latus rectum = 4a* (v) *axis : x = 0.*

Rule V. *The point* $P(x_1, y_1)$ *lies outside, on inside the parabola* $y^2 = 4ax$ *according as* $y_1^2 > 4ax_1, y_1^2 = 4ax_1, y_1^2 < 4ax_1$.

EXERCISE 23.1

SHORT ANSWER TYPE QUESTIONS

1. Find the equation of the parabola whose focus and directrix are respectively:
   (i) (2, 0) and $x = -2$
   (ii) (6, 0) and $x = -6$
   (iii) (0, -2) and y = 2
   (iv) (0, -3) and y = 3
   (v) (3, -4) and $x + y - 2 = 0$
   (vi) (6, -3) and $3x - 5y + 1 = 0$.

2. Find the equation of the parabola whose focus is (*a, b*) and directrix is $\frac{x}{a} + \frac{y}{b} = 1$.

3. Find the coordinates of a point on the parabola $y^2 = 18x$ whose ordinate is equal to three times its abscissa.

4. At what point of the parabola $x^2 = 9y$ is the abscissa three times that of the ordinate?

5. Find the position of the points (0, 3), (2, $4\sqrt{2}$), (4, 1), (5, 3) w.r.t. the parabola $y^2 = 16x$.

6. If the parabola $y^2 = 4ax$ passes through the point (9, -12), then find the value of *a*.

LONG ANSWER TYPE QUESTIONS

7. Find the vertex, focus, directrix, lature rectum and axis of the parabola :
   (i) $y^2 = 10x$ (ii) $y^2 = -8x$
   (iii) $x^2 = 6y$ (iv) $x^2 = -9y$.

8. A double ordinate of the parabola $y^2 = 4ax$ is of length $8a$. Show that the lines from the vertex to its ends are at right angle.

9. PQ is a double ordinate of a parabola $y^2 = 4ax$. Find the locus of its points of trisection.

10. Find the equation of the parabola with vertex at the origin and satisfying the additional condition :
    (i) focus at (0, 6) (ii) focus at (3, 0)
    (iii) directrix : $x + 3 = 0$ (iv) directrix : $y + 2 = 0$
    (v) axis along x-axis and passing through (2, 3)
    (vi) axis along y-axis and passing through (2, -4)
    (vii) passing through (5, 2) and symmetric with respect to y-axis.

Answers

1. (i) $y^2 = 8x$ (ii) $y^2 = 24x$ (iii) $x^2 = -8y$ (iv) $x^2 = -12y$

   (v) $x^2 - 2xy + y^2 - 8x + 20y + 46 = 0$

   (vi) $25x^2 + 30xy + 9y^2 - 414x + 214y + 1529 = 0$

2. $(ax - by)^2 - 2a^3x - 2b^3y + a^4 + a^2b^2 + b^4 = 0$ 3. (2, 6)

4. (3, 1) 5. Outside, on, inside, inside 6. 4

7. (i) (0, 0), (5/2, 0), $x + 5/2 = 0$, 10 units, y = 0

   (ii) (0, 0), (-2, 0), $x - 2 = 0$, 8 units, y = 0

   (iii) (0, 0), (0, 3/2), y + 3/2 = 0, 6 units, $x = 0$

   (iv) (0, 0), (0, -9/4) y – 9/4 = 0, 9 units , $x = 0$

9. $9y^2 = 4ax$

10. (i) $x^2 = 24y$ (ii) $y^2 = 12x$ (iii) $y^2 = 12x$ (iv) $x^2 = 8y$

    (v) $2y^2 = 9x$ (vi) $x^2 = -y$ (vii) $2x^2 = 25y$.

PROBLEMS BASED ON TRANSLATION OF AXES

Example 5. *Find the vertex, focus, directrix and axis of the parabolas :*

(i) $(y-\beta)^2 = 4a(x-\alpha), a>0$ (ii) $(x-\alpha)^2 = 4a(y-\alpha), a>0$.

Sol. (i) We have $(y-\beta)^2 = 4a(x-\alpha)$. ....(1)

Let $x = \alpha + X$ and $y = \beta + Y$. $\therefore$ (1) $\Rightarrow$ $Y^2 = 4aX, a>0$

This represents a parabola opening on the right of Y-axis.

$\therefore$ vertex = (0, 0), focus = (a, 0).

Directrix : $X + a = 0$, axis : $Y = 0$.

$\therefore$ With respect to original axis,

vertex = $(\alpha+0, \beta+0) = (\alpha, \beta)$

focus = $(\alpha+a, \beta+0) = (\alpha+a, \beta)$

Directrix : $x-\alpha+a=0$, axis : $y-\beta=0$.

(ii) We have $(x-\alpha)^2 = 4a(y-\alpha), a>0$. ....(1)

We shift the origin to (α, β).

Let $x = \alpha + X$ and $y = \beta + Y$. $\therefore$ (1) $\Rightarrow$ $X^2 = 4aY, a>0$

This represents a parabola opening above X-axis.

$\therefore$ vertex = (0, 0), focus = (0, a).

Directrix : $Y + a = 0$, axis : $X = 0$.

$\therefore$ With respect to original axis vertex = $(\alpha+0, \beta+0) = (\alpha, \beta)$.

focus = $(\alpha+a, \beta+a) = (\alpha, \beta+a)$

Directrix : $y-\beta+a=0$, axis : $x-\alpha=0$.

Example 6: *Show that the following equations represent parabolas. In each case, find vertex, axis, focus, directrix, latus rectum. Also draw rough sketch.*

$$3y^2 - 10x - 12y - 18 = 0$$

Sol. The given equation $3y^2 - 10x - 12y - 18 = 0$. ...(1)

$$\Rightarrow \quad 3y^2 - 12y = 10x + 18$$

or $3(y^2 - 4y + 4) = 10x + 18 + 12$ or $3(y-2)^2 = 10(x+3)$

or $(y-2)^2 = \frac{10}{3}(x+3)$ ....(2)

Let the origin be shifted to (- 3, 2) and let (*X, Y*) be the coordinates of the point (*x,y*) w.r.t. new axis.

$\therefore$ $x = -3 + X$ and $y = 2 + Y$

$\therefore$ (2) $\Rightarrow$ $Y^2 = \frac{10}{3}X$ or $Y^2 = 4\left(\frac{5}{6}\right)X$ or $Y^2 = 4aX$, where $a = \frac{5}{6} > 0$ ...(3)

$\therefore$ (3) represents a parabola opening on the right of Y-axis.

With respect to new axes.

vertex = (0, 0), axis is $Y = 0$, focus = $(a, 0)$ = $\left(\frac{5}{6}, 0\right)$, different is $X + a = 0$

i.e., $X + \frac{5}{6} = 0$, latus rectum $= 4a = 4\left(\frac{5}{6}\right) = \frac{10}{3}$.

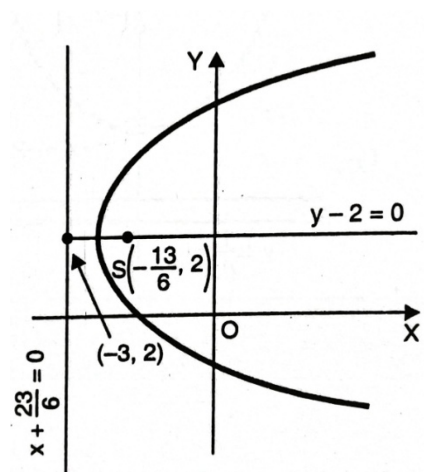
We have $x = -3 + X$ and $y = 2 + Y$.

$\therefore$ **With respect to original axis,**

vertex = (- 3 + 0, 2 + 0) or (-3, 2), axis is y – 2 = 0,

focus = $\left(-3 + \frac{5}{6}, 2 + 0\right)$ or $\left(-\frac{13}{6}, 2\right)$, directrix is

$(x+3) + \frac{5}{6} = 0$ or $x + \frac{23}{6} = 0$, latus rectum = $\frac{10}{3}$.



The rough sketch of the parabola is shown in the figure.

EXERCISE 23.2

LONG ANSWER TYPE QUESTIONS

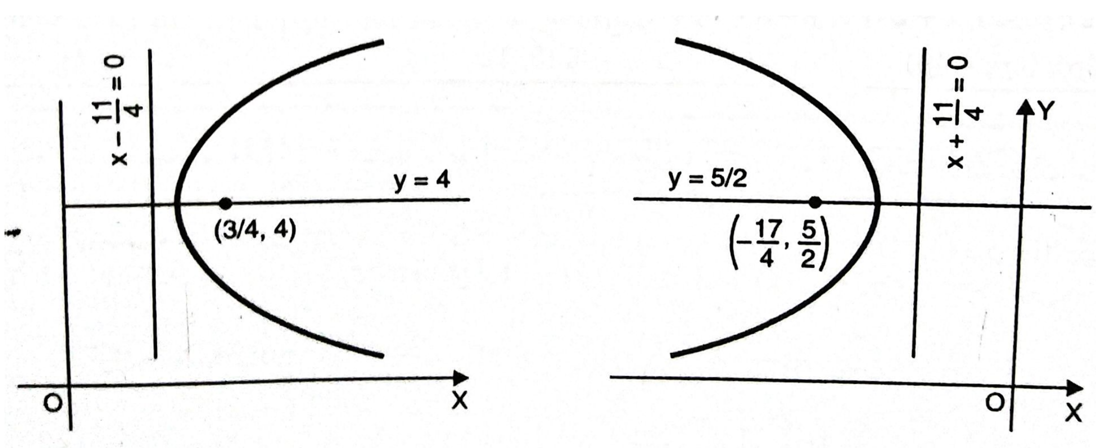
Show that the following equations represent parabolas. In each case, find vertex, axis, focus, directrix, latus rectum. Also draw rough sketches :

1. $y^2 - 8y - x + 19 = 0$
2. $4y^2 + 12x - 20y + 67 = 0$
3. $x^2 - 5y + 4x + 9 = 0$
4. $x^2 - 6x + y + 14 = 0$.

Answers

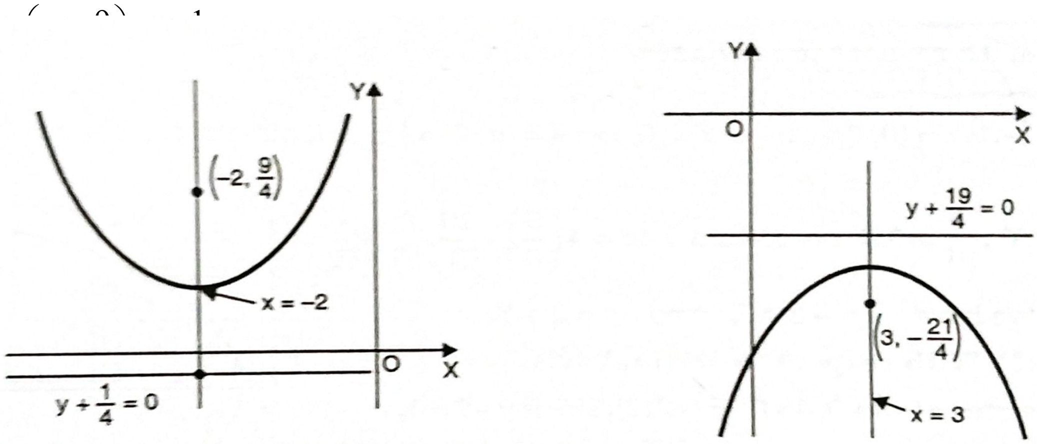
1. $(3,4) y = 4, \left(\frac{13}{4}, 4\right) x - \frac{11}{4} = 0, 1$

2. $\left(-\frac{7}{2}, \frac{5}{2}\right), y = \frac{5}{2}, \left(-\frac{17}{4}, \frac{5}{2}\right), x + \frac{11}{4} = 0.3$



3.

$(-2,1)x = -$



4.

$(3,-5), x =$

.

SUMMARY

1. A **parabola** is the locus of a point which moves so that its distance from a fixed point is equal to its distance from a fixed line.

   The fixed point and the fixed line are respectively called the **focus** and the directrix of the **parabola.**

2. A second degree equation represents a parabola if and only if the second degree terms form a perfect square.

TEST YOURSELF

1. Find the vertex, focus, directrix, axis and latus rectum of the parabola, $y^2 = 4x + 4y$.
2. For the parabola $y^2 = 4px$, find the extremities of a double ordinate of length 8p. Prove that the lines from the vertex to its extremities are at right angle.
3. At what point of the parabola x^2, 9y, the abscissa is three times that of the ordinate ?
4. Show that the line y = mx + c touches the parabola $y^2 = 4a(x+a)$, if $c = am + \frac{a}{m}$

Answers

1. (- 1, 2), (0, 2), x + 2 = 0, y – 2 = 0, 4
2. (4p, 4p), (4p – 4p)
3. (3, 1)

SECTION – D

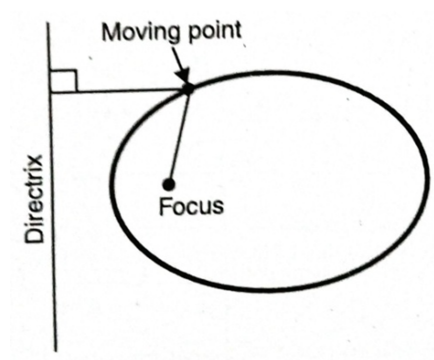
24. ELLIPSES

LEARNING OBJECTIVES

- Definition of an Ellipse
- Equation of an Ellipse in the General Form
- Equation of an Ellipse in the Standard Form
- Existence of a Second Focus and a Second Directrix for the Ellipse
- A Property of Ellipse
- Some Definitions Related to an Ellipse
- Two Standard Forms of Ellipse

DEFINITION OF AN ELLIPSE

An **ellipse** is the locus of a point which moves so that its distance from a fixed point is in a constant ratio, less than one, to its distance from a fixed line.



The fixed point is called the **focus** of the ellipse. The fixed line is called the **directrix** of the ellipse. The constant ratio (< 1) is called the **eccentricity** of the ellipse and is denoted by e.

EQUATION OF AN ELLIPSE IN THE GENERAL FORM

Let $S(h, k)$ and $ax+by+c=0$ be the focus and directrix of an ellipse respectively. Let $e(< 1)$ be the eccentricity of the ellipse. Let $P(x, y)$ be a general point on the ellipse.

$\therefore$ By definition, $PS = e$ (length of $\perp$ from P to $ax + by + c = 0$)

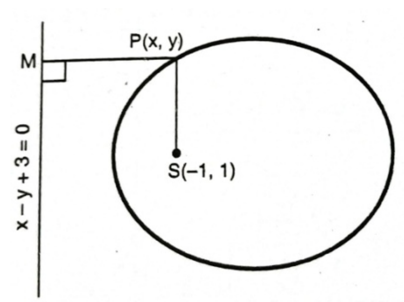
$$\Rightarrow \quad \sqrt{(x-h)^2 + (y-k)^2} = e\left|\frac{ax+by+c}{\sqrt{a^2+b^2}}\right|$$

$\Rightarrow$ **$(a^2 + b^2)[(x - h)^2 + (y - k)^2] = e^2(ax + by + c)^2$.**

This is the equation of the required ellipse.

Example 1. *Find the equation of the ellipse, whose focus, directrix and eccentricity are respectively (-1, 1), $x - y + 3 = 0$ and $\frac{1}{2}$.*

Sol. The focus, directrix and eccentricity of the ellipse are respectively, $S(-1, 1)$, $x - y + 3 = 0$ and $\frac{1}{2}$.



Let $P(x, y)$ be a general point on the ellipse.

$\therefore$ Distance of P from the focus (-1, 1) is equal to $\frac{1}{2}$ times the distance of P from the directrix $x - y + 3 = 0$.

$$\Rightarrow \quad PS = \frac{1}{2}PM$$

$$\Rightarrow \quad \sqrt{(x+1)^2 + (y-1)^2} = \frac{1}{2}\left|\frac{x-y+3}{\sqrt{1+1}}\right|$$

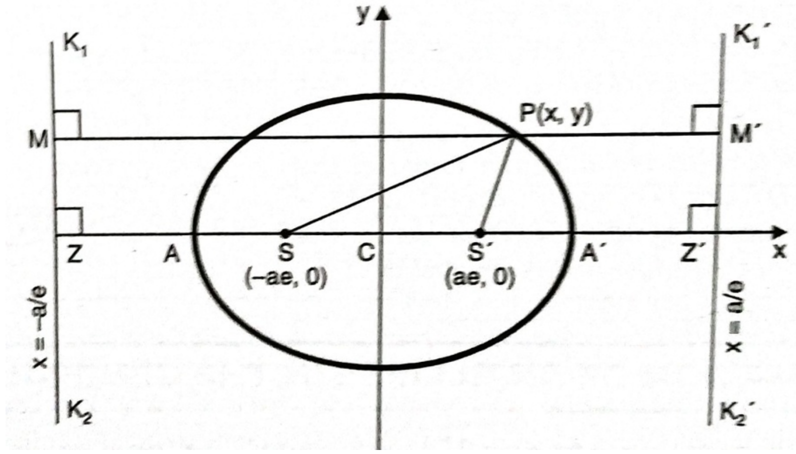
$$\Rightarrow \quad 8(x^2 + 1 + 2x + y^2 + 1 - 2y) = x^2 + y^2 + 9 - 2xy - 6y + 6x$$

$\Rightarrow$ **$7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0$.**

This is the equation of the required ellipse.

EQUATION OF AN ELLIPSE IN THE STANDARD FORM

Let S and K_1K_2 be the focus and the directrix of an ellipse. Draw $SZ \perp K_1K_2$. Let $e(< 1)$ be the eccentricity of the ellipse.



Let A be on SZ such that $SA = e\,AZ$ ....(1)

Produce ZS to A' such that $SA' = e\,A'Z$ ....(2)

By definition, A and A' lie on the ellipse.

Let $AA' = 2a$ and let C be the middle point of AA'

(2) + (1) $\Rightarrow$ $SA' + SA = eA'Z + eAZ$

$\Rightarrow$ $AA' = e[(CZ + A'C) + (CZ - CA)]$

$\Rightarrow$ $2a = e.\,2CZ$ $[\because AC = CA']$

$\Rightarrow$ $CZ = \dfrac{a}{e}$ ...(3)

(2) – (1) $\Rightarrow$ $SA' - SA = eA'Z - eAZ$

$\Rightarrow$ $(CS + CA') - (CA - CS) = eA'Z - eAZ$

$\Rightarrow$ $2CS = e[(CA + CA') - (CZ - AC)]$

$\Rightarrow$ $CS = es$ ....(4)

Let C be the origin, CA', the axis of x and a line through C perpendicular to AA', the axis of y.

$\therefore$ The coordinates of focus S are ($-ae$, 0) and the equation of the directrix K_1K_2 is

$$x = -\frac{a}{e} \quad i.e., \quad x + \frac{a}{e} = 0\,.$$

Let $P(x, y)$ be a general point on the ellipse.

$\therefore$ By definition, $PS = ePM$

$$\Rightarrow \quad \sqrt{(x+ae)^2+(y-o)^2} = e\left|\frac{x+\frac{a}{e}}{\sqrt{1^2+0^2}}\right|$$

$$\Rightarrow \quad \sqrt{x^2+a^2e^2+2aex+y^2} = e\left|x+\frac{a}{e}\right|$$

$$\Rightarrow \quad x^2+y^2+2aex+a^2e^2 = e^2\left(x+\frac{a}{e}\right)^2$$

$$\Rightarrow \quad x^2+y^2+2aex+a^2e^2 = e^2x^2+a^2+2aex \qquad \ldots(5)$$

$$\Rightarrow \quad (1-e^2)x^2+y^2 = a^2(1-e^2) \quad \Rightarrow \quad \frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)} = 1$$

$$\Rightarrow \quad \frac{x^2}{a^2}+\frac{y^2}{b^2} = 1, \text{ where } b = a\sqrt{1-e^2}.$$

This is the required equation of the ellipse in the standard form.

Remark. The derivation of this equation is not required from the examination point of view.

EXISTENCE OF A SECOND FOCUS AND A SECOND DIRECTRIX FOR THE ELLIPSE

Let S' be on the positive side of the centre C, such that $CS' = CS = ae$.

$\therefore$ The coordinate of S' are (ae, 0).

Let Z' be on the positive side of the centre C, such that $CZ' = CZ = \frac{a}{e}$.

Draw $K'_1K'_2$ perpendicular to ZZ' at Z'.

$\therefore$ The equation of the line $K'_1K'_2$ is $x = \frac{a}{e}$, *i.e.*, $x - \frac{a}{e} = 0$. Draw $PM' \perp K_1'K_2'$

Equation (5) is $x^2 + y^2 + 2aex + a^2e^2 = e^2x^2 + a^2 + 2aex$.

$\Rightarrow \quad x^2 + y^2 + a^2e^2 = e^2x^2 + a^2$

$\Rightarrow \quad x^2 + y^2 - 2aex + a^2e^2 = e^2x^2 + a^2 - 2aex$

$\Rightarrow \quad (x-ae)^2 + y^2 = e^2\left(x^2 + \frac{a^2}{e^2} - \frac{2ax}{e}\right)$

$\Rightarrow \quad (x-ae)^2 + y^2 = e^2\left(x - \frac{a}{e}\right)^2$

$\Rightarrow \quad \sqrt{(x-ae)^2 + (y-0)^2} = e\left|\frac{x-\frac{a}{e}}{\sqrt{1^2+0^2}}\right| \Rightarrow$ *PS' = ePM'*.

$\therefore$ For any point *P* on the ellipse, the distance of *P* from *S'* is *e* times its distance from $K_1'K_2'$.

$\therefore$ We would have obtained the same ellipse if we had started with focus *S'* and directrix $K_1'K_2'$ and with eccentricity *e*.

$\therefore$ **There exist a second focus and a second directrix for the ellipse.**

Remark. The *parametric equations* of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are $x = a\cos\theta$, $y = b\sin\theta$, where θ is the parameter.

A PROPERTY OF ELLIPSE

The sum of focal distances at any point on the ellipse is equal to the length of its major axis.

P(*x*, *y*) is a general point on the ellipse.

Sum of focal distance of *P*(*x*, *y*) = *PS* + *PS'* = *e PM* + *e PM'*

= *e*(*PM* + *PM'*) = *e*. *MM'* = *e*. *ZZ'*

$= e.\ 2CZ = 2e.\frac{a}{e} = 2a = AA'$

= length of major axis.

$\therefore$ **The sum of focal distance of any point on the ellipse is equal to the length of the major axis.**

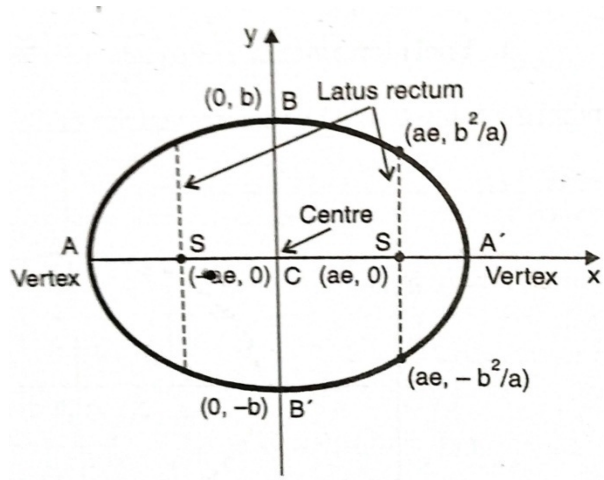
SOME DEFINITIONS RELATED TO AN ELLIPSE

The equation of an ellipse in the standard form is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a is some constant, $b=a\sqrt{1-e^2}$ and e is the eccentricity of the ellipse.

(i) *AA'* and *BB'* are respectively called the **major axis** and the **minor axis** of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Since *CA* = *a,* the major axis is equal to **2a.** Putting $x = 0$ in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get $y=\pm b$. Thus, coordinates of *B* and *B'* are (0, *b*) and (0, -*b*) respectively. Therefore, minor axis is of length **2b.** An ellipse is symmetric about its major axis and minor axis both.

(ii) The point of intersection of the major axis and the minor is called the **centre** of the ellipse. For the above ellipse, *C*(0,0) is the centre.

(iii) The points of intersection of the ellipse and its major axis are called the **vertices** of the ellipse. For the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the points *A* and *A'* are the vertices.



(iv) The double ordinate at a focus is called the **latus rectum** of the ellipse. For the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, a focus is (*ae,* 0).

Putting $x = ae$ in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get $\frac{(ae)^2}{a^2}+\frac{y^2}{b^2}=1$.

$$\Rightarrow \qquad \frac{y^2}{b^2}=1-e^2 \quad \Rightarrow \quad y=\pm b\sqrt{1-e^2}=\pm b\left(\frac{b}{a}\right)=\pm\frac{b^2}{a}.$$

∴ The coordinates of the double ordinate at the focus (ae, 0) are (ae, b^2/a) and (ae, $-b^2/a$).

∴ The latus rectum of the ellipse is equal to the distance between the points (ae, b^2/a) and (ae, $-b^2/a$) and this is equal to $2b^2/a$.

∴ Latus rectum = $\mathbf{2b^2/a}$.

Remark 1. For the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we have $b=a\sqrt{1-e^2}$.

$$\Rightarrow \quad b^2=a^2(1-e^2) \quad \Rightarrow \quad b^2=a^2-a^2e^2 \quad \Rightarrow \quad a^2e^2=a^2-b^2$$

$$\Rightarrow \quad e^2=\frac{a^2-b^2}{a^2} \quad \Rightarrow \quad e=\frac{\sqrt{a^2-b^2}}{a} \qquad (\because \ e \text{ is +ve})$$

Remark 2. Since $a\sqrt{1-e^2} < a$, we have $b < a$.

Remark 3. For any point (x, y) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

We have $\frac{x^2}{a^2}=1-\frac{y^2}{b^2}\leq 1$ *i.e.*, $x^2 \leq a^2$ or $-a \leq x \leq a$.

∴ The ellipse lies between the lines **x = -a** and **x = a** and touches these lines.

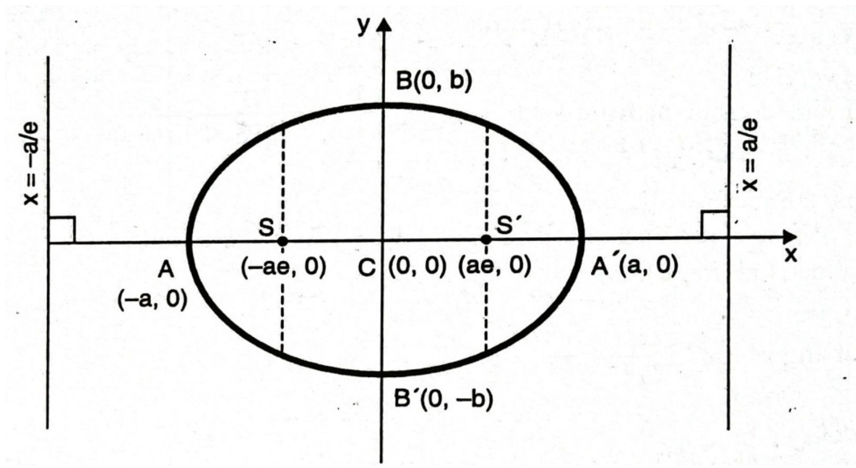
Also, $\frac{y^2}{b^2}=1-\frac{x^2}{a^2}\leq 1$ *i.e.*, $y^2 \leq b^2$ or $-b \leq y \leq b$.

∴ The ellipse lies between the lines **y = - b** and **y = b** and touches these lines.

TWO STANDARD FORMS OF ELLIPSE

There are two standard forms of ellipse with centre at the origin and axes along coordinate axes. The foci of the ellipse are either on the x-axis or on the y-axis.

1. **Foci along x-axis.** The equation of this type of ellipse is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a > b > 0$. If e be the eccentricity of this ellipse, then $b=a\sqrt{1-e^2}$.



For this ellipse :

(i) Centre : (0, 0)

(ii) Vertices : $(\pm a, 0)$

(iii) Foci : $(\pm ae, 0)$

(iv) Directrices : $x = \pm\frac{a}{e}$

(v) Major axis : $2a$

(vi) Minor axis : $2b$

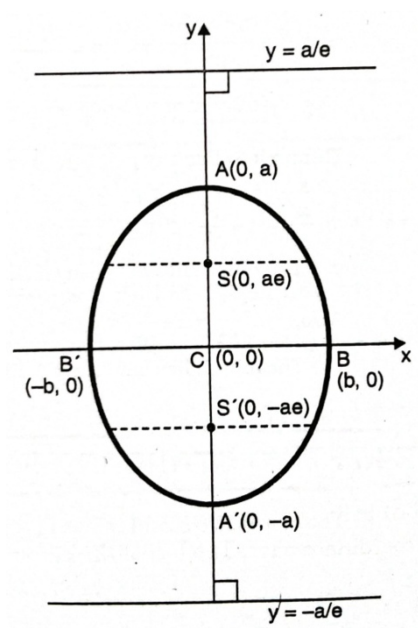
(vii) Equation of major axis : y =0

(viii) Equation of minor axis : x = 0

(ix) Latus rectum = $\frac{2b^2}{a}$

(x) Symmetry: It is symmetric about both axes.

2. **Foci along y – axis.** This equation of this type of ellipse is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a > b > 0$. If e be the eccentricity of this ellipse, then $b = a\sqrt{1-e^2}$.



For this ellipse:

i. Centre : (0, 0)
ii. Vertices : $(0, \pm\ a)$
iii. Foci : $(0, \pm\ ae)$
iv. Directrices : y = $\pm a/e$
v. Major axis :$2a$
vi. Minor axis : $2b$
vii. Equation of major axis : x = 0
viii. Equation of minor axis : y = 0

ix. Latus rectum = $\frac{2b^2}{a}$

x. Symmetry : It is symmetric about both axes.

Remark 1. The equation $\frac{x^2}{l}+\frac{y^2}{m}=1$ always represents an ellipse whenever $l \neq m$. If $l > m$, then the foci of the ellipse are along *x*- axis and if $l < m$, then foci of the ellipse are along y-axis.

Remark 2. If in a question, an ellipse is to be found out in the standard form then it is always **assumed that the foci of the ellipse are along x-axis,** unless the contrary is stated explicitly.

Example 2. *For the ellipse $x^2 + 3y^2 = a^2$, find the length of major and minor axes, foci, vertices and the eccentricity.*

Sol. The equation of the ellipse is $x^2 + 3y^2 = a^2$. $\therefore$ $\frac{x^2}{a^2}+\frac{y^2}{a^2/3}=1$. ...(1)

Since $a^2 > a^2/3$, the major axis, foci and vertices are along *x*-axis. Let $b^2 = a^2/3$.

$\therefore$ (1) $\Rightarrow$ $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\therefore$ Eccentricity, $e=\frac{\sqrt{a^2-b^2}}{a}=\frac{\sqrt{a^2-a^2/3}}{a}=\sqrt{\frac{2}{3}}=\sqrt{\frac{6}{3}}$, major axis = **2a,** minor axis $=2b=2\left(\frac{a}{\sqrt{3}}\right)=\frac{2\sqrt{3}}{3}a$, foci = $(\pm ae,0)=\left(\pm\frac{a\sqrt{6}}{3},0\right)$ and vertices = $(\pm a,0)$.

Example 3. *Find the equation of the ellipse whose foci are (-2, 3) and (2, 3) and whose semi-minor axis is $\sqrt{5}$.*

Sol. Given foci are *S* (-2, 3) and *S'*(2, 3).

$\therefore$ *SS'* = 4

Let 2*a*, 2*b*, *e* be the major axis, minor axis and eccentricity of the ellipse respectively.

$\therefore$ Using *SS'* = 2*ae*, we have 4 = 2*ae* or *ae* = 2.

Also $b^2=a^2(1-e^2)$ $\Rightarrow$ $b^2=a^2-a^2e^2$ $\Rightarrow$ $\left(\sqrt{5}\right)^2=a^2-(2)^2$

$\Rightarrow \quad a^2 = 9 \quad \Rightarrow \quad a = 3$

Let $P(x, y)$ be any point on the ellipse.

We know that the sum of focal distances of any point on the ellipse is equal to the length of the major axis.

$\therefore \quad PS + PS' = 2a$

$$\Rightarrow \quad \sqrt{(x+2)^2 + (y-3)^2} + \sqrt{(x-2)^2 + (y-3)^2} = 2(3)$$

$$\Rightarrow \quad \sqrt{x^2 + 4x + 4 + y^2 - 6y + 9} = 6 - \sqrt{x^2 - 4x + 4 + y^2 - 6y + 9}$$

$$\Rightarrow \quad x^2 + 4x + 4 + y^2 - 6y + 9 = 36 + \left(x^2 - 4x + 4 + y^2 - 6y + 9\right) - 12\sqrt{x^2 - 4x + 4 + y^2 - 6y + 9}$$

$$\Rightarrow \quad 12\sqrt{x^2 + y^2 - 4x - 6y + 13} = -8x + 36$$

$$\Rightarrow \quad 3\sqrt{x^2 + y^2 - 4x - 6y + 13} = -(2x - 9)$$

$$\Rightarrow \quad 9(x^2 + y^2 - 4x - 6y + 13) = (4x^2 - 36x + 81)$$

$\Rightarrow$ **$5x^2 + 9y^2 - 54y + 36 = 0$.**

WORKING RULES FOR SOLVING PROBLEMS

Rule I. *If l and m are unequal positive numbers, then the equation* $\frac{x^2}{l} + \frac{y^2}{m} = 1$ *always represents an ellipse.*

(i) *If $l > m$, then the foci of the ellipse are along the x-axis.*

(ii) *If $l < m$, then the foci of the ellipse are along the y-axis.*

Rule II. *For the ellipse,* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, *$a > b > 0$, we have :* $e = \frac{\sqrt{a^2 - b^2}}{a}$, *centre : (0, 0), vertices : (0, $\pm a$), foci : (0, $\pm ae$), directrices : $y = \pm a/e$, major axis =2a, minor axis = 2b, latus rectum = $2b^2/a$.*

Rule III. *For the ellipse,* $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, *a > b > 0, we have :* $e=\frac{\sqrt{a^2-b^2}}{a}$, *centre : (0, 0), vertices : (0, ±a), foci : (0, ±ae), directrices : y = ±a/e, major axis =2a, minor axis = 2b, latus rectum =* $2b^2/a$.

EXERCISE 24.1

SHORT ANSWER TYPE QUESTIONS

1. Find the equation of the ellipse whose focus, directrix and eccentricity are respectively :

   (i) (0, 3), $x + 7 = 0$ and $e = 1/3$ (ii) (4, 0), $y - 3 = 0$ and $e = 1/2$

   (iii) (-2, 3), $2x + 3y + 4 = 0$ and $e = 4/5$ (iv) (-1, 1), $x - y + 3 = 0$ and $e = 1/2$.

2. Find the equation of the set of all points whose distance from (0, 4) are $\frac{2}{3}$ of the distance from the line y = 9.

LONG ANSWER TYPE QUESTIONS

3. Find the eccentricity, foci, directrices, major axis, minor axis and latus rectum of the ellipse :

   (i) $\frac{x^2}{16}+\frac{y^2}{7}=1$ (ii) $\frac{x^2}{25}+\frac{y^2}{16}=1$ (iii) $\frac{x^2}{169}+\frac{y^2}{25}=1$.

4. Find the eccentricity, foci, directrices, major axis, minor axis and latus rectum of the ellipse :

   (i) $\frac{x^2}{4}+\frac{y^2}{9}=1$ (ii) $\frac{x^2}{7}+\frac{y^2}{11}=1$ (iii) $\frac{x^2}{225}+\frac{y^2}{289}=1$.

Answers

1. (i) $8x^2+9y^2-14x-54y+32=0$ (ii) $4x^2+3y^2-32x+6y+55=0$

(iii) $261x^2+181y^2-192xy+1044x-2334y+3969=0$

(iv) $7x^2+7y^2+2xy+10x-10y+7=0$ 2. $9x^2+5y^2=180$

3. (i) $\frac{3}{4}, (\pm 3, 0), x = \pm\frac{16}{3}, 8, 2\sqrt{7}, \frac{7}{2}$ (ii) $\frac{3}{5}, (\pm 3, 0), x = \pm\frac{25}{3}, 10, 8, \frac{32}{5}$

(iii) $\frac{12}{13}, (\pm 12, 0), x = \pm\frac{169}{12}, 26, 10\ \frac{50}{13}$

4. (i) $\frac{\sqrt{5}}{3}, (0, \pm\sqrt{5}), y = \pm\frac{9}{\sqrt{5}}, 6, 4, \frac{8}{3}$

(ii) $\frac{2\sqrt{11}}{11}, (0, \pm 2), y = \pm\frac{11}{2}, 2\sqrt{11}, 2\sqrt{7}, \frac{14}{\sqrt{11}}$

(iii) $\frac{8}{17}, (0, \pm 8), y = \pm\frac{289}{8}, 34, 30, \frac{450}{17}$

Example 4. *Find the equation of the ellipse whose axes are parallel to the coordinate axes having its centre at the point (2, -3), one focus at (3, -3) and one vertex at (4, -3).*

Sol. We have : centre = (2, -3), one focus = (3, -3), one vertex = (4, -3).

Let the origin be shifted to (2, -3) and let (X, Y) be the coordinates of the point (x, y) w.r.t. new axes.

$\therefore$ $x = 2 + X$ and $y = -3 + Y$

$\therefore$ **With respect to new axes:** centre = (2, -2, -3 + 3) = (0, 0),

one focus = (3, -2, -3 + 3) = (1, 0), one vertex = (4 – 2, - 3 + 3) = (2, 0).

Since one focus and one vertex are on X-axis, both foci are on X-axis.

Let the equation of the ellipse be $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$, where $a > b > 0$ and $b = a\sqrt{1-e^2}$.

Here foci are $(\pm ae, 0)$ and vertices are $(\pm a, 0)$.

Since $a > 0$, we have $a = 2$ and $ae = 1$.

Now $b = a\sqrt{1-e^2}$.

$\Rightarrow$ $b^2 = a^2(1-e^2) = a^2 - a^2e^2 = (2)^2 - (1)^2 = 3 \quad \Rightarrow b = \sqrt{3}$

$\therefore$ The equation of the ellipse is

$$\frac{X^2}{2^2}+\frac{y^2}{\left(\sqrt{3}\right)^2}=1 \quad \text{or} \quad \frac{X^2}{4}+\frac{Y^2}{3}=1.$$

We have $x=2+X$ and $y=-3+Y$

$\therefore$ **With respect to original axes,** the equation of the ellipse is

$$\frac{(x-2)^2}{4}+\frac{(x+3)^2}{3}=1.$$

EXERCISE 24.2

LONG ANSWER TYPE QUESTIONS

Show that the following equations represent ellipses. In each case, find centre, vertices, foci, eccentricity, directrices, latus rectum, major axis, minor axis, equation of major axis, equation of minor axis. Also rough sketches:

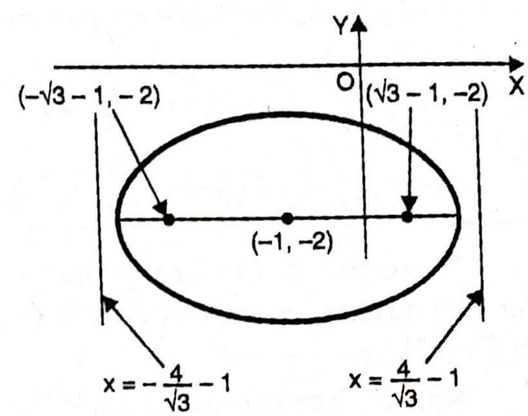
1. $x^2+4y^2+2x+16y+13=0$
2. $x^2+2y^2-2x+12y+10=0$
3. $25x^2+9y^2-150x-90y+225=0$.

Answers

1. (-1, -2), (-3, -2) and (1, -2),

(-$\sqrt{3}$ -1, -2) and ($\sqrt{3}$ -1, -2), $\frac{\sqrt{3}}{2}$,

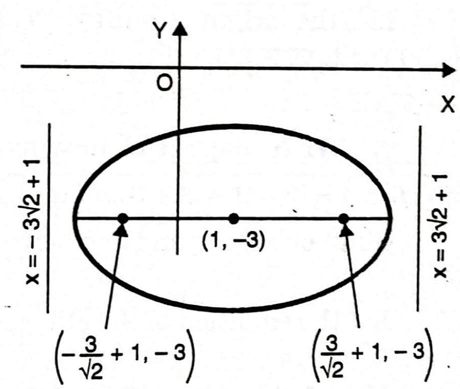
$x=\pm\frac{4}{\sqrt{3}}=-1$, 1, 4, 2, $y=-2, x=-1$



2. (1, -3), (-2, -3) and (4, -3),

$\left(-\frac{3}{\sqrt{2}}+1,-3\right)$ and $\left(\frac{3}{\sqrt{2}}+1,-3\right)\frac{1}{\sqrt{2}}$,

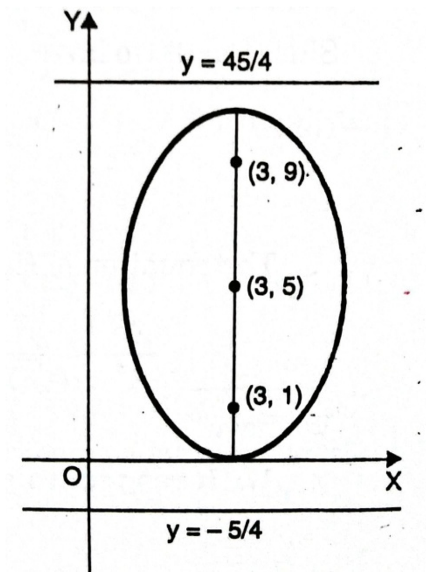
$x=\pm3\sqrt{2}+1$, 3, 6, $3\sqrt{2}$, $y=-3, x=1$



3. (3, 5), (3, 0) and (3, 10),

(3, 1) and (3, 9), $\frac{4}{5}, y = -\frac{5}{4}$ and $y = \frac{45}{4}$.

$\frac{18}{5}$, 10, 6, $x = 3$, y = 5.



SUMMARY

1. An **ellipse** is the locus of a point which moves so that its distance from a fixed point bears a constant ratio (less than one) to its distance from a fixed line.
2. The fixed point and the fixed line are respectively called the **focus** and the **directrix** of the ellipse. The constant ratio is called the **eccentricity** of the ellipse.

TEST YOURSELF

1. Find the equation of the ellipse whose focus, directrix and eccentricity are (1, -2), $3x - y + 1 = 0$ and $e = \frac{1}{\sqrt{2}}$ respectively.
2. Find the equation of the ellipse in the standard form whose minor axis is equal to the distance between foci and whose latus rectum is 10.
3. Find the equation of an ellipse, the distance between the foci is 8 units and the distance between the directrices is 18 units.

Answers

1. $11x^2 + 19y^2 + 6xy - 46x + 82y + 99 = 0$
2. $x^2 + 2y^2 = 100$
3. $5x^2 + 9y^2 = 180$.

SECTION – D

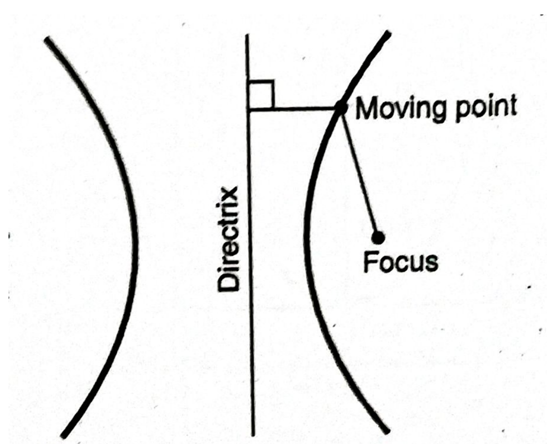
25. HYPERBOLAS

LEARNING OBJECTIVES

- Definition of a Hyperbola
- Equation of a Hyperbola in the General Form
- Equation of a Hyperbola in the Standard Form
- Existence of a Second Focus and a Second Directrix for the Hyperbola
- A Property of Hyperbola
- Some Definitions Related to a Hyperbola
- Two Standard Forms of Hyperbola
- Problem Based on Translation of Axes

DEFINITION OF A HYPERBOLA

A **hyperbola** is the locus of a point which moves so that its distance from a fixed point is in a constant ratio, greater than one, to its distance from a fixed line.



The fixed point is called the **focus** of the hyperbola. The fixed line is called the **directrix** of the hyperbola. The constant ratio (> 1) is called the **eccentricity** of the hyperbola and is denoted by e.

EQUATION OF A HYPERBOLA IN THE GENERAL FORM

Let $S(h, k)$ and $ax+by+c=0$ be the focus and directrix of a hyperbola respectively. Let $e(> 1)$ be the eccentricity of the hyperbola. Let $P(x, y)$ be a general point on the hyperbola.

$\therefore$ By definition, $PS = e$ (length of $\perp$ from P $ax+by+c=0$)

$$\therefore \quad \sqrt{(x-h)^2+(y-k)^2} = e\left|\frac{ax+by+c}{\sqrt{a^2+b^2}}\right|$$

$\Rightarrow$ $\mathbf{(a^2 + b^2) [(x\text{-}h)^2 + (y\text{-}k)^2] = e^2(ax + by + c)^2.}$

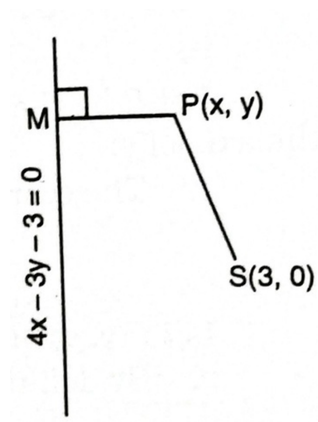
This is the equation of the required of the hyperbola,

Example 1. *Find the equation of the hyperbola, whose focus, directrix and eccentricity are respectively (3, 0), 4x – 3y = 3 and 5/4.*

Sol. The focus, directrix and eccentricity of the hyperbola are respectively, $S(3, 0)$, $4x - 3y - 3 = 0$ and5/4.

Let $P(x, y)$ be a general point on the hyperbola.

$\therefore$ Distance of P from the focus (3, 0) is equal to 5/4 times the distance of P from the directrix $4x-3y-3=0$.



$$\Rightarrow \quad PS = \frac{5}{4}PM$$

$$\Rightarrow \quad \sqrt{(x-3)^2+(y)^2} = \frac{5}{4}\left|\frac{4x-3y-3}{\sqrt{16+9}}\right|$$

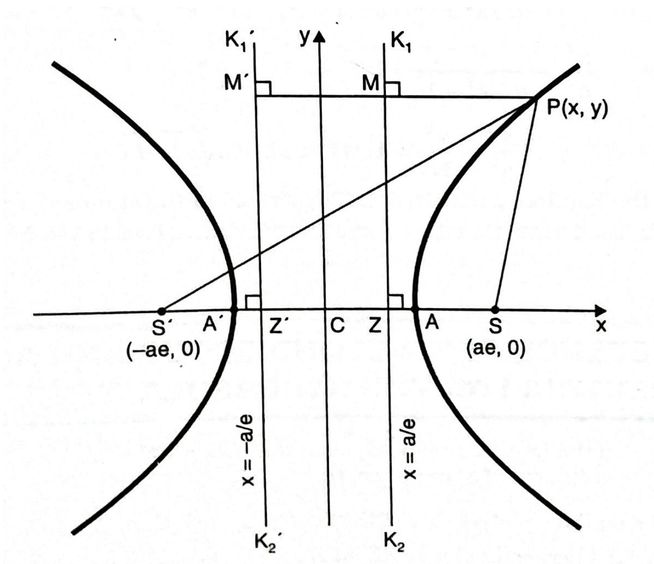
$$\Rightarrow \quad 16(x^2+9-6x+y^2) = 16x^2+9y^2+9-24xy-+18y-24x$$

$\Rightarrow$ $\mathbf{7y^2 + 24xy - 72x - 18y + 135 = 0.}$

This is the equation of the required hyperbola.

EQUATION OF A HYPERBOLA IN THE STANDARD FORM

Let S and K_1K_2 be the focus and the directrix of a hyperbola. Draw $SZ \perp K_1K_2$. Let $e(> 1)$ be the eccentricity of the hyperbola.



Let A be on SZ such that $SA = e\,AZ$ ....(1)

Produce SZ to A' such that $SA' = e\,A'Z$ ....(2)

By definition, A and A' lie on the hyperbola.

Let $AA' = 2a$ and let C be the middle point of AA'

(2) - (1) $\Rightarrow$ $SA' - SA = eA'Z - eAZ$

$\Rightarrow$ $AA' = e[(A'C + CZ) - (CA - CZ)]$ $\Rightarrow$ $2a = e.\,2CZ$

$\Rightarrow$ $CZ = \dfrac{a}{e}$ ...(3)

(2) + (1) $\Rightarrow$ $SA' + SA = eA'Z + eAZ$

$\Rightarrow$ $(A'C + CS) - (CS - AC) = e[(A'C + CZ) + (CA - CZ)]$

$\Rightarrow$ $2CS = e\,AA' = e\,.\,2a$ $\Rightarrow$ $CS = es$ ....(4)

Let C be the origin, CA, the axis of x and a line through C perpendicular to $A'A$, the axis of y.

$\therefore$ The coordinates of focus S are $(ae, 0)$ and the equation of the directrix K_1K_2 is

$$x = \frac{a}{e} \quad i.e., \quad x - \frac{a}{e} = 0 .$$

Let $P(x, y)$ be a general point on the hyperbola.

$\therefore$ By definition, $PS = ePM.$

$$\Rightarrow \qquad \sqrt{(x-ae)^2 + (y-o)^2} = e\left|\frac{x - \frac{a}{e}}{\sqrt{1^2 + 0^2}}\right|$$

$$\Rightarrow \qquad \sqrt{x^2 + a^2e^2 - 2aex + y^2} = e\left|x - \frac{a}{e}\right|$$

$$\Rightarrow \qquad x^2 + y^2 - 2aex + a^2e^2 = e^2\left(x - \frac{a}{e}\right)^2$$

$$\Rightarrow \qquad x^2 + y^2 - 2aex + a^2e^2 = e^2x^2 + a^2 - 2aex \qquad \ldots(5)$$

$$\Rightarrow \qquad (1-e^2)x^2 + y^2 = a^2(1-e^2) \quad \Rightarrow \quad (e^2-1)x^2 - y^2 = a^2(e^2-1)$$

$$\Rightarrow \qquad \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1$$

$$\Rightarrow \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ where } b = a\sqrt{e^2 - 1} .$$

This is the required equation of the hyperbola in the standard form.

Remark. The derivation of this equation is not required from the examination point of view.

EXISTENCE OF A SECOND FOCUS AND A SECOND DIRECTRIX FOR THE HYPERBOLA

Let *S'* be on the line *SC* produced, such that *CS'* = *CS* = *ae*.

∴ The coordinate of *S'* are (-ae, 0).

Let *Z'* be on the negative side of the centre *C*, such that *CZ'* = *CZ* = $\frac{a}{e}$.

Draw $K'_1K'_2$ perpendicular to *ZZ'* at *Z'*.

∴ The equation of the line $K'_1K'_2$ is $x = -\frac{a}{e}$,

i.e., $x + \frac{a}{e} = 0$. Draw $PM' \perp K_1'K_2'$

Equation (5) is $x^2 + y^2 - 2aex + a^2e^2 = e^2x^2 + a^2 + 2aex$.

$$\Rightarrow \quad x^2 + y^2 + a^2e^2 = e^2x^2 + a^2$$

$$\Rightarrow \quad x^2 + y^2 + 2aex + a^2e^2 = e^2x^2 + a^2 + 2aex$$

$$\Rightarrow \quad (x+ae)^2 + y^2 = e^2\left(x^2 + \frac{a^2}{e^2} + \frac{2ax}{e}\right)$$

$$\Rightarrow \quad (x+ae)^2 + y^2 = e^2\left(x + \frac{a}{e}\right)^2$$

$$\Rightarrow \quad \sqrt{(x+ae)^2 + (y-0)^2} = e\left|\frac{x + \frac{a}{e}}{\sqrt{1^2 + 0^2}}\right| \Rightarrow PS' = ePM'.$$

∴ For any point *P* on the hyperbola, the distance of *P* from *S'* is *e* times its distance from the line $K_1'K_2'$.

∴ We would have obtained the same hyperbola if we had started with focus *S'* and directrix $K_1'K_2'$ and with eccentricity *e*.

∴ **There exist a second focus and a second directrix for the ellipse.**

Remark. The *parametric equations* of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $x = a \sec\theta$, y = $b \tan\theta$, where θ is the parameter.

A PROPERTY OF HYPERBOLA

The difference of focal distances at any point on the hyperbola is equal to the length of its transverse axis.

P(*x, y*) is a general point on the hyperbola.

Difference of focal distance of *P*(*x, y*)

$= PS - PS' = e\,PM - e\,PM'$

$= e(PM - PM')$

$= e.\,MM' = e.\,ZZ' = e.\,2CZ = 2e.\frac{a}{e} = 2a = AA'$

= length of transverse axis.

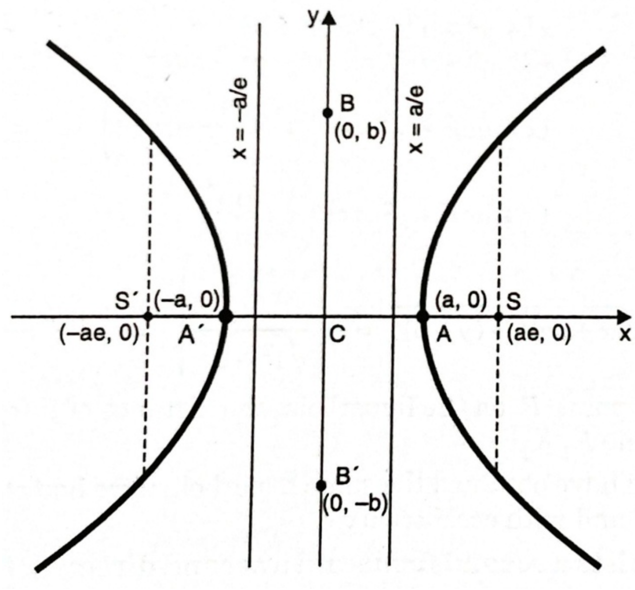
$\therefore$ **The difference of focal distance of any point on the hyperbola is equal to the length of the transverse axis.**

SOME DEFINITIONS RELATED TO A HYPERBOLA

The equation of a hyperbola in the standard form is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where *a* is some constant, $b=a\sqrt{e^2-1}$ and *e* is the eccentricity of the hyperbola.

(i) *AA'* and *BB'* are respectively called the **transverse axis** and the **conjugate axis** of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where *B*(0, *b*) and *B*(0, -*b*) are points on the y-axis.

Since *CA* = *a,* the transverse axis is equal to **2a.** Since BB' = 2b, the conjugate axis is equal to **2b.** A hyperbola is symmetric about its transverse axis and conjugate axis both.



(ii) The point of intersection of the transverse and conjugate axis is called the **centre** of the hyperbola. For the above hyperbola, *C*(0,0) is the centre.

(iii) The points of intersection of the hyperbola and its transverse axis are called the **vertices** of the hyperbola. For the hyperbola, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the points *A* and *A'* are the vertices.

(iv) The double ordinate at a focus is called the **latus rectum** of the hyperbla. For the hyperbola, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, a focus is (*ae,* 0).

Putting $x = ae$ in $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we get $\frac{(ae)^2}{a^2}-\frac{y^2}{b^2}=1$.

$$\Rightarrow \qquad \frac{y^2}{b^2}=e^2-1 \;\Rightarrow\; y=\pm b\sqrt{e^2-1}=\pm b\left(\frac{b}{a}\right)=\pm\frac{b^2}{a}.$$

∴ The coordinates of the double ordinate at the focus (*ae,* 0) are (ae, b^2/a) and (ae, $-b^2/a$).

∴ The latus rectum of the hyperbola is equal to the distance between the points (ae, b^2/a) and (ae, $-b^2/a$) and this is equal to $2b^2/a$.

∴ Latus rectum = **$2b^2/a$.**

Remark 1. For the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we have $b = a\sqrt{e^2 - 1}$.

$$\Rightarrow \quad b^2 = a^2(e^2 - 1) \quad \Rightarrow \quad b^2 = a^2e^2 - a^2 \quad \Rightarrow \quad a^2e^2 = a^2 + b^2$$

$$\Rightarrow \quad e^2 = \frac{a^2 + b^2}{a^2} \quad \Rightarrow \quad e = \frac{\sqrt{a^2 + b^2}}{a} \qquad (\because \ e \text{ is +ve})$$

Remark 2. Either $b < a$, or $b > a$.

For example, if e = 1.2, then $b = a\sqrt{(12)^2 - 1} = a\sqrt{0.44} < a$

If e = 1.8, then $b = a\sqrt{(18)^2 - 1} = a\sqrt{02.24} > a$

TWO STANDARD FORMS OF HYPERBOLA

There are two standard forms of hyperbola with centre at the origin and axes along coordinate axes. The foci of the hyperbola are either on the x-axis or on the y-axis.

1. **Foci along x-axis.** The equation of this type of hyperbola is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where a, b > 0. If e be the eccentricity of this hyperbola, then $b = a\sqrt{e^2 - 1}$.

For this hyperbola :

(i) Centre : (0, 0)

(ii) Vertices : $(\pm a, 0)$

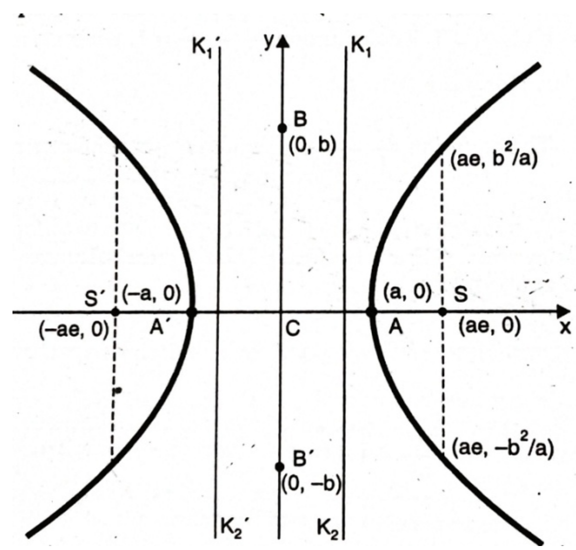
(iii) Foci : $(\pm ae, 0)$

(iv) Directrices : $x = \pm \frac{a}{e}$

(v) Transverse axis : $2a$

(vi) Transverse axis : $2b$

(vii) Equation of transverse axis : y =0



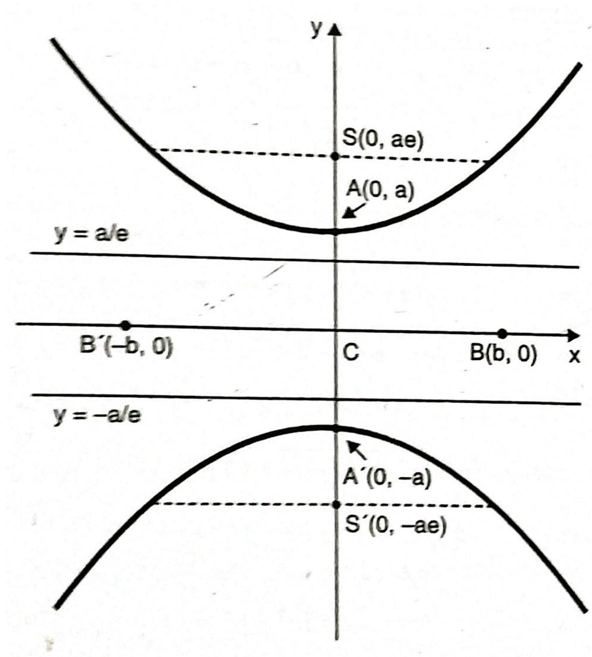
(viii) Equation of transverse axis : $x = 0$

(ix) Latus rectum = $\frac{2b^2}{a}$

(x) Symmetry: It is symmetric about both axes.

2. **Foci along y – axis.** This equation of this type of hyperbola is of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, where a ,b > 0. If e be the eccentricity of this hyperbola, then $b = a\sqrt{e^2 - 1}$.

For this hyperbola:



i. Centre : (0, 0)
ii. Vertices : $(0, \pm\ a)$
iii. Foci : $(0, \pm\ ae)$
iv. Directrices : y = $\pm a/e$
v. Transverse axis :$2a$
vi. Transverse axis : $2b$
vii. Equation of transverse axis : $x = 0$
viii. Equation of conjugate axis : y = 0
ix. Latus rectum = $\frac{2b^2}{a}$
x. Symmetry : It is symmetric about both axes.

Remark 1. The equation $\frac{x^2}{l} - \frac{y^2}{m} = 1$, where l, m > 0, always represents a hyperbola with foci x-axis.

The equation $\frac{y^2}{l} - \frac{x^2}{m} = 1$, where l, m > 0, always represents a hyperbola with foci along y – axis.

Remark 2. If in a question, a hyperbola is to be found out in the standard form then it is always **assumed that the foci of the hyperbola are along x-axis,** unless the contrary is stated explicitly.

Remark 3. For any point (x, y) on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we have $\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geq 1$ *i.e.,* $x^2 \geq a^2$ or $x \leq -a$ or $x \geq a$.

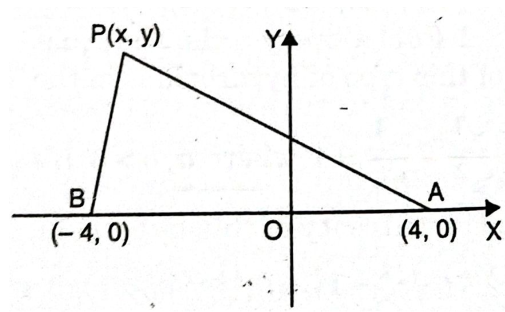
$\therefore$ The hyperbola has two branches, one in the half-plane **x ≤ - a** and the other in the half-plane **x ≥ a**.

Also, $\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1 \geq 1 - 1 = 0$ *i.e.,* $y^2 \geq 0$ or $-\infty < y < \infty$. $\left(\because x^2 \geq a^2 \Rightarrow \frac{x^2}{a^2} \geq 1\right)$

$\therefore$ The ordinate of a point on the hyperbola may have any real value.

Example 2. *For the equation of the set of all points such that the difference of their distances from (4, 0) and (-4, 0) is always equal to 2.*

Sol. Let P(x, y) be any point on the locus.



$\therefore \quad |PA - PB| = 2$

$\Rightarrow \quad PA - PB = \pm 2$

$\Rightarrow \quad PA = PB \pm 2$

$\Rightarrow$

$$\sqrt{(x-4)^2 + (y-0)^2} + \sqrt{(x+4)^2 + (y-0)^2} \pm 2$$

$$\Rightarrow \quad \sqrt{x^2 + y^2 - 8x + 16} = \sqrt{x^2 + y^2 + 8x + 16} \pm 2$$

$$\Rightarrow \quad x^2 + y^2 - 8x + 16 = \left(x^2 + y^2 + 8x + 16\right) + 4 \pm 4\sqrt{x^2 + y^2 + 8x + 16}$$

$$\Rightarrow \quad \pm 4\sqrt{x^2 + y^2 + 8x + 16} = -16x - 4$$

$$\Rightarrow \quad \pm\sqrt{x^2 + y^2 + 8x + 16} = -(4x + 1)$$

$$\Rightarrow \quad x^2 + y^2 + 8x + 16 = 16x^2 + 8x + 1$$

$$\Rightarrow \quad \mathbf{15x^2 - y^2 = 15.}$$

$\Rightarrow$ This is the equation of the locus. This represents a hyperbola.

WORKING RULES FOR SOLVING PROBLEMS

Rule I. *If l and m are positive numbers, then the equation* $\frac{x^2}{l}-\frac{y^2}{m}=1$ *and* $\frac{y^2}{l}-\frac{x^2}{m}=1$ *always represents hyperbolas.*

(i) *For the hyperbola* $\frac{x^2}{l}-\frac{y^2}{m}=1$, *the foci are along the x-axis.*

(ii) *For the hyperbola* $\frac{y^2}{l}-\frac{x^2}{m}=1$, *the foci are along the y-axis.*

Rule II. *For the hyperbola,* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, *a, b > 0, we have :* $e=\frac{\sqrt{a^2+b^2}}{a}$, *centre : (0, 0), vertices : (± a, 0), foci : (± ae, 0), directrices : x = ± a/e, transverse axis = 2a, conjugate axis = 2b, latus rectum =* $2b^2/a$.

Rule III. *For the hyperbola,* $\frac{x^2}{b^2}-\frac{y^2}{a^2}=1$, *where a > b > 0, we have :* $e=\frac{\sqrt{a^2+b^2}}{a}$,

centre : (0, 0), vertices : (0, ± a), foci : (0, ± ae), directrices : y = ± a/e, transverse axis =2a, conjugate axis = 2b, latus rectum = $2b^2/a$.

EXERCISE 25.1

SHORT ANSWER TYPE QUESTIONS

1. Find the equation of the hyperbola whose focus, directrix and eccentricity are respectively :

   (i) (0, 4), $y + 3 = 0$ and $e = 4/3$ (ii) (5, 0), $x – 4 = 0$ and $e = 2$

   (iii) (1, 2), $2x + y - 1 = 0$ and $e = \sqrt{3}$ (iv) (6, 0), $4x – 3y - 6 = 0$ and $e = 5/4$.

2. If the length of the transverse axis and conjugate axis are respectively 3 and 4, then find the equation of the corresponding hyperbola in the standard form.

LONG ANSWER TYPE QUESTIONS

3. Find the eccentricity, vertices, foci, directrices, transverse axis, conjugate axis and latus rectum of the hyperbola :

   (i) $\frac{x^2}{9}-\frac{y^2}{5}=1$ (ii) $\frac{x^2}{12}-\frac{y^2}{13}=1$ (iii) $4x^2-25y^2=100$.

4. Find the eccentricity, vertices. foci, directrices, transverse axis, conjugate axis and latus rectum of the hyperbola :

   (i) $\frac{y^2}{16}-\frac{x^2}{20}=1$ (ii) $\frac{y^2}{36}-\frac{x^2}{27}=1$ (iii) $16y^2-9x^2=144$.

Answers

1. (i) $9x^2-7y^2-168y=0$ (ii) $3x^2-y^2-22x+39=0$

   (iii) $7x^2-2y^2+12xy-2x+14y-22=0$ (iv) $7y^2+24xy-144x-36y+540=0$

2. $16x^2-9y^2=36$.

3. (i) $\sqrt{14}/3, (\pm 3,0), (\pm\sqrt{14},0), \sqrt{14}x-9=0$ and $\sqrt{14x}+9=0$, 6 units, $2\sqrt{5}$ units, 10/3 units.

   (ii) $5/2\sqrt{3}, (\pm 2\sqrt{3},0), (\pm 5,0), 5x\pm 12=0, 4\sqrt{3}$ units, $2\sqrt{13}$ units, $13/\sqrt{3}$ units.

   (iii) $\frac{\sqrt{29}}{5}, (\pm 5,0), \left(\pm\sqrt{29},0\right), x=\pm\frac{25}{\sqrt{29}}$, 10 units, 4 units, $\frac{8}{5}$ units.

4. (i) 3/2, (0, ±4), (0, ±6), 3y ±8 = 0, 8 units, $4\sqrt{5}$ units, 10 units

   (ii) $\sqrt{7}/2$ (0, ±6), (0, $\pm 3\sqrt{7}$), $\sqrt{7}$ y ±12 = 0, 12 units, $6\sqrt{3}$ units, 9 units

   (iii) $\frac{5}{3}, (0, \pm 3), (0,\pm 5), y=\pm\frac{9}{5}$, 6 units, 8 units, $\frac{32}{3}$ units.

PROBLEM BASED ON TRANSLATION OF AXES

Example 4. *Find the equation of the hyperbola whose foci are (6, 4) and (-4, 4) and eccentricity is 2.*

Sol. The foci are (6, 4) and (-4, 4).

$\therefore$ Centre = $\left(\frac{6+(-4)}{2}, \frac{4+4}{2}\right) = (1,4)$

Let the origin be shifted to (1, 4) and let (*X, Y*) be the coordinates of the point (*x, y*) w.r.t. new axes.

$\therefore$ $x = 1 + X$ and $y = 4 + Y$

$\therefore$ **With respect to new axes,**

Foci are = (6, -1, 4 – 4) = (5, 0) and (- 4 – 1, 4 - 4) = (-5, 0). These foci are on *X*-axis.

Let the equation of the hyperbola be

$$\frac{X^2}{a^2} - \frac{Y^2}{b^2} = \frac{1}{1}, \text{ where } a, b > 0 \text{ and } b = a\sqrt{e^2 - 1}.$$

$$e = 2 \quad \Rightarrow \quad b = a\sqrt{4-1} = \sqrt{3}a$$

Foci are (*ae*, 0) and (-*ae*, 0). $\therefore$ $ae = 5$ $(\because a, e > 0)$

$\Rightarrow$ $a(2) = 5 \quad \Rightarrow b = \frac{5}{2}$, Also $b = \sqrt{3}a = \sqrt{3}\left(\frac{5}{2}\right) = \frac{5\sqrt{3}}{2}$

$\therefore$ The hyperbola is $\frac{X^2}{\left(\frac{5}{2}\right)^2} - \frac{y^2}{\left(\frac{5\sqrt{3}}{2}\right)^2} = 1$ or $\frac{4X^2}{25} + \frac{4Y^2}{75} = 1$.

or $12X^2 – 4Y^2 = 75$.

We have $x = 1 + X$ and $y = 4 + Y$

$\therefore$ **With respect to original axes,** the equation of the hyperbola is

$$12(x-1)^2 - 4(y-4)^2 = 75$$

$\Rightarrow$ $12x^2 - 24x + 12 - 4y^2 + 32y - 64 - 75 = 0$

or $\mathbf{12x^2 - 4y^2 - 24x + 32y - 127 = 0.}$

EXERCISE 25.2

LONG ANSWER TYPE QUESTIONS

Show that the following equations represent hyperbolas. In each case, find centre, vertices, foci, eccentricity, directrices, latus rectum, transverse axis and conjugate axis:

1. $9x^2 - 16y^2 + 18x + 32y - 151 = 0$
2. $9x^2 - 16y^2 - 18x + 32y - 151 = 0$
3. $4x^2 - 5y^2 - 8x - 30y - 21 = 0$.
4. $4x^2 - y^2 + 8x + 6y + 11 = 0$

Answers

1. (-1, 1), (-5, 1) and (3, 1), (-6, 1) and (4, 1), $\frac{5}{4}$, $5x + 21 = 0$ and $5x - 11 = 0$, $\frac{9}{2}$ units, 8 units, 6 units
2. (1, 1), (-3, 11) and (5, 1), (-4, 1) and (6, 1), $\frac{5}{4}$, $5x + 11 = 0$ and $5x - 21 = 0$, $\frac{9}{2}$ units, 8 units, 6 units.
3. (1, -3), (1, -5) and (1, -1), (1, -8) and (1, 0), $\frac{3}{2}$, 3y + 13 = 0 and 3y + 5 = 0, 5 units, 4 units $2\sqrt{5}$ units.
4. (-1, 3), (-1, 1) and (-1, 7), (1, 3 -2$\sqrt{5}$) and (1, 3 + 2$\sqrt{5}$), $\frac{\sqrt{5}}{2}$, $y = 3 \pm \frac{8}{\sqrt{5}}$, 2 units, 8 units, 4 units.

SUMMARY

1. A **hyperbola** is the locus of a point which moves so that its distance a fixed point bears a constant ratio (greater than one) to its distance from a fixed line.
2. The fixed point and the fixed line are respectively called the **focus** and the **directrix** of the hyperbola. The constant ratio is called the **eccentricity** of the hyperbola.

TEST YOURSELF

1. Find the equation of the hyperbola whose focus, directrix and eccentricity are respectively:

   (i) (2, 0), $x - y = 0$ and $e = 2$ (ii) (2, 1), $x + 2y - 1 = 0$ and $e = 2$.

2. Find the axes, eccentricity, foci, directrices and length of latus rectum of the hyperbola $3x^2 - y^2 = 4$.
3. Find the equation of the hyperbola satisfying the following conditions :
   (i) One focus at (4, 2), centre at (6, 2) and $e = 2$.
   (ii) One focus at (5, 2), one vertex at (4, 2) and centre at (3, 2).

Answers

1. (i) $x^2 + y^2 - 4xy + 4x - 4 = 0$ (ii) $x^2 - 11y^2 - 16xy - 12x + 6y + 21 = 0$

2. $2, \left(\pm \frac{4}{\sqrt{3}}, 0\right), x = \pm \frac{1}{\sqrt{3}}, 4\sqrt{3}$

3.(i) $3x^2 - y^2 - 36x + 4y + 101 = 0$

(ii) $3x^2 - y^2 - 18x + 4y + 20 = 0$.

SECTION – D

26. POLAR COORDINATES

LEARNING OBJECTIVES

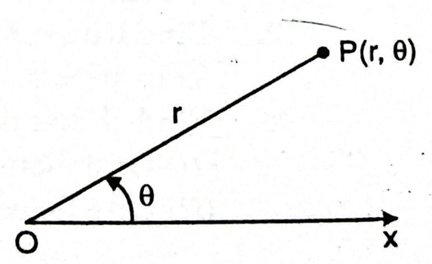
- Introduction
- Polar Coordinates System
- Conversion of Polar Coordinates to Cartesian Coordinates and Vice-Versa

INTRODUCTION

We are well versed with the rectangular system of coordinates. In this system a point is located by its distance from two perpendicular axes. There are various types of coordinate systems. In this chapter, we shall study a new type of coordinate system in which the coordinates of a point in a plane are its distance from a fixed point and its direction from a fixed line. This system of coordinates is called the **polar coordinates system.**

POLAR COORDINATES SYSTEM

Let O be a fixed point and OX a fixed line. The point O is called the **pole** (or **origin**) and the line OX is called the **initial line** (or **polar axis**). Let P be anypoint in the plane of the paper. We join OP. The position of the point P is clearly known when the directed angle XOP and the directed length OP are given. The directed angle θ is defined to be positive or negative according as it is measured counter clockwise or clockwise from the initial line OX. The directed distance



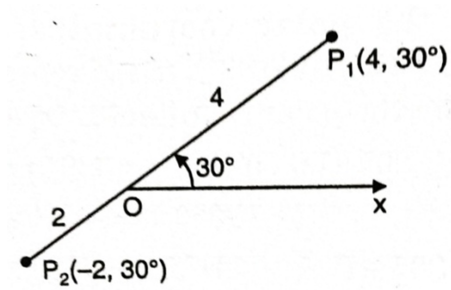
OP is defined as positive if measured from the initial OX. The directed distance

OP is defined as positive if measured from the pole along the terminal side of angle θ and negative if measured along the terminal side extended through the pole. Thus if θ and *r* be the directed angle *XOP* and length *OP* respectively then the **polar coordinates** of the point *P* are written as (r, θ). For a given point *P*, we have a pair (r, θ) of a polar coordinates. Conversely, given a pair (r, θ), we have a unique point in the plane.

The coordinates of the pole are $(0, \theta)$, where θ may be any angle. Thus there are infinitely many representations of the pole.

Illustrations:

1. In the given figure, angle *XOP* is 30^0 in the counter clockwise direction. Also, directed distance *OP* is 4.

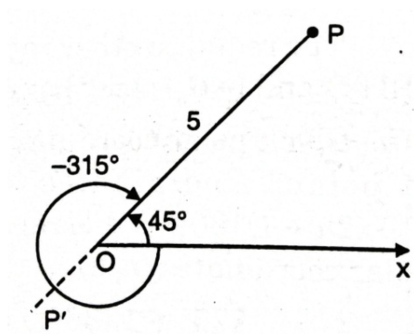


∴ The polar coordinates of the point P_1 are $(4, 30^0)$.

The directed distance OP_2 is -2 because the distance of P_2 from *O* is 2 and it lies on the terminal side of angle 30^0 extended through the pole.

∴ The polar coordinates of the point P_2 are $(-2, 30^0)$

2. In the given figure, angle *XOP* is 45^0 in the counter clockwise direction. Also, directed distance *OP* is 5.



∴ The polar coordinates of the point *P* are $(5, 405^0)$.

The angle *XOP* in the clockwise direction is 315^0.

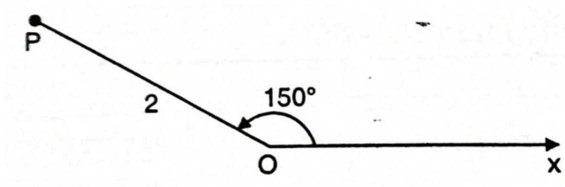
∴ The polar coordinates of the point *P* are $(5, -315^0)$.

The angle *XOP'* in the clockwise direction may also be considered as 225^0 $(=45^0 + 180^0)$.

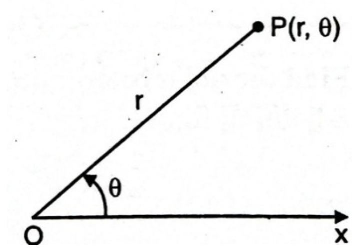
∴ The polar coordinates of the point *P* can also be written as $(-5, 225^0)$.

Thus we see that the polar coordinates of a given point are not unique.

3. Let directed angle of point P be 150^0 and the directed distance of P be 2. The polar coordinates of P can also be expressed by any of the following pairs : $(2,\ 150^0)$, $(2,\ -210^0)$, $(-2,\ 330^0)$, $(-2,\ -30^0)$.



4. Let $(r,\ \theta)$ be the polar coordinates of a point P. Adding 360^0 or any multiple of 360^0 to the directed angle θ does not alter the final position of the revolving line. Thus, the polar coordinates of P can also be given as $(r_2\ \theta + n.\ 360^0)$, where $n \in Z$.

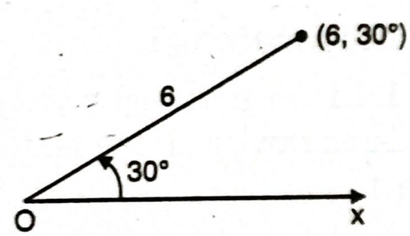


   Adding 180^0 or any odd multiple of 180^0 to the directed angle θ we get the final position of the revolving lien which is same the terminal side of θ extended through the pole.

$\therefore$ The polar coordinate of the point P can be written as $(-r,\ \theta + (2n + 1)180^0)$, where $n \in Z$.

Example 1. *Write three other pairs of polar coordinates for the points represented by the following pairs fo coordinates, restricting the directed angle to numerical values not exceeding 360^0:*

(i) *$(6,\ 30^0)$* (ii) *$(-4,\ 120^0)$.*



Sol. (i) Given polar coordinates are $(6,\ 30^0)$. We know that the polar coordinates $(r,\ \theta + n.360^0)$ and $(-r,\ \theta + (2n + 1)180^0)$, $n \in N$ represent the same point as that by the polar coordinates $(r,\ \theta)$.

$30^0 + 1.(360^0) = 390^0$ and $|\ 390\ | = 390 > 360$

$30^0 + (-1)(360^0) = -330^0$ and $|\ -330\ | = 330 < 360$

$30^0 + (2(1) + 1)180^0 = 570^0$ and $|\ 570\ | = 570 > 360$

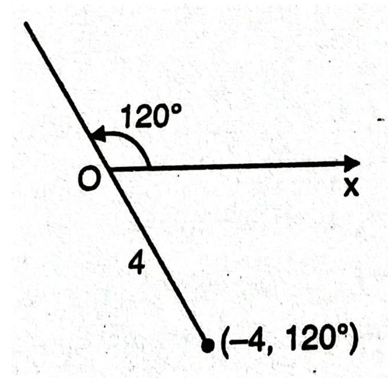
$30^0 + (2(0) + 1)180^0 = 210^0$ and $|\ 210\ | = 210 < 360$

$30^0 + (2(-1) + 1)180^0 = -150^0$ and $|-510\ | = 150 < 360$

$30^0 + (2(-2) + 1)180^0 = -510^0$ and $|-510\ | = 510 > 360$

$\therefore$ The required other representations of given polar coordinates are **(6, -330⁰), (-6, 210⁰)** and **(-6, -150⁰).**

(ii) Given polar coordinates are $(-4, 120^0)$. We know that the polar coordinates $(r, \theta + n.\, 360^0)$ and $(-r, \theta + (2n + 1)180^0)$, $n \in N$ represent the same point as that by the polar coordinates (r, θ).



$$120^0 + 1.(360^0) = 480^0 \text{ and } |480| = 480 > 360$$

$$10^0 + (-1)(360^0) = -240^0 \text{ and } |-240| = 240 < 360$$

$$120^0 + (2(1) + 1)180^0 = 660^0 \text{ and } |660| = 660 > 360$$

$$120^0 + (2(0) + 1)180^0 = 300^0 \text{ and } |300| = 300 < 360$$

$$120^0 + (2(-1) + 1)180^0 = -60^0 \text{ and } |-60| = 60 < 360$$

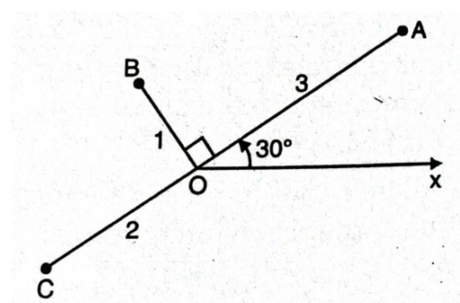
$$120^0 + (2(-2) + 1)\, 180^0 = -420^0 \text{ and } |-420| = 420 > 360.$$

$\therefore$ The required other representations of given polar coordinates are **(-4, -240⁰), (4, 300⁰) and (4, -60⁰).**

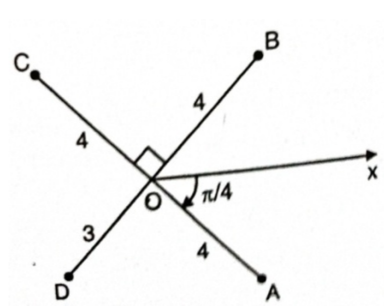
EXERCISE 26.1

SHORT ANSWER TYPE QUESTIONS

1. Find the polar coordinates of the points given in the adjoining figure.



2. Find the polar coordinates of the points given in the adjoining figure.



Answers

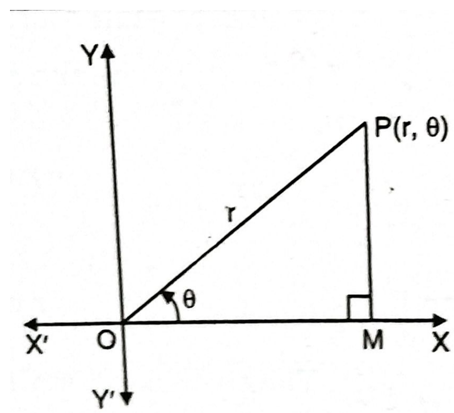
1. $A(3, 30^0)$, $B(1, 120^0)$, $C(2, 210^0)$

2. $A(4, -\pi/4)$, $B(4, \pi/4)$, $C(4, 3\pi/4)$, $D(3, 5\pi/4$.

CONVERSION OF POLAR COORDINATES TO CARTESIAN COORDINATES AND VICE-VERSA

Let (r, θ) be the polar coordinates of a point P. Draw OY perpendicular to OX. We extend XO to X' and YO to Y'.

$\therefore$ Considering XOX' as x-axis and YOY' as y-axis, we get a system of rectangular coordinates.



Draw PM perpendicular to x-axis.

$$\therefore \quad \frac{OM}{OP} = \cos\theta \quad \Rightarrow \quad OM = OP\cos\theta = r\cos\theta$$

and $$\frac{MP}{OP} = \sin\theta \quad \Rightarrow \quad MP = OP\sin\theta = r\sin\theta.$$

$\therefore$ The cartesian coordinates of the point $P(r, \theta)$ are $(r\cos\theta, r\sin\theta)$.

Thus if the cartesian coordinates of P are denoted by (x, y) then we have

$$x = r\cos\theta \qquad \ldots(1)$$

and $$y = r\sin\theta \qquad \ldots(2)$$

Relations (1) and (2) are used to find the cartesian coordinates of a point if its polar coordinates are given.

Squaring (1) and (2) and adding, we get

$$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$$

$$\therefore \quad r = \pm\sqrt{x^2 + y^2} \qquad \ldots(3)$$

Also by dividing (2) by (1), we get

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta}$$

$$\therefore \qquad \tan\theta = \frac{y}{x} \quad \Rightarrow \quad \theta = \tan^{-1} = \frac{y}{x} \qquad \ldots(4)$$

The value of θ as given by (4) is not single value. Hence it is necessary to select an appropriate value for θ when applying this formula to find this coordinate of point.

Relations (3) and (4) are used to find the polar coordinates of a point if its cartesian coordinates are given.

Example 2. *Find the cartesian coordinates of the points whose polar coordinates are :*

(i) *(3, 60^0)* (ii) *(-4, $\pi/6$)*

(iii) *(6, -2$\pi/3$)* (iv) *(-5, $-5\pi/6$).*

Sol. (i) Let (x, y) be the cartesian coordinates of the point (3, 60^0).

$$\therefore \qquad x = r\cos\theta = 3\cos 60^0 = 3\left(\frac{1}{2}\right) = \frac{3}{2}$$

$$\text{and} \qquad y = r\sin\theta = 3\sin 60^0 = 3\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2}.$$

$\therefore$ The cartesian coordinates of the given point are (3/2, $3\sqrt{3}$ /2).

(ii) Let (x, y) be the cartesian coordinates of the point (- 4, $\pi\sqrt{6}$).

$$\therefore \qquad x = r\cos\theta = -4\cos\frac{\pi}{6} = -4\left(\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$$

$$\text{and} \qquad y = r\sin\theta = -4\sin\frac{\pi}{6} = -4\left(\frac{1}{2}\right) = -2.$$

$\therefore$ The cartesian coordinates of the given point are ($-2\sqrt{3}$, -2).

(iii) Let (x, y) be the cartesian coordinates of the point (6, -2 $\pi/3$).

$$\therefore\ x = r\cos\theta = 6\cos\left(-\frac{2\pi}{3}\right) = 6\cos\frac{2\pi}{3} = 6\cos\left(\pi - \frac{\pi}{3}\right) = -6\cos\frac{\pi}{3}$$

$$= -6\left(\frac{1}{6}\right) = -3$$

and $$y = r\sin\theta = 6\sin\left(-\frac{2\pi}{3}\right) = -6\sin\frac{2\pi}{3} = -6\sin\left(-\pi - \frac{\pi}{3}\right) = -6\cos\frac{\pi}{3}.$$

$$= -6\left(\frac{\sqrt{3}}{2}\right) = -3\sqrt{3}$$

$\therefore$ The cartesian coordinates of the given point are ($-3, -3\sqrt{3}$).

(iv) Let (x, y) be the cartesian coordinates of the point (-5, -5 π/6).

$$\therefore\quad x = r\cos\theta = -5\cos\left(-\frac{5\pi}{6}\right) = -5\cos\frac{5\pi}{3} = -5\cos\left(\pi - \frac{\pi}{6}\right)$$

$$= (-5)\left(-\cos\frac{\pi}{6}\right) = 5\left(\frac{\sqrt{3}}{2}\right) = \frac{5\sqrt{3}}{2}$$

and $$y = r\sin\theta = -5\sin\left(-\frac{5\pi}{6}\right) = 5\sin\frac{5\pi}{6} = 5\sin\left(\pi - \frac{\pi}{6}\right).$$

$$= 5\sin\frac{\pi}{6} = 5\left(\frac{1}{2}\right) = \frac{5}{2}$$

$\therefore$ The cartesian coordinates of the given point are ($5\sqrt{3}$/2, 5/2).

EXERCISE 26.2

SHORT ANSWER TYPE QUESTIONS

1. Find the cartesian coordinates of the points whose polar coordinates are:

(i) (5, 0^0) (ii) (4, 90^0) (iii) (3, 30^0)

(iv) (7, 450^0) (v) (-3, 120^0) (vi) (-5, 270^0)

(vii) (2, -150^0) (viii) (3, -420^0) (ix) (-2, -135^0)

(x) (-8, -390^0).

2. Find the polar coordinates of the points whose cartesian coordinates are:

(i) (2, 0) (ii) (0, 4) (iii) $\left(\frac{3\sqrt{2}}{2}, \frac{3}{2}\right)$

(iv) (1, 1)

3. Find the distance between the given pairs of points.

(i) $(2, 30^0)$ and $(4, 120^0)$ (ii) $(-3, 45^0)$ and $(7, 105^0)$.

4. Find the area of the triangle whose vertices are:

(i) $(1, 30^0)$, $(2, 60^0)$ and $(3, 90^0)$ (ii) $(-3, -30^0)$, $(5, 150^0)$ and $(7, 210^0)$.

Answers

1. (i) (5, 0) (ii) (0, 4) (iii) $\left(\frac{3\sqrt{2}}{2}, \frac{3}{2}\right)$

(iv) (0, 7) (v) $\left(\frac{3}{2}, -\frac{3\sqrt{2}}{2}\right)$ (vi) (0, 5)

(vii) $(-\sqrt{3}, -1)$ (viii) $\left(\frac{3}{2}, -\frac{3\sqrt{2}}{2}\right)$ (ix) $(\sqrt{2}, \sqrt{2})$

(x) $(-4\sqrt{3}, 4)$

2. (i) $(2, 0^0)$ (ii) $(4, 90^0)$ (iii) $(3, 30^0)$

(iv) $(\sqrt{2}, 45^0)$

3. (i) $2\sqrt{5}$ units (ii) $\sqrt{79}$ units

4. (i) $\frac{1}{4}(8-3\sqrt{3})$ sq. units (ii) $\frac{7\sqrt{3}}{2}$ sq. units.

SUMMARY

1. In the **polar coordinates system** the coordinates of a point in a plane are its distance from a fixed point and its direction from a fixed line.
2. If the cartesian coordinates of the point $(r,\ \theta)$ are (x, y) then $x = r\cos\ \theta$ and $y = r\sin\theta$.
3. If the polar coordinates of the point (x, y) are (r, θ), then

$$r = \pm\sqrt{x^2 + y^2} \quad \text{and} \quad \theta \tan^{-1}\frac{y}{x}$$

TEST YOURSELF

1. Find the polar coordinates equation corresponding to the equation $x + 2y = 6$.
2. Find the polar coordinates equation corresponding to the equation $y^2 = 6x$.
3. Find the polar coordinates equation corresponding to the equation $xy = 4$.
4. Find the cartesian coordinates equation corresponding to the equation $r = 2$.
5. Find the cartesian coordinates equation corresponding to the equation $\theta = 60^0$.
6. Find the cartesian coordinates equation to the equation $r = \dfrac{4}{1 - 2\cos\theta}$.

Answers

1. $r(\cos\theta + 2\sin\theta) = 6$
2. $r = 6\cos\theta\cos ec^2\theta$
3. $r^2 = 8\cos ec2\theta$
4. $x^2 + y^2 = 4$
5. $y - \sqrt{3}x = 0$
6. $3x^2 - y^2 + 16x + 16 = 0$.